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A GEOMETRICAL APPROXIMATION TO THE ROOTS OF NUMBERS*

J. M. BARBOUR, Michigan State University

Origin of problem. By equal temperament is meant the division of the musical octave into twelve equal parts or semitones. Since the ratio of the octave is 2:1, the ratio of the equally tempered semitone is $2^{1/12}:1$. The piano-tuner, who operates with intervals of five and seven semitones, must be cognizant of their ratios, but only to the extent that he must know how fast one of these intervals should beat in a given part of the keyboard. He tunes by ear, not by mathematics. His problem is different in kind from that of the 16th century musical theorist, who wished to show players upon the lute and viol how to place the frets upon their instruments in order to have equal semitones. For the latter, therefore, equal temperament was the geometrical problem of inserting eleven mean proportionals between 2 and 1. A more general, but equivalent, problem was the division of the string of a monochord into equal semitones. By forming unisons with the pitches thus obtained, the strings of a harpsichord or the pipes of an organ could be tuned.

Since $12=3\times 2\times 2$, the problem of equal temperament is essentially the problem of the duplication of the cube, a problem which the ancient Greeks were unable to solve by Euclidean methods. Several of the methods which they did find suitable, such as a circle and a variable secant, a circle and a parabola, etc., were repeated by writers of the 16th and 17th centuries. The most popular of all was the mesolabium, a mechanical device with sliding parallelograms, ascribed to Archimedes [1].

Mersenne's approximation. Père Mersenne gave an ingenious and excellent geometrical approximation for equal temperament, using only the well-known Euclidean method for inserting a single mean between two lines [2], [1]. By this method he first obtained a major third with the ratio of $(3-\sqrt{2}):2$. Since the major third contains four semitones, it was then easy to insert three mean proportionals within it, and similarly to insert seven proportionals between the major third and the octave. In decimals his ratio for the major third was .7929, which is .1% shorter than the correct value for the tempered third, .7939, and this, of course, was the maximum error in the entire construction. Table 1 gives the string-lengths for equal temperament and Table 2 is Mersenne's approximation. The error is expressed as the logarithm of the quotient of the approximation by the correct value.

Galilei's approximation. Vincenzo Galilei, father of the astronomer, gave an approximation that was less accurate than Mersenne's, but much simpler to construct. He took 18:17 as the ratio of the tempered semitone on the lute, and

* The first part of this article was presented, with the title "An eighteenth century approximation to the equally tempered scale," to the Michigan Section of the Mathematical Association of America, April 24, 1951. The author wishes to thank Professor V. G. Grove for his assistance in both the initial and final stages of this study.

TABLE 1 EQUAL TEMPERAMENT			TABLE 2 MERSENNE'S APPROXIMATION		TABLE 3 GALILEI'S APPROXIMATION	
Notes	String-lengths	*	String-lengths	$10^5 \times \log$ error	String-lengths	$10^5 \times \log$ error
C	100000	2	100000	0	100000	0
C♯	94387	$2^{11/12}$	94365	−11	94444	26
D	89090	$2^{5/16}$	89045	−22	89197	52
E♭	84090	$2^{3/4}$	84028	−33	84242	79
E	79370	$2^{2/3}$	79290	−44	79562	105
F	74915	$2^{7/12}$	74850	−38	75142	131
F♯	70711	$2^{1/2}$	70658	−33	70967	157
G	66742	$2^{5/12}$	66700	−27	67024	183
G♯	62996	$2^{1/3}$	62966	−22	63301	210
A	59460	$2^{1/4}$	59439	−16	59784	236
B♭	56123	$2^{1/6}$	56110	−11	56463	262
B	52973	$2^{1/12}$	52968	−5	53326	288
C	50000	1	50000	0	50000	0

* In Table 1, the string-lengths are 50000 times the mean proportionals in the third column.

then found the remaining semitones by proportion [3], [1]. However, $(17/18)^{12} = .50363$, which is .7% too long. An easy way to reduce this error is to take twice the excess, or .00726, from the opposite end of the string, thus making Galilei's 12th fret the exact middle. With this correction, the maximum error, for the augmented fourth of six semitones, is .07%. Even without the correction, this is a very effective practical method, and, as such, it was referred to in the literature for two and a half centuries.

Strähle's approximation. In 1743 Daniel P. Strähle presented a curious and easy geometrical approximation to equal temperament [4]. This, as the present author has described it, was his method (Fig. 1): "Upon the line QR, 12 units in length, erect an isosceles triangle, QOR, its equal legs being 24 units in length. Join O to each of the 11 points of division in the base. On QO locate P, 7 units from Q, and draw RP, extending it its own length to M. Then, if RM represents the fundamental pitch and PM its octave, the points of intersection of RP with the 11 rays from O will be the 11 semitones within the octave" [1].

Strähle had apparently arrived at his approximation by intuition, for its testing was left to Jacob Faggot in the second part of Strähle's article. Faggot (1699–1777), geometer and pioneer political economist, had been a charter member of the Swedish Academy at its founding in 1739, was for three years its secretary, had 18 articles published in its *Proceedings*, the last in 1770, and in 1776 was ranked No. 4 in the Academy, with Carl Linneaus as No. 2. Faggot, the distinguished savant, proceeded to work out by trigonometry the string-lengths for Strähle's approximation.

Faggot's trigonometric solution of Strähle's construction. The trigonometric solution was not difficult: cos $\angle OQR = 6/24$, whence $\angle OQR = 75°\ 31'$. Then $\triangle$ PQR, with two sides and the included angle known, was solved by the law of tangents. All of the other angles formed between QR and one of the rays from O could also be found directly; *e.g.*, ctg $\angle(O\ \mathrm{II}\ R) = 6\sqrt{15}/5$, whence $\angle O\ \mathrm{II}\ R = 77°\ 52'$. Therefore, each of the 11 triangles having its base along QR and a leg along PR could be solved by the law of sines, since both its base and its angles were known. However, there was one little difficulty. Faggot had computed $\angle PRQ$ as $40°\ 14'$, whereas it should have been $33°\ 32'$. If he had made a mistake in any of the other angles, he might have caught it. But this was fatal, since $\angle PRQ$ was used in the solution of each of the other triangles, and exerted its baleful influence impartially upon them all. (This error in the size of $\angle PRQ$ was equivalent to using 8.605 for PQ in place of 7.)

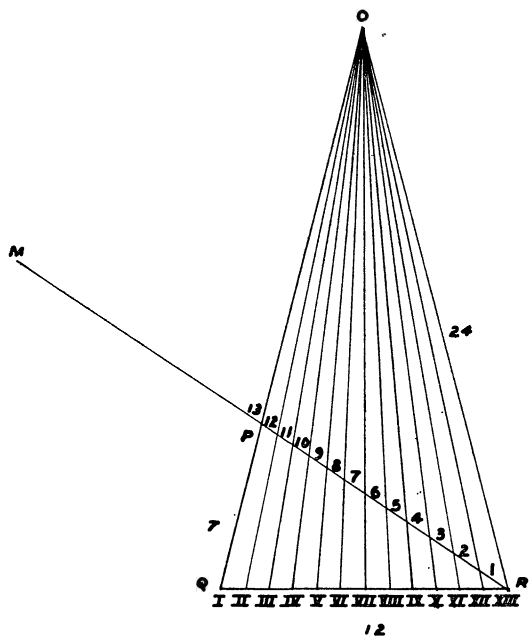

FIG. 1

After having computed all the lengths from R along MR, Faggot subtracted each of them from MR $(=2PR)$ and then divided by the length of MR. As can be seen from Table 4, Faggot did not put Strähle's method in a very good light, for the maximum error is about 1.7%, which is five times as great as a musician's ear would consider acceptable.

Algebraic solution of Strähle's construction. A corrected trigonometric solution of Strähle's approximation is unnecessary, for this problem can be

solved also by algebraic methods, and the latter solution lends itself better to generalization as well as to numerical computation. Using the same basic figure as before (Fig. 2), let C be any point on AR. Draw OC, cutting PR at D. Draw $DJ \perp AR$. Then triangles OAC and DJC are similar and so are triangles BAR and DJR. From this condition of similitude it follows that $DJ/OA = JC/AC$; $DJ/BA = JR/AR = DR/BR$. When DJ is eliminated from the above equations,

$$JR/AR = JC \times OA/BA \times AC.$$

Substitute $(JR-CR)$ for JC and $(AR-CR)$ for AC:

$$JR/AR = OA(JR - CR)/BA(AR - CR).$$

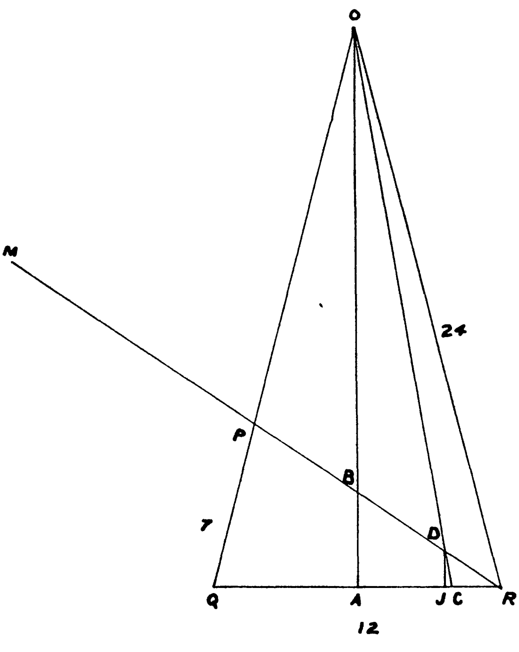

FIG. 2

Cross-multiply, collect terms, and solve for JR:

$$JR = CR \times OA \times AR/\{AR(OA - BA) + BA \times CR\}.$$

Then

$$DR/BR(= JR/AR) = CR \times OA/\{AR(OA - BA) + BA \times CR\}.$$

If m is the fractional power to be approximated, $CR = 1 - m$ and $AR = \frac{1}{2}$. Let $OA = a$ and $BA = b$.

Substituting these values in the above equation and simplifying the denominator:

$$DR/BR = 2a(1 - m)/\{a + b - 2bm\}.$$

When $m=0$, $PR/BR=2a/(a+b)$. When N (the number whose root is to be approximated) $=2$, $MP=PR$ and $MD=2PR-DR$. Hence

$$(1)\ 2^m \doteq MD/MP = \left(\frac{4a}{a+b} - \frac{2a(1-m)}{a+b-2bm}\right)\left(\frac{a+b}{2a}\right) = \frac{a+b+(a-3b)m}{a+b-2bm}.$$

To fit Strähle's values into the above formula, let us turn again to the original diagram, and drop a perpendicular PE from P to QR (Fig. 3). Then $QE=7/4$; $ER=41/4$; $PR/BR=41/24$. From this last relation, $2a/(a+b)=41/24$, $a=41$, $b=7$.

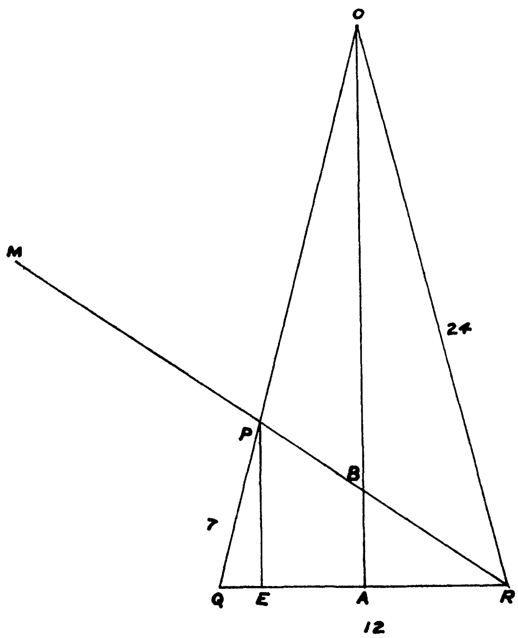

Fig. 3

Substituting these values for a and b in formula (1),

$$(2) \qquad 2^m \doteq \frac{24+10m}{24-7m}.$$

Then, for

$$m = 1,\ 11/12,\ 10/12,\ \cdots,\ 0,\ 2^m \doteq \frac{408}{204}, \frac{398}{211}, \frac{388}{218}, \cdots, \frac{288}{288}.$$

In order to compare with Faggot's incorrect table for Strähle's approximation, Table 5 shows the above fractions multiplied by 5000.

It is unfortunate that Faggot's erroneous figures were copied, with no explanation of Strähle's method itself, in Marpurg's excellent treatise on musical temperament [5]. Since Marpurg had not troubled himself to check Faggot's computations, he was naturally supercilious regarding the method. As can be seen from Table 5, Strähle's greatest errors are about $-.11\%$ and $.15\%$. It will be noted that the pattern of the errors resembles somewhat a sine curve, rising above the axis for small values of m, passing through zero almost at the midpoint, and then dipping similarly below the axis. Neither arch of the curve, however, is completely symmetrical.

TABLE 4

FAGGOT'S NUMERICAL VALUES FOR STRÄHLE'S APPROXIMATION

String-lengths	$10^5\times\log$ error
10000	0
9379	−276
8811	−479
8290	−619
7809	−706
7365	−740
6953	−732
6570	−683
6213	−601
5881	−478
5568	−344
5274	−192
5000	0

TABLE 5

CORRECTED NUMERICAL VALUES FOR STRÄHLE'S APPROXIMATION

String-lengths	$10^5\times\log$ error
10000	0
9432	−32
8899	−49
8400	−46
7931	−33
7490	−9
7073	11
6680	38
6308	58
5955	65
5620	60
5302	38
5000	0

TABLE 6

APPROXIMATION FOR EQUAL TEMPERAMENT, USING CORRECT VALUE FOR $\sqrt{2}$

String-lengths	$10^5\times\log$ error
100000	0
94305	−37
88978	−55
83981	−56
79290	−44
74874	−24
70711	0
66780	24
63061	44
59538	56
56195	55
53020	37
50000	0

Strähle's approximation generalized. Although Strähle's approximation was devised for a specific purpose, *i.e.*, to find the successive powers of the 12th root of 2, it should be possible to employ a modification of it to find the roots of numbers other than 2. To do this we shall need more than the inspired guess by which Strähle made $PQ/OQ=7/24$. The necessary clue comes from the fact that his construction is very nearly perfect for $\sqrt{2}$, his fraction being $58/41=1.41463$, which is only $.03\%$ too high. If $\angle MRQ$ can be adjusted so that MB is precisely the mean proportional between MP and MR, this will be the best approximation by this method for the roots of the number MR/MP—at least when all $(n-1)$ mean proportionals are to be found, as in Strähle's musical problem, since the negative and positive errors are then equalized.

Let us assume, then, that $MP\times MR=MB^2$. Since $MR=MP+PR$ and $MB=MR-BR$,

$$\frac{MP}{BR}\left(\frac{MP}{BR}+\frac{2a}{a+b}\right)=\left(\frac{MP}{BR}+\frac{2a}{a+b}-1\right)^2.$$

This leads to a linear equation in the variable, which simplifies to

$$\frac{MP}{BR}=\frac{a-b}{2b(a+b)}.$$

Then

$$\frac{MR}{BR}=\frac{a+b}{2b} \quad\text{and}\quad \frac{MB}{BR}=\frac{a-b}{2b}.$$

From this result,

$$N=\frac{MR}{MP}=\left(\frac{a+b}{a-b}\right)^2.$$

It follows that

$$N^m \doteq \frac{MD}{MP}=\left(\frac{a+b}{a-b}\right)\left(\frac{a-b+2bm}{a+b-2bm}\right). \tag{3}$$

It may seem a pity to change so beautifully symmetric a formula, but it will be more convenient for general use if we let $a-b=1$, whence $a+b=\sqrt{N}$. Substituting these values in formula (3) and simplifying, we obtain*

$$N^m \doteq \frac{Nm+\sqrt{N}(1-m)}{m+\sqrt{N}(1-m)}. \tag{4}$$

Table 6 shows the application of formula (4), to the solution of the musical problem, with $N=2$ and $m=1$, $11/12$, *etc.* If this table is compared with Table 5, it will be seen how closely the two approximations resemble each other (here the maximum error is less than .13%). One's admiration for Strähle increases when one contemplates the beautiful simplicity of his method.

Construction. Here is the method of constructing a geometrical approximation to the roots of numbers, in which the correct mean proportional is used;†

Problem. To construct, by approximation, $(n-1)$ geometric means between two given lines.

* **Comment by the referee.** This formula can be obtained by interpolating the exponential $y=N^x$ in the three points: $x=0$, $x=\frac{1}{2}$, $x=1$ by a linear fractional function $(A+Bx)/(C-Dx)$. Since the latter depends on exactly three parameters, the solution is unique.

† **Comment by the referee.** This is a very beautiful interpretation of formula (4). The procedure is quite natural, as the linear fractional transformation clearly calls for a perspectivity. Indeed, the author's requirement that BA be perpendicular to RQ is irrelevant, since a projective correspondence on lines is uniquely determined by three pairs of corresponding points.

Hypothesis. Let MR be the longer of the two lines; NS the shorter (Fig. 4).

Construction. Construct, by the Euclidean method, the mean proportional between MR and NS. On MR, take $MP=NS$ and $MB=$ the mean proportional. Through R draw the line QR, forming an acute angle with MR, and drop a perpendicular to it from B, which intersects it at A. Take $AQ=AR$, and draw QP intersecting AB prolonged at O. Draw OR. Divide QR into n equal parts and drop rays from O to each point of division.

Conclusion. The lines from M to the points of intersection with the $(n-1)$ rays from O will be approximations to the geometric means between MR and MP, *i.e.*, between MR and NS.

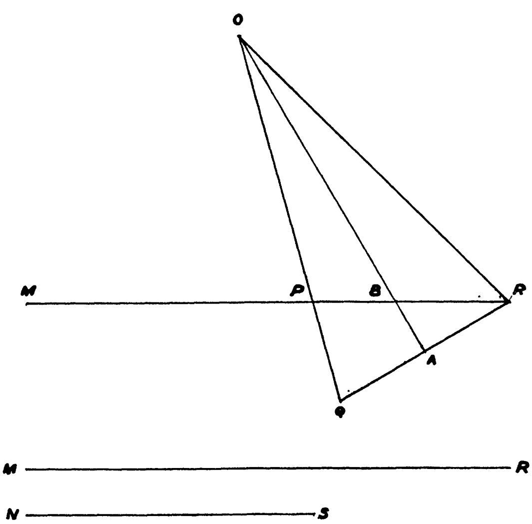

FIG. 4

Error for $N \leqq 2$. In the range $1 < N \leqq 2$, the errors for the above method of approximation are indeed small, the maximum error for $N=2$ being .13%. This error occurs when

$$m = \frac{1 \pm \sqrt{.348}}{2}.$$

When $N=1$, the maximum is at

$$m = \frac{1 \pm \sqrt{.333}}{2},$$

which is imperceptibly smaller than the previous values. The maximum for $N=1$ is precisely that of the cubic: $f(m)=m(1-m)(1-2m)$, in the range $0 \leqq m \leqq 1$. Thus the error curve can be fitted remarkably well to this cubic for

all values of N in the given range, and such a correction reduces the original very small error by something like 99%! However, a complicated third-degree formula is needed to obtain the coefficient of conversion for a particular N, and this makes correcting the error a somewhat lengthy process.

Error for $N>2$. As N becomes large, the maximum error increases very rapidly, with max $E_{N^2} \doteq 8$ max E_N. Moreover, with N large, the error curve retains the same general shape as before, but the maximum is pushed nearer the endpoints, 0 and 1. Thus the curve resembles somewhat a higher-order variant of a cubic, namely, a curve having an equation of the form: $f(x) = x(1-x^{2a})$, where $a>1$ and is not necessarily an integer. For most values of N it would be impossible to calculate the correction function by elementary methods. Thus it is not at all feasible to use this method of approximation with large numbers.

Reduction by repeated mean proportionals. The roots of large numbers can be approximated with a high degree of accuracy, however, by a simple adaptation of our version of Strähle's geometrical approximation. One of its essential features was the finding of a single mean proportional. Moreover, Mersenne has used nothing but mean proportionals in his approximation for equal temperament shown in Table 2. If mean proportionals are taken repeatedly, the problem of approximating the desired root may eventually be reduced to an interval in which the ratio is no greater than 2:1—the range in which the error is very small—after which, if desired, the cubic-curve correction may be applied.

Conclusion. As a geometrical approximation, the method shown in this article is simple and works exceedingly well for small numbers. It is especially advantageous for obtaining approximations to the successive powers of a root, as in its original application to the solution of a musical problem. For larger numbers, the extended method which uses repeated mean proportionals will also be easy to operate geometrically until the mere length of the lines becomes a burden. As a numerical approximation, the method also gives good results for small numbers, and is superfine with the correction which uses the cubic curve. For large numbers, the labor of taking repeated square roots, multiplying together the numbers thus obtained, and applying a complicated correction formula is not commensurate with the results obtained.

References

1. J. M. Barbour, Tuning and Temperament, Michigan State College Press, 1951; 2nd ed., 1953, pp. 50–53; 54 f.; 57 ff.; 65–68.
2. Marin Mersenne, Harmonie universelle, Paris, 1636–37, p. 68.
3. Vincenzo Galilei, Dialogo della musica antica e moderna, Florence, 1581, p. 49.
4. D. P. Strähle, Nytt påfund, til at finna temperaturen i stämningen för thonerne på claveret ock dylika instrumenter, Proceedings of the Swedish Academy, 1743, IV, pp. 281–291.
5. F. W. Marpurg, Versuch über die musikalische Temperatur, Breslau, 1776, p. 167 f.

PARABOLAS ASSOCIATED WITH A TRIANGLE*

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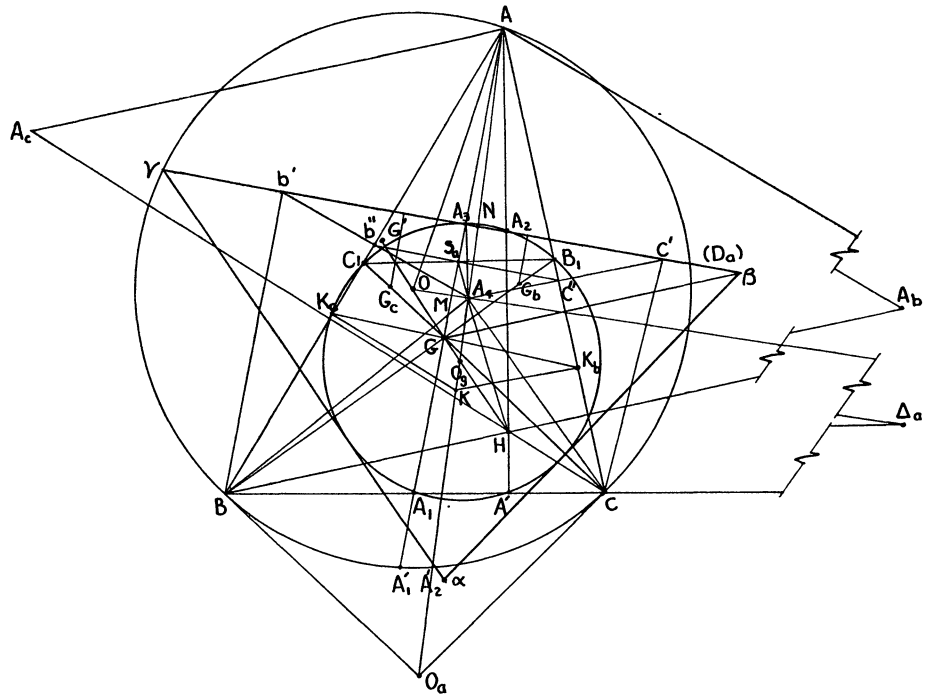
Introduction and notation. Bullard [1] cited some of the properties of the Arzt [2] parabolas (A), (B), (C) associated with the triangle $T \equiv ABC$. (A) is tangent to AB at B and to AC at C. Although these parabolas were studied a long time ago [3], we shall give some details not generally known.

Let a, b, c, S be the lengths of the sides and the area of the triangle T; m_a, m_b, m_c, h_a, h_b, h_c be the medians AA_1, BB_1, CC_1 and the altitudes AA', BB', CC'; O the center of the circumcircle (O) of radius R and O_g the center of the nine-point circle (O_g); H, G, K the orthocenter, the centroid, and the Lemoine point (symmedian point); A_2, B_2, C_2 and A_3, B_3, C_3 the midpoints (Euler points) of HA, HB, HC and the points of intersection of (O_g) with AA_1, BB_1, CC_1; (D_a), (D_b), (D_c), A_4, B_4, C_4 and p_a, p_b, p_c the directrices, the foci, and the parameters of the Arzt parabolas (A), (B), (C); V the Brocard angle of T.



1. Some known properties. The median AA_1 of T is parallel to the axis of (A), and AA_4 is antiparallel to AA_1 with respect to AB and AC. (A) is tangent to B_1C_1 at its midpoint a_1 and the focus A_4 is on the circle AB_1C_1. The directrix

* Translated from the French by W. E. Byrne.

(D_a) passes through the orthocenter A_2 of the triangle AB_1C_1. (D_a) is perpendicular to AA_1 at A_3; similar statements hold for (D_b) and (D_c).

2. The directrices of (*A*), (*B*), (*C*). THEOREM. *If the perpendiculars to AB and AC at A, to BA and BC at B, to CB and CA at C meet the altitudes BB' and CC' at A_b and A_c, CC' and AA' at B_c and B_a, AA' and BB' at C_a and C_b, the lines A_bA_c, B_cB_a, C_aC_b are the directrices (D_a), (D_b), (D_c).*

Proof. The quadrilateral AA_cHA_b is a parallelogram and its diagonals AH, A_bA_c intersect at A_2. The triangles ABC and AHA_b are similar since corresponding angles are equal; likewise ABC and AA_cH are similar. As these pairs of similar triangles have corresponding sides perpendicular, the medians A_2A_b and A_2A_c of AHA_b and AA_cH are perpendicular to AA_1, so they coincide with (D_a).

THEOREM. *The directrices (D_a), (D_b), (D_c) meet the sides BC, CA, AB at collinear points Δ_a, Δ_b, Δ_c.*

Proof. If (D_a) meets BC at Δ_a, the diagonal A_1A_2 of the quadrilateral $A_1A'A_2A_3$, inscribed in (O_g) and having two opposite right angles, is an altitude of triangle $AA_1\Delta_a$. But A_1A_2 is parallel and equal to OA, so that $A\Delta_a$ coincides with the tangent t at A to the circle (O) and Δ_a is the intersection of t and BC. Thus Δ_a is the pole of AA_4 with respect to (O). The points Δ_a, Δ_b, Δ_c are on the polar of the point of intersection K of AA_4, BB_4, CC_4 with respect to (O), called the Lemoine line of T.

COROLLARY. *The pairs of lines AA_4 and BC, BB_4 and CA, CC_4 and AB, form couples of conjugate lines with respect to (O) and respectively with respect to (A), (B), (C).*

Proof. AA_4 and BC are conjugate with respect to (O) since BC, (D_a), t meet at the pole α' of AA_4. The polar of α' with respect to (A) passes through the conjugate of α' with respect to B and C, that is the intersection of AA_4 and BC, and through A since the tangent lines of (A) at B and C meet in A. Thus the polar of α' with respect to (A) is precisely AA_4.

3. THEOREM. *The lines A_3A_4, B_3B_4, C_3C_4 are perpendicular to the lines BC, CA, AB.*

Proof. Since the midpoint a_1 of B_1C_1 is on (A), $a_1A_3 = a_1A_4$. A_3 and A_4 are symmetric with respect to B_1C_1 so that A_3A_4 is perpendicular to BC.

COROLLARY. *If the lines AA_3 and AA_4 meet OA_4 and (D_a) in M and N, respectively, MN is parallel to OA.*

Proof. The quadrilateral A_3MA_4N is inscribed in the circle described on MN as a diameter, so

$$\angle NMA_4 = \angle NA_3A_4 = \angle AA_1C.$$

But if AA_1 and AA_4 meet (O) at A_1' and A_2', then $\angle AA_1C = \angle AOA_4$ since these angles are measured by

$$\tfrac{1}{2}(\text{arc } AC + \text{arc } A_1'B) = \tfrac{1}{2}(\text{arc } AC + \text{arc } CA_2').$$

Arc $A_1'B =$ arc CA_2' since AA_1' and AA_2' are isogonal with respect to the angle BAC. Hence $\angle NMA_4 = \angle AA_1C = \angle AOA_4$ and MN is parallel to OA. This shows once more that BC, (D_a), OA_4 and t are concurrent.

4. THEOREM. *The perpendicular bisectors of the segments A_bA_c, B_cB_a, C_aC_b intercepted on the directrices (D_a), (D_b), (D_c) of (A), (B), (C) by the altitudes BB' and CC', CC' and AA', AA' and BB' meet at the midpoint of GH.*

Proof. The perpendicular bisectors of the segments in question pass through the midpoints A_2, B_2, C_2 of AH, BH, CH and are parallel to the medians AA_1, BB_1, CC_1 of T.

5. LEMMA. *The centroid and the Lemoine point of the antipedal triangle $T_1' \equiv A_1'B_1'C_1'$ of the centroid G of T, with respect to the triangle $T'' \equiv A''B''C''$ having as its vertices the points of intersection of the medians AA_1, BB_1, CC_1 with (O), coincide with G and with the symmetric of G with respect to O.*

Proof. Triangles T and T_1' are conjugate with respect to the circle of center G and of which the square of the radius is the power $\overline{GO}^2 - R^2$ of G with respect to (O). The point G has the same barycentric coordinates with respect to both T and T_1', so G is the common centroid of these triangles. T_1' is circumscribed about a conic of which the foci G and G' are symmetric with respect to O and isogonal conjugate with respect to T_1'. G' coincides with the Lemoine point of T_1'.

LEMMA. *The Lemoine point and the centroid of the antipedal triangle t_1' of G with respect to T coincide with G and with the symmetric G' of G with respect to O.*

Proof. Triangles t_1' and T_1' are symmetric with respect to O.

THEOREM. *The directrices (D_a), (D_b), (D_c) form a triangle $T_0 \equiv \alpha\beta\gamma$ of which the centroid and the Lemoine point are the centroid of T and the center of the segment GH, respectively.*

Proof. The triangles t_1' and T_0 are related by a homothetic transformation of center H and ratio 2. The centroids and Lemoine points are homologous points.

COROLLARY. *The medians of T_0 are perpendicular to the sides of T.*

6. THEOREM. *The lines A_3S_a, B_3S_b, C_3S_c meet at the orthocenter H of T* [4]. *S_a, S_b, S_c are the vertices of (A), (B), (C).*

Proof. Let b', c' be the orthogonal projections of B, C on (D_a). A_3 bisects $b'c'$; b', c' are symmetrical to A_4 with respect to the tangents AB, AC of (A).

Consequently the triangle $A_3b''c''$ of which the vertices b'', c'' are the midpoints of A_4b', A_4c' is homothetic to the triangle HA_bA_c, with the point M of intersection of A_bc'' and A_cb'' as homothetic center. Furthermore, triangles AA_cA_b and $A_4c''b''$ are also homothetic with center M. H and A_3, A_3 and S_a are homologous in these similitudes, and H, A_3, S_a are collinear, as well as H, B_3, S_b and H, C_3, S_c.

7. Metric relations. THEOREM. *The distances $G\alpha$, $G\beta$, $G\gamma$ from G to the vertices of T_0 are proportional to the lengths of the sides of T.*

Proof. T and T_0 have a common centroid G and the Lemoine point of T coincides with the centroid of the pedal triangle $K_aK_bK_c$ of K with respect to T. The homothetic ratio

$$GA_3/KK' = k,$$

where KK' is the altitude of triangle KK_bK_c, has the value (see the following theorem):

$$k = (GA - A_3A)/KK' = \left(\frac{2m_a}{3} - \frac{bc \cos A}{2m_a}\right) \Big/ KK'$$
$$= \frac{2S \cot V}{3m_a} \Big/ \left(\frac{S}{2m_a \cot V}\right) = \frac{4}{3} \cot^2 V.$$

Hence $G\alpha/KK_a = k = G\alpha \cdot 2 \cot V/a$, so $G\alpha/a = G\beta/b = G\gamma/c = \frac{2}{3} \cot V$. The triangles T and T_0 have the same Brocard angle since

$$\overline{K_bK_c}^2 + \overline{K_cK_a}^2 + \overline{K_aK_b}^2 = 3S/\cot V \text{ and } \overline{K_aK_bK_c} = 3S/4 \cot^2 V.$$

THEOREM. *The distances GA_3, GB_3, GC_3 of G to the directrices (D_a), (D_b), (D_c) are inversely proportional to the lengths m_a, m_b, m_c of the medians.*

Proof. From the relation $AA_3 \cdot AA_1 = AA_2 \cdot AA'$ it follows that

$$AA_3 = AA_2 \cdot AA'/AA_1 = Rh_a \cos A/m_a = bc \cos A/2m_a.$$

Also

$$GA_3 = GA - A_3A = 2m_a^2/3 - bc \cos A/2m_a$$
$$= (4m_a^2 - 3bc \cos A)/6m_a,$$

(1) $$GA_3 = (a^2 + b^2 + c^2)/12m_a = S \cot V/3m_a.$$

Thus $GA_3 \cdot m_a = GB_3 \cdot m_b = GC_3 \cdot m_c$.

COROLLARY. *The circles of diameters AA_3, BB_3, CC_3 are orthogonal to the orthoptic circle of the Steiner ellipse inscribed in the triangle T.*

Proof. If σ denotes the radius of the orthoptic circle (G, σ) of the Steiner ellipse inscribed in T, then

$$GA_3 \cdot GA = \frac{a^2+b^2+c^2}{12m_a}\cdot\frac{2m_a}{3} = \frac{a^2+b^2+c^2}{18} = \sigma^2 = GB_3 \cdot GB = GC_3 \cdot GC.$$

THEOREM. *The distances AA_4, BB_4, CC_4 are inversely proportional to the products am_a, bm_b, cm_c.*

Proof. Let O_a, O_b, O_c designate the poles of BC, CA, AB with respect to (O). From $AA_4/AO_a = AA_3/AA_1$ it follows that

$$(2) \qquad AA_4 = \frac{bc\cos A}{2m_a}\cdot\frac{m_a}{\cos A}\Big/ m_a = 2RS/am_a.$$

Hence

$$AA_4 \cdot am_a = BB_4 \cdot bm_b = CC_4 \cdot cm_c.$$

THEOREM. *The distances GA_4, GB_4, GC_4 are inversely proportional to the lengths of the medians.*

Proof. $\overline{A_4G}^2 = \frac{1}{3}[\overline{A_4A}^2 + \overline{A_4B}^2 + \overline{A_4C}^2 - (\overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2)]$. If M and N are the orthogonal projections of B and C on (D_a),

$$CA_c = CH + HA_c = 2R\cos C + c\cot A = 2bR/a,$$

$$CN = CA_c \sin A_1AB = b^2/2m_a,$$

$$(3) \qquad BM = c^2/2m_a.$$

Since (A) is tangent at C to AC,

$$A_4C = b^2/2m_a = CN, \qquad A_4B = c^2/2m_a = BM$$

and

$$\overline{A_4A}^2 + \overline{A_4B}^2 + \overline{A_4C}^2 = (b^2c^2 + c^4 + b^4)/4m_a^2,$$

$$\overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 = (a^2+b^2+c^2)/3,$$

we have

$$\overline{A_4G}^2 = (a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2)/36m_a^2,$$

and finally

$$GA_4 \cdot m_a = GB_4 \cdot m_b = GC_4 \cdot m_c.$$

COROLLARY. *The products*

$$m_a \cdot GA_4 = m_b \cdot GB_4 = m_c \cdot GC_4 = \tfrac{1}{6}(a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2)^{1/2}.$$

COROLLARY. *The distances A_4B_1 and A_4C_1, B_4C_1 and B_4A_1, C_4A_1 and C_4B_1 of the foci of (A), (B), (C) to the vertices of the medial triangle $A_1B_1C_1$ of T are inversely proportional, in pairs, to the lengths of the sides c and b, a and c, b and a of T.*

Proof. From the formula for the square of the median A_4A_1 of the triangle A_4AC it follows that

$$A_4B_1 = ab/4m_a, \qquad A_4C_1 = ca/4m_a. \tag{4}$$

COROLLARY. *The circles symmetric to the Apollonian circles of triangle $A_1B_1C_1$ with respect to the midpoints of B_1C_1, C_1A_1, A_1B_1 pass respectively through the foci of the Arzt parabolas of T.*

Proof. This proposition follows from (4) since $A_4B_1/A_4C_1 = b/c, \cdots$.

COROLLARY. *The distances (A_4G_b, A_4G_c) of the focus of (A) to the centroids G_b, G_c of the triangles (AGC, AGB) are proportional to the squares of the medians (m_c, m_b).*

Proof. An application of Stewart's theorem to the triangle A_4BB_1, where the side BB_1 is divided by G_b in the ratio $B_1G_b/G_bB = \frac{1}{8}$, shows that

$$A_4G_b = 2m_c^2/9m_a, \qquad A_4G_c = 2m_b^2/9m_a. \tag{5}$$

COROLLARY. *The parabolas (A), (B), (C) pass respectively through the centroids G_b and G_c, G_c and G_a, G_a and G_b of triangles GCA and GAB, GAB and GBC, GBC and GCA* [5].

Proof. This follows from (5), which expresses also the distances from G_b and G_c to (D_a). Let G_bG_b' be the distance from G_b to (D_a).

$$GA_3 = \tfrac{3}{4}G_bG_b' + \tfrac{1}{4}Bb' = \tfrac{3}{4}G_bG_b' + \tfrac{1}{4}A_4B.$$

From (1), (3) it follows that $G_bG_b' = 2m_c^2/9m_a = G_bA_4$.

THEOREM. *The parameters p_a, p_b, p_c of the parabolas (A), (B), (C) are inversely proportional to the cubes of the medians m_a, m_b, m_c.*

Proof. The triangles AA_cA_b and $A_4c'b'$ are homothetic.

$$2A_4S_a/A_3A = p_a/A_3A = AA_b/A_4b' = AA_b/2A_4b'',$$
$$2A_4S_a = p_a = AA_3 \cdot AA_b/2A_4b'',$$

but

$$A_4b'' = A_4A \sin CAA_1 = \frac{bc}{2m_a} \cdot \frac{S}{bm_a} = cS/2m_a^2,$$

so

$$p_a = bcS \sin A/2m_a^3 = S^2/m_a^3.$$

Thus

$$p_a m_a^3 = p_b m_b^3 = p_c m_c^3.$$

8. Exercises. If the vertices B and C of T are fixed, determine the geometric

locus of the focus A_4 and the envelope of the directrix (D_a) of the parabola (A) under the following hypotheses: [5]:

1) A describes a given circle passing through B and C.
2) A describes a given circle belonging to the coaxial system for which B and C are Poncelet limit points.
3) A describes a circle having A_1 as its center and radius l.
4) A describes a straight line passing through A_1.

Bibliography

1. This MONTHLY, 1935, p. 606.
2. Programme scolaire du gymnase de Recklinghausen, 1884.
3. P. Barbarin, E. Lemoine, Mathesis, 1884, pp. 142, 143, questions 341 and 350.
H. Brocard, Journal de Mathématiques Spéciales de G. de Longchamps, 1885, pp. 109, 123.
4. L. Gérard, Bulletin de Mathématiques Elémentaires, 1909, pp. 64, 196, question 2602.
5. P. Barbarin, E. Lemoine, J. Bullard, H. Brocard, *loc. cit.*
6. Agrégation feminine de Mathématiques, Paris, Concours de 1950.

SUGGESTIONS TO STUDENTS ON TALKING ABOUT MATHEMATICS PAPERS*

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The art of giving a talk is to choose and follow a dramatic technique appropriate to the occasion and to the audience. This is just as true of a mathematics talk as it is of an after-dinner speech. There are at least three types of mathematics talks which you may want to deliver: (1) a ten-minute paper before a scientific society; (2) a one-hour colloquium lecture before a department; (3) a review of a journal paper or a discussion of some problem in a working seminar.

Each of these talks has its technique, and the three techniques are different. I'd like to consider the working seminar, and give some suggestions that may help you get started. They won't fit all cases, of course, but I hope they will at least convince you of the importance of planning your talks carefully, and of looking objectively at the technique involved.

Acquiring background. As soon as you learn of a paper you will be studying, write the author for a reprint. A statement of your need may help you get it

* With many suggestions from colleagues at New York University and from the editors of the MONTHLY. My students have expressed the hope that faculty members also will pay attention to these suggestions!

† The author is now at the University of California, Los Angeles.

promptly. Most authors are delighted to receive requests for reprints, but don't be disappointed if the author has none left.

Allow a lot of time for study of the papers that you are to report on. You must know as much about a key paper as the author does; if you possibly can, you should understand it even better. Study where each hypothesis is used in the proof. (Could any hypothesis be weakened or eliminated? Here is often a place to start research of your own.) See how the results look in significant special cases. Actually compute numerical examples, if they are relevant, or special concrete cases where you can see what is going on. Not only are such cases illuminating, but they are also often extremely amusing.

Preparing the talk. Ask *early* for any suggestions the seminar instructor, or some older friend in your department, is able to give. When you have mastered the material, you are ready to draft a talk. Assume that your audience will know nothing about the subject except what may have been said in the seminar previously. But credit them with the intelligence to catch on quickly.

I would suggest writing down almost every word you expect to say, and everything you expect to write on the board. I would then reduce the above to an outline including the first and last sentences and the blackboard material verbatim, but leaving only key words for the rest. At this point I would throw away the word-for-word write-up, as it can only harm your talk henceforth! Reading a talk is utterly ineffective.

Using your outline, have a very serious rehearsal alone with a blackboard, preferably in the room where the seminar meets. To keep an orderly array, plan where each formula will be written, and plan what will be erased and when. Write everything on the board as though your audience were present. Time yourself, and allow time for heckling from the audience later. This is a good time to check up on common sense, but often forgotten, details of speaking: posture, voice (no mumbling), writing (be sure you don't hide it, and go to the back of the room to check that your writing is readable).

At this stage you will probably have to prune your talk to save time. A second rehearsal is then in order, especially if you have changed the talk substantially. If there are one or two fellow-students who are your good friends and critics, you might consider inviting them to the second rehearsal to provide audience reaction and questions.

A suggestion which seems silly but which is apparently needed is to learn the Greek letters. You will also want to pronounce names correctly. Note, for example, that "Lewy" rhymes with "gravy," and that W. E. Milne pronounces his last name in one syllable.

Content of the talk. To gain time, I would have on the side board in advance all material—like references, tables and detailed diagrams—which takes much time to write, but which is to be read rather rapidly. On the other hand, it is deadly to write in advance anything in the main development of the subject.

Your introduction is vitally important and must be most carefully planned.

You must set the stage by putting your subject in the framework of the audience's knowledge. Where do today's results fit into the general field? A historical approach is usually a good one; in this you recapitulate the main results and their discovery in chronological order. You will have to decide what definitions will first be given to make the history intelligible.

After (or before) you have traced the main developments in a general fashion, develop your basic definitions and notations carefully on the board and, if possible, leave them there for much of the talk. Lose no time here, but omit no essential detail.

A good technique is next to write down and interpret all of the theorems that you are going to cover. Include all the generality the author has contrived. Leave nothing undefined. Leave the theorems on the board (abbreviated, if necessary) for later reference.

Now discuss the implications of these results for the subjects of the seminar. Are there extensions of the theory? Do you recognize any promising unsolved problems; are there conjectures about them?

Next it is time for the key proofs. There are two good reasons for postponing them to now:

(1) you can control your time by eliminating or adding proofs, with the least harm to the total dramatic effect;

(2) you will undoubtedly lose some people in details, and you prefer to lose them as late as possible in your talk.

Select the proofs to be given on the basis of the representativeness or novelty of the methods used, the importance of the results, *etc.* When you give a proof, give it carefully. While the ten-minute talk and even the colloquium talk will ordinarily avoid details, the working seminar is expected to get to the bottom of the subject. (If they don't, who will?) So roll up your sleeves and pitch in, and don't try to prove theorems by waving your hands. You can sometimes build a sound account of a proof around a special case. But afterwards try to tie it in with the general case. Let the audience know where each hypothesis enters the proof.

Conclude with a reiteration of the nature of the contribution here and its significance. And stop. You should welcome discussion and criticism, and one way to get it is to stop early. There are many sins that mathematical speakers fall heir to. I have tried above to show you how to avoid some of them. A common and deadly sin is to talk too long. The human animal ordinarily cannot and will not listen for more than one hour. So—when your time is up (or even a bit earlier), thank your audience for their attention and SIT DOWN.

As a final suggestion, attend all the talks you can by masters of the art, and study their techniques.

A NOTE ON THE STAUDT-CLAUSEN THEOREM

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If we put

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{1}$$

so that $B_{2m+1} = 0$ for $m \leqq 1$, then according to the Staudt-Clausen theorem (see, for example [2, p. 33], [3, p. 257])

$$B_{2m} = A_{2m} - \sum_{p-1|2m} \frac{1}{p} \qquad (m \geqq 1), \tag{2}$$

where A_{2m} is an integer and the sum is over all primes p (including 2) such that $p-1|2m$. Thus (2) implies

$$pB_{2m} \equiv -1 \pmod{p} \qquad (p - 1|2m); \tag{3}$$

also the denominator of B_{2m} contains no repeated prime factors.

In view of the above it is perhaps natural to ask whether there exists a sequence of rational numbers b_{2m} defined by a simple generating function such that if $(p-1)p^r|2m$, then the denominator of b_{2m} contains p^{r+1}. It is not difficult to construct such a sequence. Indeed it follows easily from (1) that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{B_n}{n} \frac{x^n}{n!} &= \int \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{1}{x} \right) dx \\ &= \log(1 - e^{-x}) - \log x, \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} R_n \frac{x^n}{n!} = \log \frac{1 - e^{-x}}{x}, \tag{4}$$

where

$$R_n = \frac{B_n}{n} \cdot \tag{5}$$

Now in the first place it is known [3, p. 261] that if $p^r|m$ but $p-1\nmid 2m$ then the numerator of B_{2m} is divisible by p^r; consequently by (5), R_{2m} is integral (mod p) provided $p-1\nmid 2m$. On the other hand if $p-1|2m$ and p^r is the highest power of p dividing $2m$, then it is clear from (3) and (5) that the denominator of R_{2m} is divisible by exactly p^{r+1}. This result can, however, be made more precise. The writer has proved [1, Th. 3] that if $(p-1)p^r|2m$ then

$$B_{2m} + \frac{1}{p} - 1 \equiv 0 \pmod{p^r} \qquad (p \geqq 3). \tag{6}$$

This implies

$$R_{2m} = I + \frac{1}{2m}\left(1 - \frac{1}{p}\right),$$

where I is integral (mod p). Consequently if p^r is the highest power of p dividing m

$$p^{r+1}R_{2m} \equiv p^r(p-1)/2m \pmod{p^{r+1}} \qquad (p \geqq 3), \tag{7}$$

which may be compared with (3); note that both numerator and denominator of $p^r(p-1)/2m$ are prime to p.

As for the excluded case $p=2$, we recall [**2**, p. 18] that

$$B_{2m+1}(2) - B_{2m+1}(1) = 2m + 1, \tag{8}$$

where $B_{2m+1}(x)$ is the Bernoulli polynomial of degree $2m+1$. But $B_{2m+1}(1)=0$ for $m \leqq 1$; thus (8) becomes $B_{2m+1}(2)=2m+1$, that is

$$(2m+1)2B_{2m} + \binom{2m+1}{3}2^3B_{2m-2} + \cdots + \binom{2m+1}{2m-1}2^{2m-1}B_2 +$$

$$\binom{2m+1}{2m}2^{2m}B_1 + 2^{2m+1} = 2m+1, \tag{9}$$

$$(2m+1)(1-2B_{2m}) = \sum_{s=1}^{m-1}\binom{2m+1}{2s+1}2^{2s+1}B_{2m-2s} + (2m+1)2^{2m}B_1 + 2^{2m+1}.$$

Now let 2^r be the highest power of 2 dividing $2m$. We have

$$\binom{2m+1}{2s+1}2^{2s} = \frac{(2m+1)2m}{(2s+1)2s}\binom{2m-1}{2s-1}2^{2s} \equiv 0 \pmod{2^{r+1}},$$

so that

$$\sum_{s=1}^{m-1}\binom{2m+1}{2s+1}^{2s+1}B_{2m-2s} \equiv 0 \pmod{2^{r+1}}.$$

Also for $m \geqq 2$, since $2m-1 \geqq r+1$,

$$(2m+1)2^{2m}B_1 \equiv 0 \pmod{2^{r+1}}.$$

Consequently (9) becomes $(2m+1)(1-2B_{2m}) \equiv 0 \pmod{2^{r+1}}$, that is

$$2B_{2m} \equiv 1 \pmod{2^{r+1}} \qquad (m \geqq 2). \tag{10}$$

This is equivalent to

$$2^{r+1}R_{2m} \equiv \frac{2^r}{2m} \pmod{2^{r+1}} \qquad (m \geqq 2), \tag{11}$$

which is of the same form as (7). Since $B_2 = 1/6$, (10) and (11) do not hold for $m = 1$.

We may now state the following

THEOREM. *Let p^r denote the highest power of the prime p dividing $2m$. Then if $p-1 \nmid 2m$, R_{2m} is integral* (mod p). *If $p-1 \mid 2m$, then* (7) *and* (11) *hold.*

References

1. L. Carlitz, Some congruences for the Bernoulli numbers, Amer. J. Math., vol. 75, 1953 pp. 163–172.
2. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Berlin, 1924.
3. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the sixteenth William Lowell Putnam Mathematical Competition held on March 3, 1956, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were David B. Mumford, Rohitkumar Parikh, and Kenneth G. Wilson; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Columbia College, New York City. The members of the team were David M. Bloom, Jonathan Lubin, and Jerrold Rubin; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Queen's University, Kingston, Ontario. The members of the team were Donald J. C. Bures, Paul A. Herzberg, and Edward James Woods; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Richard Bumby, James Bjorken, and Peter Wolk; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Trevor H. Barker, Kenyon College; David M. Bloom, Columbia College; Richard M. Friedberg, Harvard University; David B. Mumford, Harvard University; and Kenneth G. Wilson, Harvard University. Each of these

will receive a prize of fifty dollars.

The nine succeeding persons (nine to fourteenth tied) ranking highest in the examination, named in alphabetical order, are Richard Bumby, Massachusetts Institute of Technology; Paul Henry Monsky, Swarthmore College; Rohitkumar Parikh, Harvard University; Myron Richard Porter, University of California at Berkeley; Richard Pratt, Washington University; Ian Richards, University of Minnesota; Howard C. Rumsey, California Institute of Technology; John Robert Stallings, Jr., University of Arkansas; and Edward James Woods, Queen's University.

The following teams, named in alphabetical order, won honorable mention: Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being Judith Barbara Hirschfield, Hugh N. Pendleton 3d, and George Bruno Rybicki; Cornell University, Ithaca, New York, the members of the team being Donald Kahn, Stanley Kaplan, and Robert Riley; Kenyon College, Gambier, Ohio, the members of the team being Trevor H. Barker, Thomas M. Jenkins, and Robert E. Mosher; and the University of Rochester, Rochester, New York, the members of the team being Charles McCarthy, Roy Schult, and Ronald Winkelman.

Ten individuals were given honorable mention. The names, in alphabetical order, are: David G. Cantor, California Institute of Technology; Robin Hartshorne, Harvard University; Judith Barbara Hirschfield, Carnegie Institute of Technology; Charles McCarthy, University of Rochester; N. David Mermin, Harvard University; Don Moore, University of California at Los Angeles; Paul Penfield, Jr., Massachusetts Institute of Technology, Max A. Plager, Massachusetts Institute of Technology; Paul A. Schweitzer, College of the Holy Cross; and James D. Stasheff, University of Michigan.

A total of 372 individuals from 78 institutions entered the competition this year. Of this number 81 individuals and two institutions were unable to compete, due to various reasons. Therefore, a total of 291 undergraduates from 76 institutions actually took part in the competition.

The following is a list of all colleges and universities which entered teams in the competition. The list is arranged in alphabetical order: Agricultural and Mechanical College of Texas, Alabama Polytechnic Institute, Arizona State College (Flagstaff), Arizona State College (Tempe), Brooklyn College, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Catholic University of America, College of the City of New York, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Harvard University, Haverford College, Iowa State College, Kenyon College, Knox College, Lebanon Valley College, LeMoyne College, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State College, Montana State College, Oberlin College, Occidental College, Polytechnic Institute of Brooklyn, Princeton University, Purdue University, Queen's University (Canada), Rutgers University, Saint Olaf

College, San Jose State College, Siena College, Stanford University, Swarthmore College, The Cardinal Stritch College, United States Naval Academy, University of Arizona, University of British Columbia, University of California, Berkeley, University of California, Los Angeles, University of Detroit, University of Michigan, University of Minnesota, University of Oregon, University of Pennsylvania, University of Rochester, University of Tennessee, University of Toronto, Wayne University, West Texas State College, and Yale University.

The following colleges and universities, alphabetically arranged, entered individual contestants only: Assumption College (Windsor, Ontario), Brandeis University, Brown University, Georgetown University, Gustavus Adolphus College, New York University, Ottawa University (Kansas), Purdue University Center (Fort Wayne, Indiana), Sacramento State College, Seattle University, Temple University, The Cooper Union, The Ohio State University, The University of Texas, University of Arkansas, University of Houston, University of Washington, Washington University (St. Louis, Missouri), Wesleyan College (Macon, Georgia), and Wesleyan University (Middletown, Connecticut).

Any department of mathematics may obtain the individual rankings of contestants (except for the relative ranks of the first five) for the purpose of selecting graduate students.

Those participating in the competition were given the following lists of problems:

Part I

1. Evaluate

$$\lim_{x\to\infty}\left[\frac{1}{x}\,\frac{a^x-1}{a-1}\right]^{1/x}$$

where $a>0$, $a\neq 1$.

2. Prove that every positive integer has a multiple whose decimal representation involves all ten digits.
3. A particle falls in a vertical plane from rest under the influence of gravity and a force perpendicular to and proportional to its velocity. Obtain the equations of the trajectory and identify the curve.
4. Suppose the n times differentiable real function $f(x)$ has at least $n+1$ distinct zeros in the closed interval $[a, b]$ and that the polynomial $P(z)\equiv z^n+C_{n-1}z^{n-1}+\cdots+C_0$ has only real zeros. Show that $(D^n+C_{n-1}D^{n-1}+\cdots+C_0)f(x)$ has at least one zero in the interval $[a, b]$ where D^n denotes as usual d^n/dx^n.
5. Given n objects arranged in a row. A subset of these objects is called unfriendly if no two of its elements are consecutive. Show that the number of unfriendly subsets each having k elements is

$$\binom{n-k+1}{k}.$$

6. (a) A transformation of the plane into itself preserves all rational distances. Prove that it preserves all distances.
   (b) Show that the corresponding theorem for the line is false.
7. Prove that the number of odd binomial coefficients in any finite binomial expansion is a power of 2.

Part II

1. Show that if the differential equation
$$M(x, y)dx + N(x, y)dy = 0$$
is both homogeneous and exact then the solution $y=f(x)$ satisfies $xM+yN=C$ (constant).
2. Suppose that each set X of points in the plane has an associated set $\overline{X}$ of points called its cover. Suppose further that
   (i) $\overline{X \cup Y} \supset \overline{\overline{X}} \cup \overline{Y} \cup Y$, where $\cup$ designates point set sum (or union) and $\supset$ denotes set inclusion.
   Prove: (a) $\overline{X} \supset X$, (b) $\overline{\overline{X}}=\overline{X}$, (c) $X \supset Y$ implies $\overline{X} \supset \overline{Y}$.
   Prove conversely that (a), (b) and (c) imply (i).
3. A sphere is inscribed in a tetrahedron and each point of contact of the sphere with the four faces is joined to the vertices of the face containing the point. Show that the four sets of three angles so formed are identical.
4. Prove that if A, B, and C are angles of a triangle measured in radians then $A \cos B+\sin A \cos C>0$.
5. Consider a set of $2n$ points in space, $n>1$. Suppose they are joined by at least n^2+1 segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangles being formed.
6. Given $T_1=2$, $T_{n+1}=T_n^2-T_n+1$, $n>0$. Prove:
   (a) If $m \neq n$, T_m and T_n have no common factor greater than 1.
   (b) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{T_i} = 1.$
7. The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$.
   Prove that $P(z) \equiv Q(z)$.

*Solutions of the Problems.** The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants.

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

The presentation here is intended as a brief sketch of the method of proof rather than as a model of a detailed proof such as is expected from the contestants.

Part I

1. Let $u=[(a^x-1)/(ax-x)]^{1/x}$, so that $\ln u=[-\ln x+\ln |a^x-1|-\ln |a-1|]/x$. By L'Hospital's rule, $\lim_{x\to\infty} [\ln x]/x=0$, and thus $\lim_{x\to\infty}\ln u=\lim_{x\to\infty} [\ln |a^x-1|]/x$. For $0<a<1$, this is not an indeterminate form, $\lim_{x\to\infty} [\ln u]=0$, and $\lim_{x\to\infty} u=1$. For $a>1$, $\lim_{x\to\infty} [\ln |a^x-1|]/x=\ln a$ by L'Hospital's rule and thus $\lim_{x\to\infty} u=a$ in this case.

2. Let n be a positive integer and k a positive integer such that $10^k>n$. Some multiple of n, say hn, satisfies $1230456789\times 10^k \leqq hn<1230456789\times 10^k+10^k$. Clearly every integer in this range contains all ten digits and the proof is complete.

3. Take the y-axis vertically downward in the given vertical plane with the origin at the initial position of the particle. If at time t the position of the particle is given by $(x(t), y(t))$, then the force acting on the particle at time t is the vector $[cy', \omega m-cx']$, where m is the mass of the particle and ωm is the magnitude of the gravitational force and c is a constant of proportionality. The x-axis may be chosen to make $c>0$. Thus the equations of motion are $mx''=cy'$ and $my''=\omega m-cx'$. Integrating the first equation and applying the initial conditions we have that $mx'=cy$. Substituting for x in the second equation we get $m^2y''=\omega m^2-c^2y$. Solving for y and x by standard methods we get $y=A(1-\cos (at))$, $x=A(at-\sin (at))$, where $A=\omega m^2/c^2$ and $a=c/m$. This is the equation of a cycloid.

4. Let $n=1$ and suppose $f(z_1)=f(z_2)=0$, $z_1\neq z_2$, in $[a, b]$ with ω any real constant. If the zeros of f are dense in the interval $[z_1, z_2]$, then f vanishes identically in the interval and so does $f'-\omega f$. If the zeros of f are not dense in $[z_1, z_2]$ there is a subinterval in which f does not vanish except at the end points. We shall designate such an interval by $[u_1, u_2]$. Thus $\ln |f(x)|$ attains a maximum at some point b interior to $[u_1, u_2]$. It follows that $[\ln |f(x)|-\ln |f(b)|]/(x-b)$ attains all positive values as x varies in $[u_1, b)$ and attains all negative values as x varies in $(b, u_2]$. By the mean value theorem, it follows that $f'(x)/f(x)$ attains all real values in (u_1, u_2) and thus $f'-\omega f$ vanishes in $[z_1, z_2]$ in this case also.

The induction follows, since if $P(z)=\prod_{j=1}^n (z-\omega_j)$, then

$$P(D)f=\left[\prod_{j=1}^{n-1}(D-\omega_j)\right][(D-\omega_n)f]=\left[\prod_{j=1}^{n-1}(D-\omega_j)\right][f'-\omega_n f]\cdot$$

5. Let S be an unfriendly subset containing k elements. For each element a_j contained in S let n_j be the number, possibly zero, of elements a_h of $a_1, a_2, \cdots, a_n$ such that $h<j$ and a_h does not belong to S. By this rule each unfriendly subset determines k distinct non-negative integers in $[0, n-k]$ and it is easily seen that the correspondence is biunique. Thus the number of unfriendly

subsets required is the same as the number of combinations of $n-k+1$ elements taken k at a time.

6. Let P and Q be distinct points of the plane and $\overline{P}$ and $\overline{Q}$ their images under the given transformation. Given $\omega>0$, there is a point R such that the distances $d(P, R)$ and $d(R, Q)$ are both rational and $d(R, Q)<\omega$. Indeed a circle with center Q and rational radius $r<\omega$ intersects infinitely many circles which have center at P and rational radius. Then $d(P, Q)-2\omega\leqq d(P, Q)-2d(R, Q)\leqq d(P, R)-d(R, Q)=d(\overline{P}, \overline{R})-d(\overline{R}, \overline{Q})\leqq d(\overline{P}, \overline{Q})\leqq d(\overline{P}, \overline{R})+d(\overline{R}, \overline{Q})=d(P, R)+d(R, Q)\leqq d(P, Q)+2d(R, Q)\leqq d(P, Q)+2\omega$. Since ω is an arbitrary positive number, this proves (a).

Take any point on the line as origin of a coordinate system. Let all rational points be fixed under the transformation and let all irrational points be translated one unit in the positive direction. Then all rational distances are preserved but not all distances are preserved.

7. Suppose the proposition false and let n be the smallest positive integer such that the number of odd coefficients in the expansion of $(x+y)^n$ is not a power of 2. By inspection n is not 1, nor is n equal to 2.

Case 1. Let $n=2k$, with k a positive integer. If m is odd then, writing ${}_nC_m$ for binomial coefficients, ${}_nC_m=\{(n-m+1)/m\}{}_nC_{m-1}$ is an even integer since $n-m+1$ is even. Thus all the odd coefficients must occur for even values of m. Let $m=2q$. Then ${}_nC_m$ is congruent (mod 2) to

$$\frac{2\times 4\times\cdots\times n}{2\times 4\times\cdots\times m\times 2\times 4\times\cdots\times(n-m)}$$

by removing all odd factors and divisors. That is, if x and y are congruent (mod 2) then so are hx and ky for any odd integers h and k, and conversely. But the latter fraction is evidently ${}_kC_q$. Thus ${}_nC_m={}_{2k}C_{2q}$ is odd if, and only if, ${}_kC_q$ is odd and so the number of odd coefficients in $(x+y)^n$ is the same as the number of odd coefficients in $(x+y)^k$, which contradicts the assumption that n was the least positive integer for which the proposition fails.

Case 2. Let $n=2k+1$, with k a positive integer. Since ${}_nC_m={}_nC_{n-m}$ and, since n and $n-m$ are not congruent (mod 2), the number of odd coefficients is twice the number of odd coefficients obtained by considering only even values of m in the symbols ${}_nC_m$. But as before we find ${}_{2k+1}C_{2q}\equiv{}_kC_q$ (mod 2). Thus the number of odd coefficients in $(x+y)^n$ is just twice the number of odd coefficients in $(x+y)^k$, again giving a contradiction. Thus the proposition is true.

Part II

1. Let M and N be homogeneous of degree k. Then, by Euler's theorem, $xM_x+yM_y=kM$, where M_x denotes the partial derivative of M with respect to x. Let y denote a solution of the equation in some interval. Then, in this interval, $d(xM+yN)=(M+xM_x+yN_x)dx+(N+xM_y+yN_y)dy=(M+xM_x+yM_y)dx$

$+(N+xN_x+yN_y)dy=(1+k)(Mdx+Ndy)=0$ and so $xM+yN$ is constant in this interval.

2. (a) Take $Y=X$. Then $X=Y\subset\overline{\overline{X}}\cup\overline{Y}\cup Y\subset\overline{X\cup Y}=\overline{X}$. (b) By (a), $\overline{X}\subset\overline{\overline{X}}$ and the relation displayed above shows that $\overline{\overline{X}}\subset\overline{X}$. (c) If $Y\subset X$, then $\overline{Y}\subset\overline{\overline{X}}\cup\overline{Y}\cup Y\subset\overline{X\cup Y}=\overline{X}$. Conversely, $Y\subset X\cup Y\subset\overline{X\cup Y}$ by (a). But then $\overline{Y}\subset\overline{X\cup\overline{Y}}=\overline{X\cup Y}$ by (c) and (b). Also $\overline{\overline{X}}=\overline{X}\subset\overline{X\cup Y}$ by (b) and (c). The last three relations prove the converse statement.

3. Consider the faces with vertices ABC and ABD, which thus have edge AB in common. Let P and Q be the points of contact of the sphere with ABC and ABD respectively. Then sides AQ and AP are congruent, as are BQ and BP. (Tangents to a sphere from an external point.) Thus the triangles APB and AQB are congruent and $\angle APB=\angle AQB$. Let the points of contact of the sphere with faces ACD and BCD be R and S respectively. Then $\angle APB=\pi-\angle APC-\angle BPC=\pi-\angle ARC-\angle BSC=\pi-(\pi-\angle CRD-\angle ARD)-(\pi-\angle CSD-\angle BSD)=\angle CRD+\angle AQD-\pi+\angle CSD+\angle BQD=\angle CRD+\angle CSD-\angle AQB=2\angle CRD-\angle APB$. Thus the angles subtended by the opposite edges are equal and the theorem follows readily.

4. Since $B<\pi-C$, we have $\cos B>\cos(\pi-C)$ and $\cos B+\cos C>0$. Hence $[\sin A][\cos B+\cos C]>0$. If $B\leqq\pi/2$, then $\cos B$ is non-negative and, since $A>\sin A$, $A\cos B+\sin A\cos C>0$. If $B>\pi/2$, $-\cos B=+\cos(A+C)=\cos A\cos C-\sin A\sin C<\cos A\cos C$. Thus $[\tan A][\cos B+\cos A\cos C]=\tan A\cos B+\sin A\cos C>0$. Since $A<\tan A$ and $\cos B<0$, $A\cos B>\tan A\cos \mathrm{B}$ and so $A\cos B+\sin A\cos C>0$.

5. Consider $2n$ points numbered from 1 to $2n$. If each odd numbered point is joined to every even numbered point by a segment then no triangles are formed and there are n^2 segments used.

Suppose $2n$ points are joined by n^2+1 segments, and no triangles are formed. Take two points x and y that are joined by a segment. Then no point is joined to both x and y. Thus there can be no more than $2n-2$ segments joining x to points other than y or joining y to points other than x. Thus not more than $2n-1$ segments join to either of x or y. Thus there must be $n^2+1-(2n-1)$ lines joining the points other than x or y. Since no triangles are formed it follows that if $2(n-1)$ points cannot be joined by $(n-1)^2+1$ segments without forming triangles then neither can $2n$ points be joined by n^2+1 segments without forming triangles. But the result is obviously true for $n=1$ and thus it holds for all positive integers n.

6. By induction, $T_{n+1}=1+\prod_{j=1}^{n}T_j$ which implies (a) and also that $T_n\geqq 2^n$. Induction also establishes that $\sum_{j=1}^{n}1/T_j=1-(T_{n+1}-1)^{-1}$ and this implies (b).

7. Let the degree of P be n and suppose the degree of Q does not exceed n. If P has k zeros z_i, $i=1, 2, \cdots, k$, all distinct, and if $P+1$ has h distinct zeros z_j^*, $j=1, 2, \cdots, h$, then P', the derivative of both P and $P+1$, has $n-k$ zeros from the z_i and $n-h$ zeros from the z_j^*, multiplicity counted. Thus $n-1\geqq n-k+n-h$ or $h+k\geqq n+1$. But every z_i and every z_j^* is a zero of $P-Q$ and this implies $P-Q$ is identically zero.

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ANNIHILATORS IN POLYNOMIAL RINGS

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It is known that if f is a divisor of zero in the polynomial ring $R[x]$, where R is a commutative ring, there exists a non-zero element c of R such that $cf=0$. Proofs of this theorem have been given by McCoy [3], Forsythe [2], Cohen [1], and Scott [4]. It is clear that the theorem as stated does not immediately generalize to polynomials in more than one indeterminate. Moreover, it has been pointed out in Problem 4419 of this MONTHLY (1950, p. 692 and 1952, p. 336) that the theorem itself is not necessarily true for noncommutative rings. The purpose of this note is to obtain a suitable generalization of this result to the case of an arbitrary polynomial ring in any finite number of indeterminates.

Let R be an arbitrary ring and $R[x_1, \cdots, x_k]$ the ring of polynomials in the indeterminates $x_1, \cdots, x_k$, with coefficients in R. If A is a right ideal in $R[x_1, \cdots, x_k]$, let us denote by A^r the set of right annihilators of A, that is, the ideal consisting of all elements h of the ring $R[x_1, \cdots, x_k]$ such that $Ah=0$. We shall prove the following theorem.

THEOREM. *If R is an arbitrary ring and A is a right ideal in $R[x_1, \cdots, x_k]$ such that $A^r \neq 0$, then $A^r \cap R \neq 0$.*

The proof is by induction on k, so we begin by considering the case in which $k=1$. Throughout this part of the proof we shall write x in place of x_1.

We assume that $A^r \cap R=0$, and shall obtain a contradiction. From this assumption it follows that if c is a non-zero element of R, then $fc\neq 0$ for some $f \in A$. Let g be a non-zero element of A^r of minimum degree m, and let us set

$$g = b_0x^m + \cdots + b_m \qquad (b_0 \neq 0).$$

Then $m>0$, since $A^r \cap R=0$. Let f be an element of A such that $fb_0 \neq 0$, and let

$$f = a_0x^n + \cdots + a_n \qquad (a_0 \neq 0).$$

Since $fb_0 \neq 0$, we have $a_jg \neq 0$ for some $j=0, 1, \cdots, n$. Choose p as the smallest integer such that $a_pg \neq 0$. Then $fg=0$ yields $a_pb_0=0$, and it follows that the degree of a_pg is less than m. Moreover, $a_pg \in A^r$ since A^r is an ideal in $R[x]$. We therefore have a contradiction, and the theorem is established for the case in which $k=1$. This part of the proof was suggested by the referee and is a simplified version of our original proof.

To complete the proof we must establish the theorem for $k>1$ indeterminates under the assumption that it holds for $k-1$ indeterminates. Any element f of $R[x_1, \cdots, x_k]$ may be considered as a polynomial in x_k with coefficients

that are polynomials in $x_1, \cdots, x_{k-1}$. That is, we may write f in the form

$$f = h_0 x_k^n + \cdots + h_n,$$

where the h_i are elements of the ring $R[x_1, \cdots, x_{k-1}]$. For convenience, we may refer to the h_i as the *coefficients* of f. Since $A^r \neq 0$, the case already proved (with R replaced by $R[x_1, \cdots, x_{k-1}]$) shows that there exists a non-zero element g of $R[x_1, \cdots, x_{k-1}]$ such that $Ag = 0$. However, since g does not contain x_k, this implies that $hg = 0$ for *every* coefficient h of an element of A. Hence if B is the right ideal in the ring $R[x_1, \cdots, x_{k-1}]$ generated by all coefficients of elements of A, it follows that $g \in B^r$ and hence $B^r \neq 0$. By the induction hypothesis, we then have $B^r \cap R \neq 0$. But clearly $B^r \subseteq A^r$, and so $A^r \cap R \neq 0$, completing the proof.

COROLLARY. *If R is a commutative ring and f is a divisor of zero in the polynomial ring $R[x_1, \cdots, x_k]$, then there exists a non-zero element c of R such that $cf = 0$.*

This follows at once from the observation that if R is commutative and A is the principal ideal generated by f in $R[x_1, \cdots, x_k]$, then $g \in A^r$ if and only if $fg = 0$.

References

1. I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc., vol. 59, 1946, pp. 54–106.
2. A. Forsythe, Divisors of zero in polynomial rings, this MONTHLY, vol. 50, 1943, pp. 7–8.
3. N. H. McCoy, Remarks on divisors of zero, this MONTHLY, vol. 49, 1942, pp. 286–295.
4. W. R. Scott, Divisors of zero in polynomial rings, this MONTHLY, vol. 61, 1954, p. 336.

ON A CHARACTERIZATION OF SOME ORTHOGONAL FUNCTIONS

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1. Introduction. Nanjundiah proved [1] that if $P_n(x)$ is the Legendre polynomial of degree n then, $(1-x^2)D_n(x) = n(n+1)\Delta_n(x)$ where

$$D_n(x) = [P_n'(x)]^2 - P_{n+1}'(x)P_{n-1}'(x) \quad \text{and} \quad \Delta_n(x) = P_n^2(x) - P_{n-1}(x)P_{n+1}(x).$$

L. Carlitz [2] proved that this relation actually characterized the Legendre polynomials. M. S. Webster [3] obtained a similar characterization for the ultraspherical polynomials based on a relation derived in [4].

The purpose of this note is to point out a similar characterization for the Hermite and Bessel functions and derive a simpler one for the Tchebycheff polynomials.

In the following we define, for the sequence of functions $\{f_n(x)\}$

$$\Delta_n(f) = f_n^2 - f_{n+1}f_{n-1}, \qquad D_n(f) = f_n'^2 - f_{n+1}'f_{n-1}'.$$

2. Hermite polynomials. The Hermite polynomial of degree n is defined as

$$(2.1) \qquad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

These polynomials satisfy the recurrence formula

$$(2.2) \qquad H_{n+1}(x) - xH_n + nH_{n-1} = 0,$$

from which we obtain

$$(2.3) \qquad H_n(x) - xH_{n-1} + (n-1)H_{n-2} = 0.$$

From (2.2) and (2.3) we obtain

$$(2.4) \qquad \Delta_n(H) = n\Delta_{n-1}(H) + H_nH_{n-2}.$$

From (2.1) we get

$$(2.5) \qquad D_n(H) = n^2\Delta_n(H) + H_nH_{n-2}.$$

From (2.4) and (2.5) we get

$$(2.6) \qquad D_n(H) = \Delta_n(H) + n(n-1)\Delta_{n-1}(H).$$

THEOREM. *Let* $\{f_n(x)\}$ *be a sequence of polynomials such that deg* $f_n = n$, $f_0(x) = 1$, $f_1(x) = x$, *and*

$$(2.7) \qquad D_n(f) = \Delta_n(f) + n(n-1)\Delta_{n-1}(f).$$

Then $f_n(x) = H_n(x)$.

Proof: Let $f_k(x) = H_k(x)$ for $0 \leqq k \leqq n$ and let $f_{n+1} = H_{n+1}(x) + g(x)$ where $g(x)$ is a polynomial of degree $m \leqq n+1$. Using (2.7) and (2.6) we obtain the relation $H'_{n-1}g'(x) = H_{n-1}g(x)$ which is impossible unless $g(x) \equiv 0$, since the left hand side is of degree $n+m-3$ and the right hand side is of degree $n+m-1$.

In a similar fashion one can show that solutions of Weber's equation

$$\frac{d^2W}{dx^2} + \left\{n + \frac{1}{2} - \frac{x^2}{4}\right\} W = 0$$

satisfy

$$\frac{1}{4} D_n(W) = \Delta_n(W) + n(n-1)\Delta_{n-1}(W).$$

(See [5] p. 360 for definition and recurrence formulas used in obtaining the above relation.)

3. Tchebycheff polynomials. Defining the Tchebycheff polynomials [6] as

$$(3.1) \qquad T_n(x) = 2^{-(n-1)} \cos (n \cos^{-1} x)$$

one can show easily that

$$(3.2) \qquad \Delta_n(T) = 2^{-2(n-1)}(1 - x^2).$$

THEOREM. *Let $\{f_n(x)\}$ be a sequence of functions such that $f_0(x)=1$, $f_1(x)=x$ and such that*

$$\Delta_n(f) = 2^{-2(n-1)}(1 - x^2). \tag{3.3}$$

Then $f_n(x)=T_n(x)$.

For the case of the Tchebycheff polynomials of the second kind

$$Q_n(x) = \sin [(n + 1) \cos^{-1} x]/(1 - x^2)^{1/2} \tag{3.4}$$

we get

$$\Delta_n(Q) = 1. \tag{3.5}$$

THEOREM. *Let $\{f_n(x)\}$ be a sequence of functions such that $f_0(x)=1$, $f_1(x)=2x$ and $\Delta_n(f)=1$. Then $f_n(x)=Q_n(x)$.*

The proofs of the above two theorems are similar to that of section 2 except that the assumption that f_n be a polynomial is no longer necessary.

4. Bessel functions. Let $J_n(x)$ and $I_n(x)$ be respectively the Bessel functions of the first kind and the modified Bessel Functions of the first kind with usual notations (see [5] Chap. 17). Let us define

$$\phi_n = x^n J_n(x). \tag{4.1}$$

Hence

$$\phi_n'(x) = x^n J_{n-1}(x) = x\phi_{n-1}(x). \tag{4.2}$$

Now it is easily seen that

$$D_n(\phi) = x^2\Delta_{n-1}(\phi). \tag{4.3}$$

THEOREM. *Let $\{f_n(x)\}$ be a sequence of differentiable functions such that*

$$D_n(f) = x^2\Delta_{n-1}(f), \qquad f_n(0) = 0 \qquad (n > 1). \tag{4.4}$$

Then

(a) $f_n(x) = x^n J_n(x)$ *if* $f_0(x) = J_0(x)$, $f_1(x) = - J_0(x)$, $f_2(x) = x^2 J_2(x)$,

(b) $f_n(x) = x^n I_n(x)$ *if* $f_0(x) = I_0(x)$, $f_1(x) = xI_0(x)$, $f_2(x) = x^2 I_2(x)$.

Proof: We shall prove part (a) only. Part (b) follows in the same fashion. Assume

$$f_k(x) = x^k J_k(x) \qquad (0 \leqq k \leqq n),$$
$$f_{n+1}(x) = x^{n+1}J_{n+1}(x) + g(x).$$

Substituting in (4.4) and using (4.3) we get $g'(x)=0$, and hence $g(x)=$constant$=c_{n+1}$,

$$f_{n+1} = x^n J_n(x) + c_{n+1}.$$

But

$$f_{n+1}(0) = 0, \qquad c_{n+1} = 0.$$

Szasz [7] defined $\Lambda_n^{(x)} = (2/x)^n \Gamma(n+1) J_n(x)$ and obtained

$$D_n(\Lambda) = \frac{4n(n-1)}{x^2} \Delta_{n-1}(\Lambda).$$

This relation can be used for a similar characterization.

References

1. T. S. Nanjundiah, Half-Yearly J. Mysore Univ. Sect. B. vol. 11, 1950.
2. L. Carlitz, Bull. Calcutta Math. Soc. vol. 46, 1954.
3. M. S. Webster, Bull. Amer. Math. Soc. 61, 1955.
4. V. R. Thuruvenkatachar and T. S. Nanjundiah, Proc. Indian Acad. Sci., Sect. A, vol. 33, 1951.
5. E. T. Whittaker and G. N. Watson, Modern Analysis Cambridge, 1950.
6. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 1, New York, 1953.
7. Otto Szasz, Proc. Amer. Math. Soc. vol. 1, 1950, pp. 256–267.

NOTE ON A ONE DIMENSIONAL NON-CONSERVATIVE SYSTEM

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1. The system of two linear differential equations

$$x' = f(t)y, \qquad y' = -f(t)x \qquad (' = d/dt), \tag{1}$$

in which $f(t)$ denotes a continuous function on $-\infty < t < \infty$, can be regarded as the linear canonical equations belonging to a one dimensional dynamical system with Hamiltonian $\frac{1}{2}f(t)(x^2+y^2)$. Only real-valued non-trivial ($\not\equiv 0$) solutions of (1) will be considered.

Suppose that $f(t)$ is periodic with a period p, so that

$$f(t) = f(t+p), \qquad -\infty < t < \infty. \tag{2}$$

Then, since any solution (x, y) of (1) satisfies $x^2+y^2 \equiv \text{const.}$ (whether or not (2) holds) and hence is bounded on $-\infty < t < \infty$, it follows from the standard Floquet theory for systems of equations that every solution (x, y) is almost periodic.

THEOREM. *If $f(t)$ possesses a single (proper, relative) maximum and a single (proper, relative) minimum over a period p and if (x, y) is any periodic solution of (1) having p as a period, then either (i) at least one of the components x or y has a non-zero constant term in its Fourier expansion or (ii) the curve defined by*

$$u(t) = \int_0^t x(s)ds, \qquad v(t) = \int_0^t y(s)ds \tag{3}$$

is a closed curve (of class C^2) consisting of exactly two simple loops.

In (ii), a closed curve with two simple loops is defined as in [3, p. 573], to be a curve consisting of two simple closed curves, neither of which crosses the other.

Remarks. In the theorem above, it is clear that the maximum and minimum of $f(t)$ are actually absolute extrema. Moreover, if (x, y) is a solution of (1) so also is $(-y, x)$ and the two solutions are linearly independent. Thus, it is clear that if either (i) or (ii) holds for some solution of (1), the corresponding assertion holds for every solution. That the assertion of the theorem can become false if $f(t)$ does not satisfy the condition that it possess a single maximum and minimum is readily seen by a consideration of the (linear oscillator) example in which $f\equiv 1$. In this case $(\sin t, \cos t)$ is a periodic solution of (1) and neither (i) nor (ii) holds; in fact, the curve defined by (3) is a circle.

The proof of the theorem, given below, will be seen to be an easy consequence of known facts on the differential geometry of curves.

2. *Proof.* Suppose that (i) fails to hold; hence $\int_0^p x(s)ds = \int_0^p y(s)ds = 0$. Then the functions u, v of (3) have a period p, are twice differentiable, and the corresponding curve in the u, v plane is closed. It remains to be shown that this curve has exactly two loops.

The system (1) can be expressed in the vector form

$$X' = f(t)Y, \qquad Y' = -f(t)X, \tag{4}$$

where $X=(x, y)$ and $Y=(-y, x)$. Since (1) is linear and homogeneous, the orthogonal vectors X and Y can be normalized to unit length and thus can be made to correspond to the unit tangent and principal normal to a plane curve with curvature $f(t)$ (not necessarily positive) given by the (Frenet) equations (4).

According to a generalization (*cf.* [1], [2]) of the standard four vertex theorem [4] of differential geometry (in which the usual convexity assumption, corresponding to the positivity of the curvature f, is dropped), it follows from the present assumption on $f(t)$ (namely, that it possesses exactly two extrema) that the locus (3) cannot be a simple closed curve. In fact, as was shown by Jackson [3], this locus consists of exactly two loops, and the proof of the theorem is now complete.

References

1. D. Fog, Über den Vierscheitelsatz und seine Verallgemeinerungen, S.-B. Deutsch. Akad. Wiss. Berlin Math.-Nat. Kl., 1933, pp. 251–254.

2. W. C. Graustein, Extensions of the four-vertex theorem, Trans. Amer. Math. Soc., vol. 41, 1937, pp. 9–23.

3. S. B. Jackson, Vertices of plane curves, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 564–578.

4. S. Mukhopadhyaya, New methods in the geometry of a plane arc, Bull. Calcutta Math. Soc., vol. 1, 1909, pp. 31–37.

A PROOF OF THE COMPOSITE FUNCTION THEOREM FOR MATRIC FUNCTIONS

D. W. Robinson, Case Institute of Technology

The composite function theorem for matric functions was stated and demonstrated by R. F. Rinehart [1] in vol. 62, 1955, this Monthly. The proof contained in that paper seems to be rather lengthy and tedious. The present note provides a simpler proof, which depends upon the combinatorial properties enjoyed by matric functions.

First, let A be a square matrix over the field of complex numbers. Let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of A of respective multiplicities $s_1, \cdots, s_k$, called indices, in the minimum polynomial of A. Let f be a complex-valued function of a complex variable that is defined at each λ_i, $i=1, \cdots, k$, and analytic at those λ_i whose $s_i > 1$. Then, using the definition of H. Schwerdtfeger [2], let the matrix

$$f(A) \equiv \sum_{i=1}^{k} A_i \sum_{m=0}^{s_i-1} \frac{f^{(m)}(\lambda_i)}{m!} (A - \lambda_i I)^m$$

correspond to A under f, where A_i is the Frobenius covariant (projection) of A associated with λ_i, $i=1, \cdots, k$. (See [1] for equivalent definitions).

It is readily demonstrated that this defined composition satisfies the following combinatorial requirements, which were first stated by L. Fantappiè. If the composition of A with f and g is defined, then it is also defined for $f+g$ and $f \cdot g$, and

$$(f+g)(A) = f(A) + g(A), \tag{1}$$

$$(f \cdot g)(A) = f(A) \cdot g(A). \tag{2}$$

Further,

$$\text{If} \quad f(z) = k, \quad \text{then} \quad f(A) = kI. \tag{3}$$

$$\text{If} \quad f(z) = z, \quad \text{then} \quad f(A) = A. \tag{4}$$

A further property enjoyed by matric functions, with which this note is concerned, is the composite function theorem for single-valued functions.

Theorem. *Let A have distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ with associated indices $s_1, \cdots, s_k$. Let g and f be single-valued functions, respectively defined at λ_i and $g(\lambda_i)$, $i=1, \cdots, k$, and analytic at the λ_i and $g(\lambda_i)$ for which $s_i > 1$. Let $h(z) \equiv f(g(z))$. Then $f(g(A))$ and $h(A)$ are both defined and are equal.*

Proof. By the requirements of composition, it is immediate that both $g(A)$ and $h(A)$ are defined. It is also observed, by the properties of similarity transformations, that for P non-singular, $g(PAP^{-1})$ is defined and given by $P\,g(A)P^{-1}$. Using the Jordan canonical form of A, it then follows from direct evaluation of $g(PAP^{-1})$ that the eigenvalues (not necessarily distinct) of $g(A)$ are

$g(\lambda_1), \cdots, g(\lambda_k)$, where the index of $g(\lambda_i)$ is at most the maximum of the s_j such that $g(\lambda_j) = g(\lambda_i)$. Hence, $f(g(A))$ is defined.

Let F be a polynomial, such that $F^{(m)}(g(\lambda_i)) = f^{(m)}(g(\lambda_i))$, $m=0, \cdots, s_i-1$, $i=1, \cdots, k$. Then, by the definition of composition, $F(g(A)) = f(g(A))$. Also, define $H(z) \equiv F(g(z))$. Then $H^{(m)}(\lambda_i) = h^{(m)}(\lambda_i)$, $m=0, \cdots, s_i-1$, $i=1, \cdots, k$. Hence, $H(A) = h(A)$. But by the preceding combinatorial properties, since F is a polynomial, $H(A) = F(g(A))$. The final conclusion is a consequence of the last three results.

References

1. R. F. Rinehart, The equivalence of definitions of a matric function, this MONTHLY, vol. 62, 1955, pp. 395–414.
2. H. Schwerdtfeger, Les Fonctions de Matrices I. Les Fonctions Univalentes, Actualitiés Scientifiques et Industrielles, No. 649, Paris, 1938.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

NOTE ON LINEAR DIFFERENTIAL EQUATIONS

R. STEINBERG, University of California, Los Angeles, and the Institute for Advanced Study

Let D designate differentiation with respect to x, and let f and g be polynomials with constant (complex) coefficients. Consider the equation

$$f(D)z = u, \tag{1}$$

where u is a given function of x, which satisfies the equation

$$g(D)u = 0. \tag{2}$$

Suppose that we wish to find a particular solution of (1) by the method of undetermined coefficients. The author has perused many textbooks on differential equations and has not been able to find a precise statement as to what should be tried. The object of this note is to supply such a statement.

THEOREM. *Let h and k be polynomials such that $f=hk$ and k is relatively prime to g. Then there exists a solution of* (1) *which is also a solution of*

$$g(D)h(D)z = 0. \tag{3}$$

Such a solution can be obtained by applying a polynomial differential operator to any solution v of the equation

$$h(D)v = u. \tag{4}$$

Proof. Since k and g are relatively prime, there exist polynomials p and q such that

$$pk + qg = 1. \tag{5}$$

Now, let v be any solution of (4) and set $z=p(D)v$. To complete the proof, we need only show that z is a solution of both (1) and (3). First, multiply (5) by h and apply the corresponding differential operators to the function v. This gives

$$p(D)k(D)h(D)v + q(D)g(D)h(D)v = h(D)v.$$
$$f(D)p(D)v + q(D)g(D)h(D)v = h(D)v.$$
$$f(D)z = u, \text{ by (2) and (4).}$$

Thus z is a solution of (1). Finally, we have

$$\begin{aligned} g(D)h(D)z &= g(D)h(D)p(D)v \\ &= p(D)g(D)u, \text{ by (4)} \\ &= 0, \text{ by (2);} \end{aligned}$$

so that (3) is also fulfilled.

COROLLARY. *If f and g are relatively prime, then a solution of* (1) *can be obtained by applying a polynomial differential operator to u.*

Proof. In this case, we can take $h=1$. Then $v=u$ and $z=p(D)v=p(D)u$.

Example. Consider the equation

$$(D^2 - 1)z = x \sin x.$$

It is easily seen that

$$(D^2 + 1)^2(x \sin x) = 0,$$

and that $(D^2+1)^2$ is relatively prime to D^2-1. Hence the corollary applies. Now, Euclid's algorithm gives the identity

$$-\tfrac{1}{4}(D^2 + 3)(D^2 - 1) + \tfrac{1}{4}(D^2 + 1)^2 = 1.$$

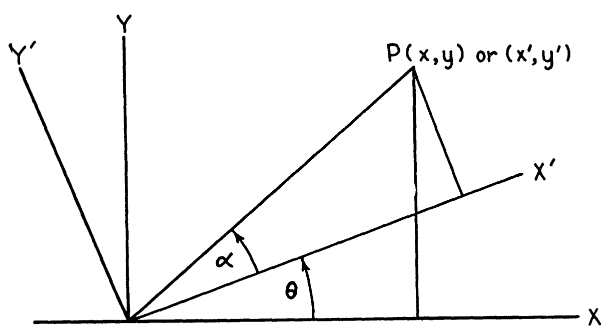
Thus $p(D) = -\frac{1}{4}(D^2+3)$, and a solution is $z=p(D)u= -\frac{1}{4}(D^2+3)(x \sin x) = -\frac{1}{2}x \sin x-\frac{1}{2} \cos x$.

THE ROTATION OF AXES

ARTHUR PORGES, Los Angeles City College

In deriving the equations for the rotation of axes in plane analytic geometry, some textbook writers become needlessly involved in complicated relations based on similar triangles. A better method, easier and more useful in recalling a topic of mathematical importance to the minds of the students, makes use of De Moivre's Theorem. This variant has the additional advantage of giving the inverse transformations with a minimum of extra computation.

Given a point P with coordinates (x, y) referred to the X- and Y-axes, and coordinates (x', y') referred to new axes, X' and Y', where the X'-axis makes a positive angle $\theta < 90°$ with the X-axis, we may write P in the forms $x+iy$ and $x'+iy'$, where the two complex numbers differ only in amplitude according to the axes of reference implied.



From the diagram it is evident that the following relations are valid:

(1) $$Z = x + iy = r[\cos(\alpha + \theta) + i \sin(\alpha + \theta)].$$

(2) $$Z = x' + iy' = r(\cos\alpha + i \sin\alpha).$$

From De Moivre's Theorem we may write (1) in the form

(3) $$Z = r(\cos\alpha + i\sin\alpha)(\cos\theta + i\sin\theta).$$

Replacing the first binomial factor of the right member of (3) by its value from (2), we have

(4) $$Z = (x' + iy')(\cos\theta + i\sin\theta).$$

Equating the real and imaginary parts in (4), we find

(5) $$x = x'\cos\theta - y'\sin\theta,\ y = x'\sin\theta + y'\cos\theta,$$

the standard rotation formulas.

If now we multiply both sides of (4) by $\cos\theta - i\sin\theta$, and again equate the real and imaginary parts, we derive, without explicitly solving the system (5), the inverse equations $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$.

RAPID SKETCHING OF A CONIC

ARTHUR PORGES, Los Angeles City College

Rotation of the axes in discussing the equation

$$AX^2 + BXY + CY^2 + DX + EY + F = 0, \qquad (B \neq 0), \tag{1}$$

is of considerable interest as mathematical theory, but as a practical aid in making a rapid sketch of a conic leaves much to be desired.

The following method, based on oblique coordinates, seems to have many advantages, since for conics with rational coefficients it involves no irrational transformations at any stage.

The discussion will be restricted to the case in which either A or C is different from zero, since if both vanish, a rotation through 45 degrees obviously simplifies the equation, which represents, if not degenerate, a hyperbola easily graphed.

By completing the square and writing (1) in the form

$$\left(X + \frac{BY}{A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{A^2}\right)Y^2 + \frac{DX}{A} + \frac{EY}{A} + \frac{F}{A} = 0, \tag{2}$$

we may make the substitutions

$$U = X + \frac{BY}{A}, \qquad V = Y. \tag{3}$$

With reference to an oblique coordinate system having a V-axis $AX+BY=0$ and U-axis $V=Y=0$, (1) becomes

$$U^2 + \left(\frac{C}{A} - \frac{B^2}{A^2}\right)V^2 + \frac{DU}{A} + \left(\frac{E}{A} - \frac{BD}{A^2}\right)V + \frac{F}{A} = 0, \tag{4}$$

a simple conic quickly sketched by at most one translation.

As an example, consider the equation,

$$2X^2 + 5XY - 4Y^2 + 8X - 3Y + 20 = 0. \tag{5}$$

We have, after completing the square,

$$\left(X + \frac{5}{4}Y\right)^2 - \frac{57}{16}Y^2 + 4X - \frac{3Y}{2} + 10 = 0. \tag{6}$$

Letting $U = X + \frac{5}{4}Y$ and $V = Y$, (6) becomes

$$U^2 - \frac{57}{16}V^2 + 4U - \frac{13}{2}V + 10 = 0. \tag{7}$$

Another completion of the square gives

$$(8)\qquad (U+2)^2 - \frac{57}{16}\left(V+\frac{52}{57}\right)^2 = -\frac{511}{57}\,.$$

Since the X-axis always remains fixed, although called the U-axis for symmetry of notation, we need only draw in the line $4X+5Y=0$ as our V-axis. Referred to these oblique axes, the conic is obviously a hyperbola with its center at $U=-2$ and $V=-\frac{52}{57}$. We may use the asymptotes in the usual fashion to "box in" the curve, but even with this refinement a good graph may be completed in less time than most students take to determine the equations of rotation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1246. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relation between the radius of the base and the altitude of a right circular cone in which a trihedral angle can be inscribed whose face angles are all equal to a given angle 2α.

Show that if two trihedral angles whose face angles are all equal to 2α and $2\alpha'$, respectively, are inscribed in two right circular cones having a common base, then a necessary and sufficient condition for the radius of the common base to be a mean proportional between the altitudes of the cones is that

$$\sin^2\alpha + \sin^2\alpha' = 3/4.$$

E 1247. *Proposed by C. W. Topp, Fenn College*

Prove that

$$\sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\binom{kn}{n} = (-1)^{n+1}n^n.$$

E 1248. *Proposed by Leo Moser, University of Alberta*

(a) The ten numbers $s_1 \leqq s_2 \leqq \cdots \leqq s_{10}$ are the sums of the five unknown numbers $x_1 \leqq x_2 \leqq \cdots \leqq x_5$ taken two at a time. Determine the x's in terms of the s's.

(b) Show that if $s_1 < s_2 < \cdots < s_6$ are six distinct numbers formed by taking the sums of four numbers two at a time, then there exist four other numbers which give the same sums when added in pairs.

E 1249. *Proposed by C. M. Sandwick, Sr., Easton High School, Easton, Pa.*

Find an integer less than 1000, the cube of which may be represented as the sum of the cubes of three positive integers in five distinct ways.

E 1250. *Proposed by N. A. Court, University of Oklahoma*

Through a point G two secants GAD, GBC are drawn meeting a given circle (H) in the points A, D; B, C. Show that the points $E=(AB, CD)$, $F=(AC, BD)$ are the centers of similitude of the two circles orthogonal to (H) and having for centers the harmonic conjugates of G for the pairs of points A, D; B, C, respectively.

SOLUTIONS

A Theorem of Van Aubel

E 1216 [1956, 342]. *Proposed by N. A. Court, University of Oklahoma*

The area of the triangle formed by the midpoints of three (not necessarily concurrent) cevians drawn through the three vertices of a given triangle is equal to one fourth of the area of the triangle determined by the feet of those cevians.

I. *Solution by Chih-yi Wang, University of Minnesota.* Let the coordinates of the vertices and the feet of the cevians be

$$A(-2a, 0), \qquad B(2b, 0), \qquad C(0, 2c),$$
$$X(2bd, 2c(1-d)), \qquad Y(-2ae, 2c(1-e)), \qquad Z(2f, 0).$$

Then the midpoints D, E, F of AX, BY, CZ are, respectively,

$$(bd-a, c(1-d)), \qquad (b-ae, c(1-e)), \qquad (f, c).$$

Using the determinant expression for the area of a triangle we easily obtain

$$(XYZ) = 2\,|\,c(bd+ef+ae-df-bde-ade)\,| = 4(DEF),$$

where (XYZ) and (DEF) represent the areas of triangles XYZ and DEF respectively.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* Let the triangle have vertices A, B, C; denote the feet of the cevians by X, Y, Z (X on BC, Y on CA, Z on AB) and their respective midpoints by D, E, F. We assume a coordinate system with $A=(a, a')$, $B=(b, b')$, *etc.* The absolute value of

$$(1/2)\begin{vmatrix} (a+x)/2 & (a'+x')/2 & 1 \\ (b+y)/2 & (b'+y')/2 & 1 \\ (c+z)/2 & (c'+z')/2 & 1 \end{vmatrix}$$

gives the area (DEF). But this determinant may be decomposed into

$$(1/8)\left\{\begin{vmatrix} a & a' & 1 \\ b & b' & 1 \\ c & c' & 1 \end{vmatrix} + \begin{vmatrix} a & x' & 1 \\ b & y' & 1 \\ c & z' & 1 \end{vmatrix} + \begin{vmatrix} x & a' & 1 \\ y & b' & 1 \\ z & c' & 1 \end{vmatrix} + \begin{vmatrix} x & x' & 1 \\ y & y' & 1 \\ z & z' & 1 \end{vmatrix}\right\},$$

where the sum of the first three determinants is equivalent to the sum

$$\begin{vmatrix} a & a' & 1 \\ b & b' & 1 \\ z & z' & 1 \end{vmatrix} + \begin{vmatrix} a & a' & 1 \\ y & y' & 1 \\ c & c' & 1 \end{vmatrix} + \begin{vmatrix} x & x' & 1 \\ b & b' & 1 \\ c & c' & 1 \end{vmatrix}.$$

But A, B, Z; A, Y, C; X, B, C are each collinear triples, whence the latter sum is zero and all that remains is

$$(1/8)\begin{vmatrix} x & x' & 1 \\ y & y' & 1 \\ z & z' & 1 \end{vmatrix},$$

the absolute value of which is $(XYZ)/4$.

III. *Solution by Azriel Rosenfeld, Columbia University.* We employ triangular coordinates, taking the given triangle ABC as base triangle (with area normalized as 1). Then the feet X, Y, Z of the cevians have the form

$$(0, a, 1-a), \qquad (b, 0, 1-b), \qquad (c, 1-c, 0),$$

and the midpoints D, E, F of the cevians are

$$(1/2, a/2, (1-a)/2), \qquad (b/2, 1/2, (1-b)/2), \qquad (c/2, (1-c)/2, 1/2).$$

Then

$$(XYZ)/(DEF) = \begin{vmatrix} 0 & a & 1-a \\ b & 0 & 1-b \\ c & 1-c & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} 1/2 & a/2 & (1-a)/2 \\ b/2 & 1/2 & (1-b)/2 \\ c/2 & (1-c)/2 & 1/2 \end{vmatrix} = 4.$$

IV. *Solution by O. J. Ramler, The Catholic University of America.* Let the given triangle ABC have cevians AX, BY, CZ meeting the sides of the medial triangle of ABC in the cevian midpoints D, E, F. Since the medial triangle is homothetic to the given triangle in the ratio 1:2 with respect to their common centroid, it follows that the points X', Y', Z' on sides BC, CA, AB of the given triangle, corresponding to the points D, E, F on the sides of the medial triangle, are isotomically situated on those sides to the points X, Y, Z. Since the areas (XYZ) and $(X'Y'Z')$ are equal (see *e.g.*, Nathan Altshiller-Court, *College Geometry*, 2nd ed., art. 324, p. 157), and because of the homothecy of DEF and $X'Y'Z'$, area (DEF) is one fourth area $(X'Y'Z')$, or one fourth area (XYZ).

Also solved by Leon Bankoff, A. P. Boblétt, A. E. Danese, Michael Goldberg,

Edgar Karst, Sam Kravitz, M. A. Laframboise, Josef Langr, Beckham Martin, Paul Payette, Nathaniel Queen, J. T. Robertson and Dale Woods (jointly), G. B. Robison, Victor Thébault, Alan Wayne, David Zeitlin, and the proposer.

Goldberg employed an attack based on the fact that, since the theorem is an affine one, it suffices to consider the case where the given triangle is equilateral.

Thébault remarked that the theorem is due to Van Aubel (*Mathesis*, 1881, p. 202), that the same problem also appears in *Journal de Vuibert*, vol. 43, p. 100, and that the problem has been generalized by H. L. Meunessier in the form:

If on the sides BC, CA, AB of a triangle ABC points X, Y, Z are chosen such that $CX/CB=m$, $AY/AC=n$, $BZ/BA=p$, then the algebraic area (DEF) of the triangle formed by the points D, E, F of the lines AX, BY, CZ defined by the relation $AD/AX=BE/BY=CF/CZ=\lambda$ has for value

$$(DEF) = (XYZ)\frac{[(mn+np+pm)-(m+n+p)+3]\lambda^2-3\lambda+1}{(mn+np+pm)-(m+n+p)+1}.$$

E 1216 is the case where $\lambda=\frac{1}{2}$. If $\lambda=\frac{1}{2}$ and X, Y, Z are collinear, we obtain the classical theorem: *The midpoints of the three diagonals of a complete quadrilateral are collinear.* When D, E, F are taken on CA, AB, BC such that $AD/AC=BE/BA=CF/CB=\lambda$, we find

$$(DEF) = (3\lambda^2-3\lambda+1)(ABC).$$

In terms of the above notation, Martin showed that

$$(DEF) = \lambda^2(XYZ)+(2\lambda^2-3\lambda+1)(ABC).$$

A Property of Euler's Function

E 1217 [1956, 342]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Evaluate

$$\prod_{d|n} d^{\phi(n/d)+\phi(d)}.$$

Solution by J. B. Johnston, Cornell University. Let f be any function defined on the integers. Then

$$\begin{aligned}\prod_{d|n} d^{f(d)+f(n/d)} &= \prod_{d|n} d^{f(d)}\prod_{d|n} d^{f(n/d)}\\ &= \prod_{d|n} d^{f(d)}\prod_{d|n}(n/d)^{f(d)}\\ &= \prod_{d|n} n^{f(d)} = n^{\sum_{d|n} f(d)}.\end{aligned}$$

Since

(1) $$\sum_{d|n}\phi(d) = n,$$

the answer to the given problem is n^n.

Also solved by W. J. Buckingham, Leonard Carlitz, A. E. Danese, M. P. Drazin, L. T. Gardner, A. J. Goldman, D. S. Greenstein, Cornelius Groenewoud, Emil Grosswald, Virginia Hanly, A. R. Hyde, Richard Kelisky, Sidney Kravitz, R. G. McDermot, D. C. B. Marsh, Leo Moser, J. B. Muskat, F. R. Olson, Hiram Paly, M. Perisastri, Azriel Rosenfeld, A. V. Sylwester, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by M. S. Klamkin.

Editorial Note. For a proof of (1) see, *e.g.*, Uspensky and Heaslet, *Elementary Number Theory*, p. 113. As another application of the general result established above we have

$$\prod_{d|n} d^{(n/d)+d} = n^{\sigma(n)},$$

where $\sigma(n)$ is the sum of the divisors of n.

A Proposition Equivalent to Dirichlet's Theorem

E 1218 [1956, 342]. *Proposed by Robert Spira, Berkeley, Calif.*

Consider the two propositions:

I. If $(a, b)=1$, then $ax+b$ assumes infinitely many prime values.
II. If $(a, b)=1$, then $ax+b$ assumes at least one prime value.

I is Dirichlet's theorem. Clearly I implies II. Show that II implies I.

Solution by David Zeitlin, University of Minnesota. By II there exists x_1 such that ax_1+b is prime. Then $(a, ax_1+b)=1$ and so, again by II, there exists x_2 such that $ax_2+(ax_1+b)=a(x_2+x_1)+b$ is prime. Continuing this process we see that $ax+b$ is prime for infinitely many values of x, and I follows.

Also solved by S. B. Adam and Givat Brenner (jointly), W. J. Cody, D. S. Greenstein, Virginia Hanly, P. J. McCarthy, D. C. B. Marsh, Leo Moser, Hiram Paley, Azriel Rosenfeld, R. E. Williamson, Jr., and the proposer.

Limit of a Sequence

E 1219 [1956, 342]. *Proposed by D. A. Robinson, University of Wisconsin*

The sequence $\{s_n\}$, where $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$, has a finite limit. Find this limit in closed form.

Solution by David Freedman, Montreal, Quebec. Since $s_1<2$ and since $s_k<2$ implies that $s_{k+1}<\sqrt{2+\sqrt{2}}<2$, it follows that $\{s_n\}$ is bounded above. Also, since $s_2>s_1$ and $s_{k+1}^2-s_k^2=\sqrt{s_k}-\sqrt{s_{k-1}}$, we see that $s_k>s_{k-1}$ implies $s_{k+1}>s_k$, and it follows that $\{s_n\}$ is monotone increasing. Hence $\{s_n\}\to s\leqq 2$. But then $s=\sqrt{2+\sqrt{s}}$, or $s^4-4s^2-s+4=(s-1)(s^3+s^2-3s-4)=0$. Since $s=1$ is extraneous for the problem, we seek the real root of the cubic factor and obtain, by Cardan's formula,

$$s = [\{(79 + 3\sqrt{249})/2\}^{1/3} + \{(79 - 3\sqrt{249})/2\}^{1/3} - 1]/3 = 1.8311771+.$$

Also solved by A. P. Boblétt, Julian Braun, A. E. Danese, L. T. Gardner,

D. S. Greenstein, Cornelius Groenewoud, Emil Grosswald, A. R. Hyde, J. B. Johnston, P. G. Kirmser, Sidney Kravitz, D. C. B. Marsh, Erich Michalup, K. W. Miller, J. B. Muskat, C. S. Oglivy, O. W. Rechard, Azriel Rosenfeld, C. M. Sandwick, Sr., Robert Sibson, and the proposer.

Muskat called attention to Rudin, *Principles of Mathematical Analysis*, prob. 3, p. 62, and Rechard called attention to a more general result found on p. 868 in C. J. Everett, "Representations for real numbers," *Bull. Amer. Math. Soc.*, vol. 52, 1946, pp. 861–869.

A Bound for the Absolute Error in Partially Summing a Series

E 1220 [1956, 342]. *Proposed by Judith Blankfield, University of Illinois*

It is well known that if $r_1 > r_2 > r_3 > \cdots \to 0$, then

$$\left| \sum_{t=n+1}^{\infty} (-1)^t r_t \right| \leqq r_n.$$

Prove the following generalization which reduces to the above case when $\theta = \pi$.

$$\left| \sum_{t=n+1}^{\infty} r_t e^{it\theta} \right| \leqq r\ /\sin(\theta/2), \qquad 0 < \theta < 2\pi.$$

I. *Solution by Michael Goldberg, Washington, D. C.* Using the Argand diagram, the partial sums of

$$\sum_{t=n+1}^{\infty} r_n e^{it\theta}$$

are represented by the vertices of a polygon which lie on a circle through the origin and whose diameter is $r_n/\sin(\theta/2)$. However, if $r_n > r_{n+1} > r_{n+2} > \cdots \to 0$, the vertices of the broken line spiral which represent the partial sums of the given sum all lie within the circle. Hence the theorem.

II. *Solution by David Zeitlin, University of Minnesota.* If $0 < \theta < 2\pi$ and $q \geqq 1$, then

$$\left| \sum_{t=n+1}^{n+q} e^{it\theta} \right| = \left| e^{i(n+1)\theta}(1 - e^{iq\theta})/(1 - e^{i\theta}) \right|$$
$$\leqq 2/\left| 1 - e^{i\theta} \right| = 1/\sin(\theta/2).$$

Therefore, by Abel's inequality,

$$\left| \sum_{t=n+1}^{n+q} r_t e^{it\theta} \right| \leqq r_{n+1}/\sin(\theta/2) < r_n/\sin(\theta/2).$$

Letting $q \to \infty$ we obtain the desired result.

III. *Solution by Chih-yi Wang, University of Minnesota,* If $0 < \theta < 2\pi$ and $q \geqq 0$, then

$$\left|\sum_{t=n+1}^{n+q} r_t e^{it\theta}\right| = \left|\sum_{t=n+1}^{n+q} r_t e^{it\theta}(e^{i\theta}-1)\right| / \left|e^{i\theta}-1\right|$$

$$= \frac{\left|-r_n e^{i(n+1)\theta} + \sum_{j=n}^{n+q-1}(r_j - r_{j+1})e^{i(j+1)\theta} + r_{n+q}e^{i(n+q+1)\theta}\right|}{\left|e^{i\theta}-1\right|}$$

$$\leqq \left(r_n + \sum_{j=n}^{n+q-1}(r_j - r_{j+1}) + r_{n+q}\right)/2\sin(\theta/2)$$

$$= r_n/\sin(\theta/2).$$

Letting $q\to\infty$ we obtain the desired result.

IV. *Solution by M. Perisastri, Vizianagram, India.* Let

$$C = \sum_{t=n+1}^{m} \cos t\theta, \qquad S = \sum_{t=n+1}^{m} \sin t\theta.$$

Then it is known that, for $0<\theta<2\pi$,

$$C = \cos(m+n+1)\theta/2 \sin(m-n)\theta/2 \csc(\theta/2),$$

$$S = \sin(m+n+1)\theta/2 \sin(m-n)\theta/2 \csc(\theta/2),$$

whence $(C^2+S^2)^{1/2}\leqq\csc(\theta/2)$. Therefore

$$\left|\sum_{t=n+1}^{m} r_t e^{it\theta}\right| = \left|\sum_{t=n+1}^{m} r_t\cos t\theta + i\sum_{t=n+1}^{m} r_t \sin t\theta\right|$$

$$= \left[\left(\sum_{t=n+1}^{m} r_t\cos t\theta\right)^2 + \left(\sum_{t=n+1}^{m} r_t \sin t\theta\right)^2\right]^{1/2}$$

$$< r_n(C^2+S^2)^{1/2} \leqq r_n/\sin(\theta/2).$$

Letting $m\to\infty$ we obtain the desired result.

Also solved by A. P. Boblétt, A. E. Danese, L. T. Gardner, D. S. Greenstein, P. G. Kirmser, M. S. Klamkin, Paul Payette, Michael Skalskyj, and the proposer.

Editorial Note. Abel's inequality, referred to in solution II, says that if $\{u_n\}$ is any sequence of complex quantities, and if $\{v_n\}$ is a positive monotone decreasing sequence, then

$$\left|\sum_{n=1}^{p} u_n v_n\right| < Bv_1,$$

where B is an upper bound of $|u_1|$, $|u_1+u_2|, \cdots, |u_1+u_2+\cdots+u_p|$. See, *e.g.*, Smail, *Elements of the Theory of Infinite Processes*, 1st ed., 1923, p. 31, or Franklin, *Treatise on Advanced Calculus*, 1940, pp. 314, 315.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4718. *Proposed by V. F. Ivanoff, San Carlos, California*

The six points of intersection of a conic and a cubic determine a Pascal hexagon. Show that the residual six points of intersection of the sides of the hexagon with the cubic form two collinear sets, and the lines determined by these sets meet on the Pascal line.

4719. *Proposed by L. Carlitz, Duke University*

Let p be a prime >2. Show that the determinant of order $p-1$

$$\Delta_p = \left| \left(\frac{r-s}{p} \right) \right| \qquad (r, s = 0, 1, \cdots, p-2),$$

where (r/p) is the Legendre symbol, satisfies

$$\Delta_p = p^{(p-3)/2}.$$

4270. *Proposed by S. W. Golomb, University of Oslo, Norway*

Show that in the power series expansion

$$\frac{1}{1-2x-2x^2+x^3} = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < 1,$$

the coefficients are given by $a_n = \sum_{k=0}^{n} f_k^2$, where $f_0 = 1, f_1 = 1, f_2 = 2, \cdots$ is the Fibonacci sequence.

4721. *Proposed by D. J. Newman, AVCO Research and Advanced Development Division, Lawrence, Mass.*

Let $u(x, y)$ be continuous and summable ($\int\int |u| dxdy < \infty$) over the entire plane. Suppose the line integral, $\int_L u(x, y)ds$ vanishes for all straight lines L infinite in both directions. Prove that $u(x, y)$ is identically zero.

4722. *Proposed by R. C. Warner, Toronto, Canada*

Establish the following inequality

$$\left\{ \sum_{i=1}^{n} \prod_{j=1}^{r} a_{ij}^{m/r} \right\}^r \leq \prod_{j=1}^{r} \sum_{i=1}^{n} a_{ij}^m.$$

SOLUTIONS

A Trigonometric Sum

4602 [1954, 477]. *Proposed by C. M. Ablow and D. L. Johnson, Seattle, Washington*

Show that

$$f(t) = \sum_{i=1}^{n} A_i \cos (B_i t + C_i)$$

changes sign as t varies from zero to positive infinity. The A_i, B_i, C_i are real constants; $A_i B_i \neq 0$.

II. *Solution by G. Lorentz, Wayne University.* The fact that if $f(t)$ is not identically zero it changes sign, is an immediate consequence of elementary theorems about continuous almost periodic (c.a.p.) functions. Each of the functions $\cos (B_i t+C_i)$, and therefore $f(t)$, is c.a.p. with vanishing mean value M. Our statement then follows from the fact that a non-negative c.a.p. function f is identically zero whenever $M(f)=0$.

Editorial Note. See Bohr, *Almost Periodic Functions*, theorem p. 63. A. Oppenheim points out that there is a gap in the argument of an earlier proof [1955, 736] where the interchange of summation and integration is not justified under the circumstance that the limits of the integration depend on i.

A Diagonal Property

4658 [1955, 659]. *Proposed by P. R. Halmos, University of Chicago*

If X is a set, let X^2 be the Cartesian product of X with itself. Call a subset D of X^2 a *diagonal* if for every x in X there exists a unique y in X and there exists a unique z in X such that $(x, y) \in D$ and $(z, x) \in D$. Prove that there exists a mapping from X^2 onto X such that the inverse image of every point is a diagonal.

Solution by D. J. Foulis, University of Chicago. We can endow the abstract set X with the structure of an abelian group; in fact, if X is a finite set containing n elements, we may use the cyclic group Z_n to induce a group structure on X, while if X is an infinite set, it can be put into 1–1 correspondence with the set of all finite subsets of itself, and the latter set is an abelian group under the operation of symmetric difference. In either case, let $\phi: X^2 \rightarrow X$ be the group operation induced on X. That ϕ is the desired mapping follows immediately from the fact that the equations $\phi(a, x)=b$ and $\phi(x, a)=b$ are uniquely soluble in a group. Even more, since the group is abelian, the inverse image of every point is a *symmetric* diagonal in the sense that $(x, y) \in \phi^{-1}(a)$ if and only if $(y, x) \in \phi^{-1}(a)$ for each element $a \in X$.

Also solved by S. K. Berberian, H. H. Corson, R. O. Davies, Walter Feit and Jean-Pierre Meyer, Lawrence Glasser, A. J. Goldman, A. S. Hendler, A. H. Kruse, Joachim Lambek, O. W. Rechard, and the proposer.

Attraction under an Inverse Square Law

4659 [1955, 659]. *Proposed by R. C. Lyness, Preston, England*

A solid of uniform density is bounded by those halves of the surfaces $x^2+y^2+z^2=a^2$ and $x^2+y^2+c^2z^2=a^2$ for which $z \geqq 0$. Find c so that under an inverse square law the solid attracts a particle at the origin as if its mass were concentrated at a point on its surface.

Solution by Norman Miller, Queen's University, Kingston, Canada. There are two cases according as $c>1$ or $c<1$. (1) Take $c>1$. The number of units of volume of the solid in question is $2\pi a^3(c-1)/3c$. Suppose this mass were concentrated at the point $(0, 0, a)$. (By symmetry the point must be on the z-axis, and it cannot be on or beyond the outer surface.) Then the attraction of the mass on a particle of unit mass at the origin would be $K2\pi a(c-1)/3c$ units, K being a constant. The actual force of attraction is F units, where

$$dF = K2\pi xz(x^2 + z^2)^{-3/2}dxdz,$$

and

$$F = K2\pi \int_0^a xdx \int_{\sqrt{a^2-x^2/c}}^{\sqrt{a^2-x^2}} z(x^2 + z^2)^{-3/2}dz.$$

This is easily evaluated with the result $F=K\pi a(c-1)/(c+1)$. The problem requires that

$$2(c - 1)/3c = (c - 1)/(c + 1)$$

whence $c=2$.

(2) Take $c<1$. The point at which the mass should be concentrated is then $(0, 0, a/c)$. Obvious modifications of the foregoing solution lead to the equation

$$2c(1 - c)/3 = (1 - c)/(1 + c)$$

whence $c=(\sqrt{7}-1)/2$.

Also solved by W. A. Al-Salam, A. J. Goldman, A. R. Hyde, Gene Pulley, and the proposer.

Expected Number of Correct Guesses

4661 [1955, 659]. *Proposed by J. B. Kelly, Michigan State University*

From an urn containing m black balls and m white balls, balls are drawn one at a time without replacement. An observer guesses the outcome (black or white) of each drawing. It is assumed that at any stage he will guess black if there remain more black balls than white ones and vice versa. Determine $E(m)$ the expected value of the number of correct guesses made during the entire procedure until all the balls have been withdrawn. Give an asymptotic formula for $E(m)-m$.

I. *Solution by Max Wyman and Leo Moser, University of Alberta.* Let $E(n, m)$ denote the expectation for the number of correct guesses starting with n white and m black balls. An elementary consideration yields the recurrence

$$E(n, m) = \frac{\max(n, m)}{n+m} + \frac{n}{n+m} E(n-1, m) + \frac{m}{n+m} E(n, m-1), \tag{1}$$

$$E(0, m) = E(m, 0) = m. \tag{2}$$

Let

$$E(n, m) = \max(n, m) + f(n, m)/\binom{n+m}{n}. \tag{3}$$

Then (1) and (2) take the form

$$f(n, m) = f(n-1, m) = f(n, m-1) \qquad n \neq m, \tag{4}$$

$$f(m, m) = 2f(m, m-1) + \binom{2m-1}{m}, \tag{5}$$

$$f(0, m) = f(m, 0) = 0. \tag{6}$$

For each r, $0 \leqq r \leqq \min(n, m)$, the function

$$\binom{n+m-2r}{n-r}$$

satisfies (4). Hence for arbitrary constants, b_r, the function

$$\phi(n, m) = \sum_{r=0}^{\min(n,m)} b_r \binom{n+m-2r}{n-r} \tag{7}$$

will satisfy (4). Further, $\phi(n, m)$ can also be made to satisfy (6) by taking $b_0 = 0$, and then to satisfy (5) by taking

$$b_m = \binom{2m-1}{m} = \frac{1}{2}\binom{2m}{m}.$$

Hence the solution of (4), (5) and (6) is given by

$$f(n, m) = \frac{1}{2} \sum_{r=1}^{\min(n,m)} \binom{2r}{r}\binom{n+m-2r}{n-r} = \sum_{r=1}^{\min(n,m)} \binom{m+n}{r-1}, \tag{8}$$

where the last form results from a familiar identity in binomial coefficients (easily established by induction).

Finally from (3) and (8) we have

$$E(m, m) = E(m) = m + \sum_{r=1}^{m} \binom{2m}{r-1} \Big/ \binom{2m}{m} = m - \frac{1}{2} + 2^{2m-1}/\binom{2m}{m}. \tag{9}$$

Stirling's formula for $n!$ gives the asymptotic value $E(m)-m\sim\frac{1}{2}\sqrt{\pi m}$.

II. *Solution by H. Kesten and J. Th. Runnenburg, Mathematical Centre, Amsterdam, Holland.* Consider the lattice points (i, k) with $0\leqq i, k\leqq m$. We can represent every sequence of drawings by means of a path joining the point $(0, 0)$ to the point (m, m). Every path consists of horizontal and vertical segments of unit length connecting lattice points. A horizontal (vertical) segment corresponds to the drawing of a white (black) ball. In a point of the diagonal (*i.e.*, a point (r, r) with $0\leqq r<m$) the probability of a correct guess is $\frac{1}{2}$. In a point not on the diagonal a correct guess corresponds to a step towards the diagonal. For every complete path exactly m steps are towards the diagonal and m steps away from it. Therefore $E(m)=m+\frac{1}{2}\sum_{r=0}^{m-1} P_r$, where P_r is the probability of a path meeting the diagonal at the point (r, r) and is given by

$$P_r = \binom{2r}{r}\binom{2m-2r}{m-r}\Big/\binom{2m}{m}.$$

After simple reductions we have

$$E(m) - m = -\tfrac{1}{2} + \tfrac{1}{2}\cdot 4^m\Big/\binom{2m}{m} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\pi m} + O(1/\sqrt{m}).$$

Also solved by A. E. Currier, and the proposer.

Complete Sequences

4662 [1955, 660]. *Proposed by J. E. Wilkins, Jr., Nuclear Development Corporation, White Plains, N. Y.*

If X is a measure space and if $\phi_i(x)$ and $\psi_j(x)$ are two sequences of functions in $L_2(X)$ which are complete in $L_2(X)$, then the sequence $\phi_i(x)\psi_j(y)$ is complete in $L_2(X, X)$.

Solution by the proposer. Suppose $f(x, y)$ is in $L_2(X, X)$, and that

$$\int\int f(x, y)\phi_i(x)\psi_j(y)dxdy = 0 \qquad (i, j = 1, 2, \cdots).$$

For almost all x, the function $f(x, y)$ is in $L_2(X)$ when considered as a function of y, and so the function

$$f_j(x) = \int f(x, y)\psi_j(y)dy$$

exists almost everywhere. This function is clearly measurable and has a summable square, and so is in $L_2(X)$. Moreover, by Fubini's theorem,

$$\int f_j(x)\phi_i(x)dx = 0.$$

Hence $f_j(x)=0$ almost everywhere, and the exceptional set E may be chosen independent of j. It is then clear that if x is not in E there exists a set $F(x)$ contained in X such that $F(x)$ has measure zero and $f(x, y)=0$ whenever (x, y) is not in the set A consisting of all points (x, y) for which either X is in E, or x is not in E and y is in $F(x)$. The set A need not be measurable, but the set B of all points (x, y) for which $f(x, y)\neq 0$ is a measurable subset of A. If B_x is the set of points y for which (x, y) is in B, then B_x is a subset of $F(x)$ if x is not in E, and so B_x has measure zero for almost all x. By Fubini's theorem, the set B has measure zero and so $f(x, y)=0$ almost everywhere.

Also solved by R. O. Davies.

Editorial Note. It seems that the hypothesis should require X to be a σ-finite measure space in order to permit the use of Fubini's theorem as given above.

A Diophantine Equation

4666 [1955, 734]. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p=p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x}+\frac{1}{T_y}=\frac{1}{T_z}\cdot$$

Editorial Note. By the substitution $a=2x+1$, $b=2y+1$, $c=2z+1$, the given equation becomes

$$(a^2-1)^{-1}+(b^2-1)^{-1}=(c^2-1)^{-1}. \tag{1}$$

In special cases, (1) is easy to solve. Thus, if $b=a$, (1) becomes

$$a^2-2c^2=-1,$$

and if $b=a+2$, (1) becomes

$$c^2-2\{(a-1)/2\}^2=-1.$$

Now, as is well known, all solutions of the "Pell" equation

$$m^2-2n^2=-1$$

are given by

$$2m=(1+\sqrt{2})^{2r-1}+(1-\sqrt{2})^{2r-1},\qquad 2\sqrt{2}n=(1+\sqrt{2})^{2r-1}-(1-\sqrt{2})^{2r-1}.$$

From these, infinitely many solutions of the original equation are obtained, *e.g.*, $(x, y, z)=(3, 3, 2)$, $(20, 20, 14)$, $(119, 119, 84)$, $\cdots$, $(4, 5, 3)$, $(28, 29, 20)$, $(168, 169, 119)$, $\cdots$.

No reason why solutions corresponding to $|b-a|\geqq 2$ may not exist has been suggested, but no particular solutions have appeared except such as come under the two special cases cited.

Partially solved by F. A. Homann, Edgar Karst, J. M. Khatri, E. M. Michalup, Walter Penney, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Modern Trigonometry. By W. A. Rutledge and J. A. Pond. New York, Prentice-Hall, Inc. 1956. ix+243 pages. $3.95.

At first glance one may be impressed by the unusual format of the book. Each chapter is preceded by a full page on the left devoted to the title and chapter number, the latter being superimposed on a small figure designed to portray some phase of the material in the chapter. Most of the figures are quite large, many occupying half a page or more. One must keep these facts in mind when evaluating the actual number of pages devoted primarily to text material.

The book consists of the following chapters with their summaries: Chapter 1—Angles. This chapter introduces various ideas involving angles and angle measure. Chapter 2—Logarithms. A discussion of logarithms occupies approximately twenty-six pages and as a result it is not until Chapter 3, page 51, that definitions of the trigonometric functions are introduced. Perhaps the best excuse for the arrangement can be supplied by quoting directly from the preface—"The order of Chapters 1 and 2 is arbitrary. It is undesirable for the study of logarithms to interrupt the development of trigonometry yet we feel that in the unsettled opening days of the course the ideas of Chapter 1 can be mastered more easily than can the comprehension of logarithms." In spite of this, the reviewer cannot help but feel that the chapter on logarithms might well have been postponed.

Chapter 3—Trigonometric Functions. Functions of real variables, ranges and domains are discussed and the trigonometric functions are introduced as special examples.

Chapter 4—Trigonometric Solutions of Plane Triangles. Numerical solutions of triangles including the law of sines and the law of cosines together with their applications are treated.

Chapter 5—Cos $(\alpha-\beta)$ and Related Functional Values. The formula for cos $(\alpha-\beta)$ is derived in an unconventional but elegant manner by use of the distance formula. From this the formulas for cos $(\alpha+\beta)$, sin $(\alpha\pm\beta)$, tan $(\alpha\pm\beta)$, *etc.*, are deduced.

Chapter 6—Related Angles and the Law of Tangents. Functional values of a number θ are expressed in terms of functions of a number between 0 and $\pi/2$ by use of the formulas developed in Chapter 5. In addition the law of tangents is derived and a formula for the radius of a circle inscribed in a triangle is obtained.

Chapter 7—Variations and Graphs of the Functions. Graphs of the trigonometric functions are discussed. Great care is given to emphasize the distinction

between $y = A \sin x$ and $y = \sin Ax$.

Chapter 8—Inverse Functions and Trigonometric Equations. The inverse functions (written arc sin, $\cdots$ instead of $\sin^{-1}$, $\cdots$) are discussed. Care is given in the explanation of principal values. Use of the logical symbolism $x \in \text{arc sin } S$ for x is an element of the set arc sin S or x belongs to the set arc sin S appears throughout the chapter.

Chapter 9—Complex Numbers. The usual operations with complex numbers are discussed. De Moivre's theorem for positive integers is proved by use of mathematical induction. A proof that the equation $z^n = w$, where w is a given complex number, has n distinct roots, is presented.

Problems are divided into two types called ORALS which are essentially sight problems and EXERCISES which are of the usual type. Answers are not given for the ORALS but answers are given for the odd-numbered exercises.

The book has many favorable features to recommend it, not the least being the functional concept which pervades the text and the clear unhurried manner in which the subject matter is developed.

M. R. SPIEGEL
Rensselaer Polytechnic Institute

Plane Trigonometry. By A. Spitzbart and R. H. Bardell. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1955. viii+205 pages. $3.75.

The book is designed, as the authors state in the preface, to emphasize "the analytical rather than computational trigonometry." The following are the chapter headings with their summaries:

Chapter 1—The Trigonometric Functions. This chapter treats degree and radian measure of an angle, functions in general and the trigonometric functions in particular. The various trigonometric functions of general angles in terms of functions in the first quadrant are taken up. Use of tables is adequately emphasized.

Chapter 2—Circle Relations. Right Triangles. Formulas for the length of an arc of a circle and the area of a sector of a circle are derived. The concept of uniform angular speed and its relation with linear speed is introduced. Interpolation in trigonometric tables is presented. Solutions of right triangles are given together with applications among which are problems involving vectors such as computation of tensions in cables, *etc.* An added feature is a section on significant figures.

Chapter 3—Graphs of the Trigonometric Functions. Inverse Trigonometric Functions. The chapter deals with the graphs of all the trigonometric functions including contractions, expansions, translations and reflections of the graphs. Definitions of odd and even functions are given. Inverse trigonometric functions and their graphs are introduced here together with the concept of principal-values.

Chapter 4—Properties of the Trigonometric Functions. Various identities

are treated. The addition formulas sin $(A \pm B)$ *etc.*, the double and half angle formulas, and the product and factor formulas are derived. A proof of the addition formula for general angles utilizes the law of cosines which was established in an exercise of a previous chapter. Trigonometric equations are solved.

Chapter 5—Logarithms. This is a purely computational chapter dealing with the usual properties of logarithms. Trigonometrical functions are not mentioned in this chapter.

Chapter 6—The Solution of Oblique Triangles. In this chapter are derived the law of sines and the law of cosines and tangents. Applications involving the use of logarithms are included. The area of a triangle is determined in terms of two sides and the included angle and also in terms of two angles and a side.

Chapter 7—Complex Numbers. Here are discussed the customary operations with complex numbers. De Moivre's theorem is derived for a few special cases. The proof by mathematical induction is left to the student. The various nth roots of a complex number are obtained.

There are several features of this book which are noteworthy. There seems to be a constant attempt toward unification of the various concepts of trigonometry. As one example of this is the almost simultaneous treatment of the trigonometric and inverse trigonometric functions. The various physical applications should prove useful to the scientifically-minded. The exercises are well chosen and answers to the odd numbered ones are presented. The book seems well-organized and the format is good.

In spite of the relatively short space of text material, the reviewer feels that there is a large amount of subject matter present, much of which is quite valuable. As a consequence the student who uses the whole text should be kept busy but his efforts will probably be proportionately rewarded.

M. R. SPIEGEL
Rensselaer Polytechnic Institute

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

SHELL MERIT FELLOWSHIPS

Shell Companies Foundation has announced that seminars for teachers of high school physics, chemistry and mathematics will be conducted at Cornell University and Stanford University during the summer of 1957. Teachers with five years experience and

known leadership ability are eligible for fellowships covering travel costs, tuition fees, living expenses, and $500 in cash.

Teachers living west of the Mississippi should send requests for fellowship applications to School of Education, Stanford University, Stanford, California. Teachers east of the Mississippi should write to the Department of Education, Cornell University, Ithaca, New York.

ORACLE APPLICATIONS PROGRAM

The following information supplements an announcement of this program which was published in this MONTHLY, vol. 63, 1956, pp. 594–595. The ORACLE Applications Program is conducted by the University Relations Division of the Oak Ridge Institute of Nuclear Studies and the Mathematics Panel of Oak Ridge National Laboratory. Dr. T. W. Hildebrandt, staff member of the Institute and director of the program, is working with the Mathematics Panel of the Laboratory, headed by Dr. A. S. Householder, in preparing university-originated problems for ORACLE computation. The ORACLE is the high-speed electronic digital computer of the Oak Ridge National Laboratory. The program is designed to make available to university personnel not only computer time but also the combined experience and knowledge of members of the Mathematics Panel. Further information may be obtained from Dr. T. W. Hildebrandt, ORACLE Applications, University Relations Division, Oak Ridge Institute of Nuclear Studies, P. O. Box 117, Oak Ridge, Tennessee.

PERSONAL ITEMS

Mr. Richard M. Friedberg, Harvard University, has been awarded the William Lowell Putnam Prize Scholarship.

Professor Jewell H. Bushey of Hunter College represented the Association at the Twenty-First Educational Conference which was held in New York City on November 1–2, 1956.

Professor Izaak Wirszup, University of Chicago, was the representative of the Association at the Thirty-Ninth Annual Meeting of the American Council on Education on October 11–12, 1956 in Chicago.

Alabama Polytechnic Institute: Associate Professor Wilfrid Wilson, University of Illinois, has been appointed to a professorship; Associate Professor W. D. Peeples, Jr., Howard College, has been appointed to an assistant professorship; Mr. D. H. Clanton, University of South Carolina, Mr. E. R. Ivey, Mrs. Ella S. Lindsey, Mr. J. L. Locker, and Mrs. Christine Parker, Northwest Junior College, have been appointed to instructorships; Associate Professor Ernest Ikenberry has been promoted to Research Professor; Assistant Professor N. C. Perry has been promoted to a research associate professorship; Assistant Professors R. W. Ball and L. P. Burton have been promoted to associate professorships.

Brooklyn College: Dr. Meyer Jordan, Lecturer at the College, has been appointed to an instructorship; Assistant Professor A. J. Maria has been promoted to an associate professorship; Professor Walter Prenowitz lectured on vector analysis at the Institute for Teachers, Williams College during the summer of 1956.

California Institute of Technology: Assistant Professor Robert Finn, University of Southern California, has been appointed to an associate professorship; Dr. Basil Gordon has been appointed to an instructorship.

Carnegie Institute of Technology: Dr. Walter Noll, University of Southern California, has been appointed to an associate professorship; Dr. A. D. Martin, Institute for Advanced Study, Dr. R. C. MacCamy, University of California, and Dr. R. A. Moore, Yale University, have been appointed to assistant professorships; Mr. J. O. Montague, Teaching Assistant at the Institute, and Mr. E. P. Shelley have been appointed to instructorships; Mr. J. S. Rustagi has been promoted to an assistant professorship.

Columbia University: Dr. H. C. Wang, University of Washington, is Visiting Associate Professor; Dr. Jacob Feldman, Institute for Advanced Study, has been appointed Visiting Assistant Professor; Mr. E. J. Taft, Assistant, Yale University, has been appointed to a research instructorship; Dr. Pinchas Mendelson, a graduate student, Princeton University, has been appointed to an instructorship; Assistant Professor R. V. Kadison has been promoted to an associate professorship.

Hampton Institute: Assistant Professor S. R. Beyma has been promoted to an associate professorship; Miss Rosalind Eagleson has returned to the mathematics staff after two years of graduate work at the University of Michigan.

Illinois Institute of Technology: Associate Professor Haim Reingold has been promoted to a professorship; Assistant Professors Gerald Berman and R. A. Struble have been promoted to associate professorships; Dr. C. A. Nicol and Dr. Alfonso Shimbel have been promoted to assistant professorships.

Iowa State College: Dr. R. E. Gaskell, a mathematician, Boeing Aircraft Corporation, Seattle, Washington, is Visiting Professor for the first semester; Dr. R. L. Dunn, Graduate Assistant, University of California, Los Angeles, has been appointed to an instructorship; Assistant Professors Frank Bortle, George Seifert, and H. J. Weiss have been promoted to associate professorships; Dr. D. A. Storvick has been promoted to an assistant professorship.

Lehigh University: Mr. Paul Axt, Teaching Assistant, University of Wisconsin, and Mr. Constantine Kassimatis, Lecturer, Queen's University, have been appointed to instructorships; Assistant Professor S. G. Bourne, University of California, has been appointed to an assistant professorship; Dr. R. C. Carson has been promoted to an assistant professorship.

Montana State College: Mr. Wilbur Sims, Mrs. Patricia Peckenpaugh, and Mr. Homer Terwilliger have been appointed to instructorships; Dr. Hans Sagan has been promoted to an assistant professorship.

Montana State University: Dr. L. A. Schmittroth, Harvard University, has been appointed to an assistant professorship; Associate Professor T. G. Ostrom has been promoted to a professorship; Assistant Professors W. M. Myers and J. Hashisako have been promoted to associate professorships.

Northwestern University: Assistant Professors J. C. E. Dekker and Alex Rosenberg, and Associate Professor Daniel Zelinsky are on leaves of absence for the year 1956–57 at the Institute for Advanced Study.

Occidental College: Associate Professor P. B. Johnson has been promoted to a professorship; Assistant Professor Mabel S. Barnes has been promoted to an associate professorship.

Oklahoma Agricultural and Mechanical College: Dr. Jeanne L. Agnew has been appointed to an assistant professorship; Mr. J. C. Hetrick, Continental Oil Company, Ponca City, Oklahoma, and Dr. D. R. Shreve, Research Mathematician, Carter Research Laboratories, Tulsa, Oklahoma, have been appointed Adjunct Professors; Mr. M. A. Albright, Graduate Assistant at the College, has been appointed to an instructorship; Associate Professor C. E. Marshall has been promoted to a professorship; Professor E. F. Allen has retired with the title Professor Emeritus.

Pomona College: Mr. Jay Davis has been appointed to an instructorship; Miss Jean B. Walton, Dean of Women, spent the year in Japan as a Fulbright scholar; Professor C. G. Jaeger worked as a senior research engineer at Consolidated-Vultee Aircraft Corporation during the summer.

Rutgers University: Assistant Professor R. E. Heaton, University of Richmond, and Dr. Abe Shenitzer, a member of the Bell Telephone Laboratories, Murray Hill, New Jersey, have been appointed to instructorships; Dr. R. K. Brown has been promoted to an assistant professorship.

State College of Washington: Dr. O. W. Rechard, staff member, Los Alamos Scientific Laboratory, New Mexico, has been appointed Associate Professor and Director of the Computing Center; Dr. A. P. Hillman, Associate, Columbia University, Dr. C. T. Long, Teaching Fellow, University of Oregon, Dr. P. L. Meyer, Assistant Professor G. K. Overholtzer, Georgia Institute of Technology, and Dr. Hans Schneider, Queen's University, Belfast, have been appointed to assistant professorships; Mr. C. D. Aronson has been appointed to an acting instructorship.

State University of Iowa: Dr. Steve Armentrout, Jr., University of Texas, has been appointed to an instructorship; Assistant Professor R. V. Hogg, Jr., has been promoted to an associate professorship; Professor Emeritus E. W. Chittenden has a leave of absence for 1956–57 to continue his appointment as a mathematician, Diamond Fuse Laboratory, Washington, D. C.; Associate Professor Byron Cosby, Jr., is on leave of absence for the academic year 1956–57 as an analyst, Office of the Secretary of Defense, Weapons Systems Evaluation Group, Washington, D. C.

Syracuse University: Dr. Bruce Gilchrist, Mathematician, Institute for Advanced Study, Dr. A. G. Kostenbauder, University of Connecticut, and Dr. R. J. Lundegard, Research Fellow, Purdue University, have been appointed to assistant professorships; Dr. Lily Seshu, Research Assistant, University of Illinois, has been appointed to an instructorship; Associate Professors Albert Edrei and Abe Gelbart have been promoted to professorships; Assistant Professors R. M. Exner and W. A. Pierce have been promoted to associate professorships; Dr. G. F. Leger has been promoted to an assistant professorship; Assistant Professor H. C. Bennett has retired with the title of Assistant Professor Emeritus; Associate Professor Ruth Stokes is on leave of absence for 1956–57.

University of Akron: Assistant Professor Louis Ross has been promoted to an associate professorship; Assistant Professor R. H. Davis spent the summer at the Watson Computing Laboratory, Columbia University.

University of Alabama: Professor Rudolf Festa, University of Vienna, Austria, has been appointed Visiting Professor; Mr. Henry Miller, a graduate student at the University of North Carolina, Mr. Richard McCourt, a graduate student at Cornell University, Mr. Arnold Milner, a graduate student at the University, and Miss Emma J. Pitts, Alabama College, have been appointed to instructorships.

University of Alberta: Dr. O. P. Aggarwal, Purdue University, has been appointed Visiting Associate Professor of Mathematical Statistics for 1956–57; Associate Professor Max Wyman has been promoted to a professorship; Assistant Professors G. K. Horton and R. Jacka have been promoted to associate professorships.

University of Arizona: Assistant Professor E. D. Nering of the University of Minnesota has been appointed to an associate professorship; Mrs. Georgia T. Hart, Assistant at the University, Mrs. Caroline M. Jensen, University of Colorado, and Mr. L. D. McLain have been appointed to instructorships; Dr. A. H. Steinbrenner has been promoted to an assistant professorship.

University of Arkansas: Assistant Professor O. P. Sanders, Southeastern State College, has been appointed to an assistant professorship; Miss Isabella K. Smith, Fort Smith Junior College, Miss Joyce Caraway, and Mr. R. M. Coulter have been appointed to instructorships.

University of British Columbia: Dr. P. S. Bullen, Lecturer, University of Natal, Durban, South Africa, has been appointed to an instructorship; Dr. R. A. Restrepo, Research Fellow, California Institute of Technology, has been appointed Lecturer; Dr. M. D. Marcus has received a Postdoctoral Research Associateship from the National Research Council for 1956–57 and is on leave of absence for this period.

University of California, Berkeley: Professor Alfred Tarski has been elected President of the International Union for the Philosophy of Science; Associate Professor Bertram Yood, the University of Oregon, has been appointed Visiting Associate Professor;

Dr. James Eells, Jr., Institute for Advanced Study, Dr. Bertram Kostant, formerly Higgins Lecturer at Princeton University, and Dr. P. E. Thomas, Associate, Columbia University, have been appointed to assistant professorships; Mr. Leslie Fox, on leave from the National Physical Laboratory, England, has been appointed Lecturer; Dr. A. B. J. Novikoff, Associate, Radiation Laboratory, Johns Hopkins University, and Mr. Mishael Zedek, a graduate student at Harvard University, have been appointed to instructorships; Mr. Gonzlo Zubieta, National University of Mexico, is spending a few months at the University; Associate Professor M. H. Protter is on leave of absence for 1956–57 in Europe; Professors Hans Lewy and R. M. Robinson are on sabbatical leave.

University of California, Los Angeles: Professor P. J. Laasonen, Finland Institute of Technology, Helsinki, Finland, has been appointed Visiting Professor; Assistant Professor P. K. Henrici, American University, has been appointed Visiting Associate Professor; Assistant Professor B. W. Volkmann, University of Utah, has been appointed Visiting Assistant Professor; Dr. O. B. Helman, Finland Institute of Technology, and Dr. T. S. Ferguson, a graduate student at the University of California, have been appointed to instructorships; Assistant Professors E. A. Coddington, R. M. Redheffer, and J. D. Swift have been promoted to associate professorships.

University of Colorado, Department of Applied Mathematics: Professor K. H. Stahl has been named Assistant Dean of the College of Engineering; Mr. C. E. Aull, University of Arizona, Mr. J. D. DePree, Mr. J. R. Florence, Jr., Mr. K. L. Hillam, Assistant, University of Utah, Miss Anna M. Merrill, Mr. P. F. Smith, and Mr. R. G. Thompson have been appointed to instructorships; Assistant Professor P. F. Hultquist is on leave of absence at Lockheed Aircraft Corporation, Palo Alto, California; Assistant Professor J. F. Wagner is on leave of absence at Control Cells, Inc.

University of Connecticut: Dr. E. T. Wong, University of Rochester, and Mr. Arthur Radin have been appointed to instructorships; Dr. E. S. Wolk has been promoted to an assistant professorship; Miss Fredrika H. Kilbourn, instructor at the Waterbury Branch, has retired.

University of Florida: Colonel F. C. Myers of the U. S. Air Force has been appointed to an assistant professorship; Assistant Professor E. H. Lehman of the Department of Statistics has been appointed Interim Assistant Professor; Mr. J. L. Savige, a mathematics teacher at New Port Richey, Florida, has been appointed Interim Instructor.

University of Idaho: Assistant Professors H. W. Crowley, State College of Washington, and A. E. Labarre, Jr., University of Wyoming, have been appointed to assistant professorships; Mr. J. R. Eno, Mr. Donald Muir, and Mr. John Reay have been appointed to acting instructorships.

University of Illinois: Dr. H. A. Osborn, University of California, and Dr. Eugene Paige have been appointed to assistant professorships; Mr. Tyler Allhands and Mr. T. T. Robinson, Research Assistant at Princeton University, have been appointed to instructorships; Dr. D. B. Lowdenslager, Research Associate, University of California, has been appointed Research Associate; Assistant Professors Alex Heller and Irving Reiner have been promoted to associate professorships; Associate Professor P. T. Bateman and Professor G. P. Hochschild are on leaves of absence for 1956–57 at the Institute for Advanced Study; Professor J. L. Doob is on sabbatical leave in Geneva, Switzerland; Professor M. M. Day is on sabbatical leave for the academic year 1956–57; Professor Reinhold Baer is on leave of absence for 1956–57 at Mathematisches Seminar der Universität, Frankfurt, Germany; Assistant Professor N. S. Hawley, Jr., is on leave of absence at Stanford University; Professor W. G. Madow, on leave during 1956–57, is at the Center for Advanced Study in the Behavioral Sciences, Stanford, California; Assistant Professor Michio Suzuki is on leave during 1956–57 at Harvard University.

University of Kansas: Dr. D. R. Truax, Research Associate, California Institute of Technology, and Dr. E. A. Walker, Mathematician, National Security Agency, Washing-

ton, D. C., have been appointed to assistant professorships; Miss Frances L. Wolfe, Assistant to the Chancellor of the Woman's College of the University of North Carolina, has been appointed to an instructorship; Assistant Professor K. T. Smith has been promoted to an associate professorship; Professor G. W. Smith has retired with the title Professor Emeritus; Dr. Avner Friedman, Hebrew University, Jerusalem, Irsael, is spending the academic year 1956–57 at the University as a research associate.

University of Maryland: Dr. Werner Greub of Zurich University, Switzerland, Dr. Harold Holmann, Assistant Professsor Dora E. Kearney, Iowa Wesleyan College, Mrs. Mary B. McClay, Mr. T. A. Paley, the University of Connecticut, and Dr. John Raleigh, University of Pennsylvania, have been appointed to instructorships; Miss Dagmar Henney and Mr. C. L. Tibery have been appointed to junior instructorships; Dr. Gertrude Ehrlich and Dr. W. G. Rosen have been promoted to assistant professorships; Assistant Professor G. L. Spencer, II, is on leave of absence for one year on a National Science Foundation Post-doctoral Fellowship at the Institute for Advanced Study.

University of Mississippi: Assistant Professor R. A. Stokes has returned after graduate work at University of Texas; Dean Fred Fulton, East Mississippi Junior College, and Lt. Col. Alliston Slade, U. S. Air Force, have been appointed to instructorships; Assistant Professor N. A. Childress has been promoted to an associate professorship; Associate Professor L. L. Scott, recipient of a Ford Foundation Grant for 1955–56, has returned after a year of study at the University of California.

University of Montreal: Dr. Istvan Fary, Research Associate at the University, has been appointed to an associate professorship; Assistant Professor Maurice L'Abbé has been promoted to a professorship.

University of New Mexico: Mr. P. G. Carr, University of Oregon and Mr. R. K. Scheer, University of Nebraska, have been appointed to instructorships.

University of Oklahoma: Mr. L. L. Koontz, University of Arkansas, has been appointed to an instructorship; Associate Professor A. A. Grau has been promoted to a professorship; Mr. J. C. Bradford, Graduate Assistant, has been promoted to an instructorship; Professor Grau is on leave during the first semester of 1956–57 to work at the Oak Ridge National Laboratory; Associate Professor R. V. Andree is on leave for the academic year 1956–57 and will teach at Oklahoma Agricultural and Mechanical College in the National Science Foundation program during the first semester; Assistant Professor J. R. Foote is on leave of absence for the year 1956–57 and is at the Institute of Mathematical Sciences, New York University.

University of Pennsylvania: Dr. Louis Auslander, Institute for Advanced Study, has been appointed to an assistant professorship; Professor Peter Scherk, University of Saskatchewan, has been appointed Visiting Professor; Associate Professors N. J. Fine and W. H. Gottschalk have been promoted to professorships; Dr. Pincus Schub has been promoted to Associate; Professor I. J. Schoenberg is on leave of absence at Stanford University.

University of Rochester: Dr. J. C. Mairhuber, Graduate Assistant, University of Pennsylvania, and Dr. M. H. Pearl, Mathematical Analyst, Department of Defense, Washington, D. C., have been appointed to instructorships; Dr. Mary E. Rudin is part-time Assistant Professor for 1956–57; Mr. E. H. Batho has been promoted to an assistant professorship; Assistant Professor R. A. Raimi has returned after a year's leave of absence at Yale University under an Alfred H. Lloyd Postdoctoral Fellowship; Professor J. F. Randolph has returned from a sabbatical leave as Visiting Professor at the American University, Beirut, Lebanon; Associate Professor Dorothy Bernstein was Acting Chairman during Professor Randolph's absence; Associate Professor Walter Rudin has received an Alfred P. Sloan Research Fellowship for 1956–57; Associate Professor N. G. Gunderson and Assistant Professor R. W. MacDowell received summer research fellowships from the University; Professor Rudin participated in the special analysis program

at the University of Chicago during the summer; Professors Raimi and Rudin were members of the Conference on Harmonic Analysis and Integral Transforms at Cornell University. A Computing Center with a Burroughs E101 and an IBM 650 has been established at the University and is directed by Dr. T. A. Keenan, Purdue University.

University of Saskatchewan: Assistant Professor G. F. D. Duff, University of Toronto, has been appointed Visiting Professor; Dr. Richard Blum has been promoted to an associate professorship; Dr. G. H. M. Thomas has been appointed to an assistant professorship.

University of South Dakota: Dr. Marjorie H. Beaty has been appointed Assistant Professor; Mrs. Maxine Stewart has been appointed to an instructorship; Mr. Paul Haeder is on leave of absence at Iowa State College.

University of Tennessee: Dr. Smbat Abian, University of Cincinnati, has been appointed Acting Assistant Professor; Mr. R. D. McWilliams, Mrs. Irene P. Millsaps, Mr. C. C. Oehring, Virginia Military Institute, Mrs. Evelyn Sharp, and Mr. R. G. Vinson, Florida State University, have been appointed to instructorships; Associate Professors G. E. Albert, D. D. Miller, and W. S. Snyder have been promoted to professorships; Professor O. G. Harrold is principal investigator on a project supported by a National Science Foundation grant for a three year period.

University of Utah: Assistant Professor J. H. Barrett, University of Delaware, has been appointed to an associate professorship; Dr. Lida K. Barrett, University of Connecticut, Waterbury Branch, has been appointed Lecturer; Dr. W. J. Coles, Analyst, National Security Agency, Washington, D. C., and Dr. E. E. Kohlbecker, Assistant, University of Illinois, have been appointed to assistant professorships; Associate Professors F. C. Biesele, R. E. Chamberlin, and James Wolfe have been promoted to professorships; Professor C. R. Wylie, Jr., received one of the three annual alumni awards from Wayne State University.

University of Wisconsin: Assistant Professor C. E. Burgess, University of Utah, has been appointed Visiting Lecturer; Dr. Joshua Chover, Institute for Advanced Study, has been appointed to an instructorship; Dr. J. B. Kruskal, Jr., Princeton University, has been appointed Research Instructor; Professor Rafael Artzy, Israel Institute of Technology, has been appointed Research Associate; Dr. Klaus Krickeberg, Research Associate, University of Illinois, has been appointed a project associate; Dr. E. R. Fadell has been promoted to an assistant professorship; Associate Professor W. F. Eberlein is on leave at the Institute of Mathematical Sciences, New York University; Professor S. C. Kleene is on leave of absence at Princeton University; Associate Professor Jacob Korevaar will be on leave during the second semester on a grant from the Office of Naval Research.

University of Wyoming: Miss Dorothy J. Stodola, Marquette University, has been appointed to an instructorship; Assistant Professor M. J. Walsh, Florida State University, has been appointed to an assistant professorship.

Virginia Polytechnic Institute: Professor S. T. Gormsen, Rollins College, has been appointed to a professorship; Capt. E. W. Lamons, U. S. Navy, and Mr. R. H. Riffenburgh have been appointed to assistant professorships; Mr. J. G. Moore has been appointed to an instructorship.

Wayne State University: Mrs. Ellen Dunlap, Graduate Assistant, Syracuse University, and Mr. Max Krolik, Graduate Assistant at the University, have been appointed to instructorships: Associate Professors Y. W. Chen and A. W. Jacobson have been promoted to professorships; Assistant Professors S. D. Conte and B. J. Eisenstadt have been promoted to associate professorships; Professors Y. W. Chen and Benjamin Epstein have returned from leaves of absence; Professor K. W. Folley is on leave of absence for one year at the U. S. Naval Postgraduate School, Monterey, California; Associate Professor Samuel Kaplan is on leave of absence at the Institute for Advanced Study.

Wisconsin State College, River Falls: Associate Professor Lillian Gough has been

appointed Chairman of the Mathematics Department; Dr. J. J. McLaughlin, previously Chairman of the Department, is now Director of the School of Arts and Sciences.

Yale University: Professor A. A. Albert, University of Chicago, is Visiting Professor for 1956–57; Assistant Professors I. N. Herstein, University of Pennsylvania, and Erwin Kleinfeld, Ohio State University, have been appointed Visiting Lecturers; Assistant Professor F. E. Browder, Brandeis University, has been appointed to an assistant professorship; Dr. G. B. Seligman, Princeton University, has been appointed to an instructorship; Dr. F. D. Quigley has been promoted to an assistant professorship.

Dr. R. E. Block has been appointed to an instructorship at Indiana University.

Miss Jessie W. Boyce, who retired from the chairmanship of the Mathematics Department of Wayne State Teachers College, Nebraska, in 1955, is teaching at Augustana College, South Dakota.

Assistant Professor H. E. Campbell, Emory University, has been appointed to an assistant professorship at Michigan State University.

Capt. L. G. Campbell, U. S. Air Force, has been appointed Assistant Professor at the U.S. Air Force Academy.

Dr. Peter Chiarulli, National Bureau of Standards, Washington, D. C., has been appointed Chairman of the Department of Mechanics, Illinois Institute of Technology.

Mr. G. L. Crumley, Coffeyville College, has been appointed to an assistant professorship at The Citadel.

Assistant Professor M. L. Curtis, Northwestern University, has been appointed to an associate professorship at the University of Georgia.

Mr. L. W. Ehrlich, Graduate Assistant, University of Maryland, is employed by the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. Isidore Fleischer, Lecturer, Northwestern University, has a position at Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. Aubyn Freed, Assistant, University of Illinois, has been appointed to an instructorship at Smith College.

Mr. W. V. Gamzon, a teacher at Woodrow Wilson High School, Long Beach, California, has been appointed to an instructorship at East Los Angeles Junior College.

Associate Professor B. H. Gundlach, University of Arkansas, has been appointed to an assistant professorship at Bowling Green State University.

Dr. H. M. Gurk, Moore School of Electrical Engineering, University of Pennsylvania, is employed now in the Advanced Development Section, Defense Electronic Products, Radio Corporation of America, Camden, New Jersey.

Assistant Professor P. W. Healy, University of New Mexico, has accepted a position with Phillips Petroleum Company, Division of Research and Development, Idaho Falls, Idaho.

Associate Professor R. F. Jackson, University of Delaware, has been promoted to a professorship.

Associate Professor H. G. Jacob, Jr., Louisiana State University, has been appointed to an assistant professorship at Johns Hopkins University.

Dr. J. P. Jans, Yale University, has been appointed to an assistant professorship at Ohio State University.

Assistant Professor John Jewett, University of Alabama, has been appointed to an assistant professorship at the University of Georgia.

Assistant Professor Ralph Johnson, Iowa State College, has retired from active teaching.

Mr. Thomas Kampe, Pomona College, has a position as Mathematician, Librascope Corporation, California.

Assistant Professor L. H. Kanter, Clarkson College of Technology, has been appointed to an assistant professorship at Drexel Institute of Technology.

Mr. Marvin Karlin, University of Arizona, is teaching now at Pueblo High School, Tucson, Arizona.

Professor L. M. Kelly, Michigan State University, is on leave of absence for the year at Harvard University.

Mr. D. B. Kirk, research analyst, Mutual Benefit Life Insurance Company, Newark, New Jersey, has a position as Mathematician, Curtiss-Wright Research Division, Quehanna, Pennsylvania.

Assistant Professor Anne L. Lewis, Woman's College, University of North Carolina, has been promoted to an associate professorship.

Mr. H. L. Lewis, Southwest Texas State Teachers College, has accepted a position with the Rand Corporation, Santa Monica, California.

Associate Professor H. M. Linnette of Virginia State College has been granted a leave of absence for one year to receive a special appointment to government service.

Associate Professor G. H. Lundberg, Vanderbilt University, has been promoted to the position of Professor of Applied Mathematics in the School of Engineering.

Dr. E. J. Lytle, University of Florida, is employed by the I.B.M. Corporation, Poughkeepsie, New York.

Associate Professor G. W. Mackey, Harvard University, has been promoted to a professorship.

Mr. F. H. McGar, Jr., has been appointed to an instructorship in physics at Sweet Briar College.

Dr. A. D. Miller, Santa Barbara Public Schools, California, has been appointed to an assistant professorship at Long Beach State College.

Assistant Professor R. A. Miller, University of Mississippi, has a position with Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. B. G. Mullen, University of Arkansas, has been appointed to an instructorship at Arkansas Polytechnic College.

Dr. Mary M. Neff, University of Florida, has been appointed to an instructorship at John Carroll University.

Mr. A. C. Nelson, University of Delaware, is employed by the Westinghouse Corporation, Pittsburgh, Pennsylvania.

Mr. E. D. Nichols, University of Illinois, has been appointed to an associate professorship at Florida State University.

Assistant Professor R. H. Oehmke, Butler University, has been appointed to an assistant professorship at Michigan State University.

Associate Professor W. H. Pell, Brown University, has a position as Mathematician, National Bureau of Standards, Washington, D. C.

Associate Professor W. M. Perel, Georgia Institute of Technology, has been appointed to an assistant professorship at Texas Technological College.

Dr. W. E. Perrault, St. Louis University, has been appointed to an assistant professorship at Boston College.

Mr. C. F. Pinzka, Statistician, Educational Testing Service, Princeton, New Jersey, has been appointed to an instructorship at Xavier University.

Mrs. Jean B. Richmond, Texas State College for Women, has been appointed to an instructorship at Texas Christian University.

Mr. G. W. Ricker, Teaching Assistant, Rutgers University, is teaching at Lakewood High School, New Jersey.

Associate Professor Saul Rosen, Wayne State University, has accepted a position with the Burroughs Adding Machine Corporation, Philadelphia, Pennsylvania.

Associate Professor H. J. Ryser, Ohio State University, has been promoted to a professorship.

Dr. W. C. Sangren, Chief of the Computing and Mathematics Division, Curtiss-

Wright Research, Clifton, New Jersey, is employed now by the General Dynamics Corporation, San Diego, California.

Associate Professor J. W. Sawyer of the University of Richmond has been appointed to an associate professorship at Wake Forest College.

Associate Professor Alfred Schild, Carnegie Institute of Technology, has a position at Westinghouse Research Laboratories, Pittsburgh, Pennsylvania.

Assistant Professor Lowell Schoenfeld, University of Illinois, is employed in the Mathematics Division, Westinghouse Electric Corp., Pittsburgh, Pennsylvania.

Dr. Frank Stallard, Iowa State College, has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. R. H. Stark, Mathematician, Knolls Atomic Power Laboratory, Schenectady, New York, is now Manager of Mathematics & Computation, Vallecitos Atomic Laboratory, General Electric Company, San Jose, California.

Mr. R. K. Stump, Teaching Assistant, Rutgers University, is now in the U. S. Army.

Mr. R. J. Wagner, Teaching Assistant, Rutgers University, has been appointed to an instructorship at Upsala College.

Assistant Professor R. Dorsett, University of Oklahoma, died on August 6, 1956.

Associate Professor J. C. Layman, Virginia Polytechnic Institute, died on May 18, 1956.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 102 persons have been elected to membership by the Board of Governors on applications duly certified.

W. R. Abel, A.M. (South Dakota) Instr., University of Missouri.

R. L. Arms, B.S. (Stanford) Pvt., United States Army, Dugway, Utah.

Lt. W. R. Ballard, M.S. (Chicago) Asst. Professor, Air Force Institute of Technology, Dayton, Ohio.

Arne Benson, Mech. Engr., Sanders Associates, Nashua, N. H.

Rev. J. E. Billotti, S.J., B.A. (Fordham) Grad. Student, Fordham University.

T. H. Blackburn, M.A. (Western Reserve) Instr., Case Institute of Technology.

Salomon Bochner, Ph.D. (Berlin) Professor, Princeton University.

R. E. Bowland, Student, Kent State University.

Ralph Byrd, Student, De La Salle Military Academy, Kansas City, Mo.

Helen I. Carbonneau, B.A. (New Hampshire) Junior Engr., Philco Corp., Philadelphia, Pa.

J. B. Chassan, M.A. (George Washington) Chief Statistician, St. Elizabeths Hospital, Washington, D. C.

L.T. Claiborne, Student, Baylor University.

C. J. Clark, Ph.D. (Oklahoma A.&M.C.) Res. Scientist, Lockheed Aircraft Corp., Palo Alto, Calif.

Marie B. Cook, B.A. (Colorado) Math., Holloman Air Force Base, N. Mex.

Anthony D'Antuono, M.Ed. (Boston U.) Principal, Cohassit High School, Mass.; Instr., Northeastern University.

H. A. Dennis, M.A. (Illinois) Asso. Professor, Lamar State College of Technology.

A. J. DiPalo, Student, Manhattan College.

W. G. Dotson, Jr., Student, Wake Forest College.
J. A. Dudman, B.A. (Reed) Asst. Professor, Reed College.
O. O. Duncan, B.A. (Wyoming) Instr. and Department Head, American River Junior College.
C. K. Fendall, B.A. (Puget Sound) Grad. Student, Reed College.
D. J. Fieldhouse, Student, Queen's University.
T. E. Finley, Jr., A.B. (Harvard) Chm., Department of Mathematics, Loomis School, Windsor, Conn.
Mrs. Wanda W. Fleming, M.A. (Chicago) Asst. Professor, Jackson College.
E. L. Gans, Student, Bronx High School of Science, New York.
R. L. Garrett, M.A. (North Carolina) Professor, Middle Georgia College.
Mrs. Daisy Gogan, M.A. (Montclair S.T.C.) Teacher, Clifton High School, N. J.
L. D. Goldstone, B.C.E. (Rensselaer) Chemist, N. Y. State Department Public Works Laboratory, Albany, N. Y.
Frances Gooen, M.A. (Montclair S.T.C.) Teacher, Central Evening High School, Newark, N. J.
Nicholas Grant, M.A. (Pennsylvania) Teacher, Central High School, Philadelphia, Pa.
E. E. Grate, Asst. Mgr., Feature Products, Columbus, Ohio.
Rev. F. A. Greene, S.J., M.A. (Fordham) Instr., Gonzaga High School, Washington, D. C.
E. A. Gunter, M.A. (Columbia Teachers C.) Head, Department of Mathematics, North Plainfield High School, N.J.
H. D. Hairfield, Jr., Student Asst., Oklahoma Agricultural and Mechanical College.
L. W. Halbe, L.L.B. (Kansas City School of Law) Owner, L. W. Halbe Agency, Fort Pierce, Fla.
E. F. Harding, Student, King's College, Cambridge, England.
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THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The fifteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Stevens Institute of Technology, Hoboken, New Jersey on April 28, 1956. Dr. J. H. Davis, President of Stevens Institute of Technology, gave the address of welcome. Dr. Barnett Rich, High School Vice-Chairman of the Section, presided at the morning session and Professor A. B. Brown, Chairman of the Section, at the afternoon session.

There were 158 persons in attendance, including 73 members of the Association.

The following officers were elected for the year 1956–57: Chairman, Dean Mina S. Rees, Hunter College; Collegiate Vice-Chairman, Professor J. N. Eastham, Cooper Union; High School Vice-Chairman, Dr. Irving Dodes, Bronx High School of Science; Secretary, Dr. Azelle B. Waltcher, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting reports were given by the Treasurer, the Committee on Membership, and the Committee on Contests and Awards. Professor C. G. Salkind, Co-Chairman of the latter committee reported, among other things, that 502 schools including 14,013 contestants had registered with the Section for the current contest. A motion that Professor W. H. Fagerstrom receive a vote of thanks and appreciation for his years of work as Chairman of the Committee on Contests and Awards was passed unanimously.

The following papers were presented:

1. *Constructions with ruler and divider*, by Professor Emil Artin, Princeton University.

The elementary constructions with ruler and divider were given as well as some examples of constructible and non-constructible problems. The analysis of constructibility led to the condition that all solutions of the problem should be real for all values of the parameters.

The next three papers were presented as a panel discussion on the topic: "Does the High School Curriculum in Mathematics Provide the Optimum College Preparation for the Brighter Pupil?"

2. *What are the high schools doing for the abler mathematics student?*, by Mr. S. L. Greitzer, Bronx High School of Science.

Most schools can do little or nothing for the abler mathematics student because there are too few per school, and because of lack of personnel and courses. Most of the offerings for these students are to be found in a few specialized schools. At these schools, there are offered, in addition to advanced algebra and solid geometry, a variety of courses now considered on the college level. These include analytic geometry, calculus, statistics, mathematical applications to science, *etc.* Because of the varied nature of the first year college courses, it is obviously impossible to prepare students so that they will show the same degree of ability at all colleges. It is suggested that the colleges would do well to standardize their offerings at the first year level.

3. *Does the high school curriculum in mathematics provide the optimum college preparation for the brighter pupil?*, by Professor W. H. Fagerstrom, The City College of New York.

The speaker presented the College's idea of what the preparatory curriculum should be. He warned that the success of any education program is hinged on the curiosity and creativeness of the pupils themselves. The teachers, he said, do not seem to encourage the students to work enough on their own. Many seem content to impart only a "smattering" rather than a mastery of a subject in their classes. He gave a list of about fifty topics including concepts and techniques from the field of secondary school mathematics in which the entering freshmen seem poorly prepared. He did not blame the teachers for the lack of knowledge on the part of the students, but attributed it to the trend of the times.

4. *What should be taught to bright high school students who intend to go to college?*, by Mr. I. M. Rothman, Brooklyn Technical High School and Brooklyn College.

The tentative course of study for the "Five Years in Four" mathematics honor classes at Brooklyn Technical High School was described. Then the speaker discussed concepts and topics that should be stressed, those that should be eliminated or minimized, and those that should be considered for inclusion in the curriculum for bright students. It was pointed out that we need

special textbooks for these students. Since these are not available at present, supplementary material should be prepared.

5. *Applications of non-euclidean geometry to some technological problems in waveguides*, by Mr. G. A. Deschamps, Federal Telecommunication Laboratories, Nutley, New Jersey.

The paper illustrated a practical use of non-euclidean geometry. In the first part some properties of non-euclidean geometry of the hyperbolic type were reviewed using the well known conformal and projective models of Klein, Poincaré, and Beltrami. It was shown that on the projective model, geometrical constructions were greatly simplified by means of a "hyperbolic protractor" which gave the hyperbolic distance between two points by direct reading. A simple construction for the angles was also described. In the second part of the paper applications to waveguide technology were given. They were based essentially on the fact that the transformation of the reflection coefficient through a waveguide junction could be represented by a congruent transformation in the hyperbolic space. Some specific problems that were simplified by this interpretation are: the description of a junction by a minimum number of parameters, the experimental determination of these parameters, the interpretation of measurements taken on one side of the junction in terms of what they mean for the other side, the composition of junctions in cascade, and the design of matching elements.

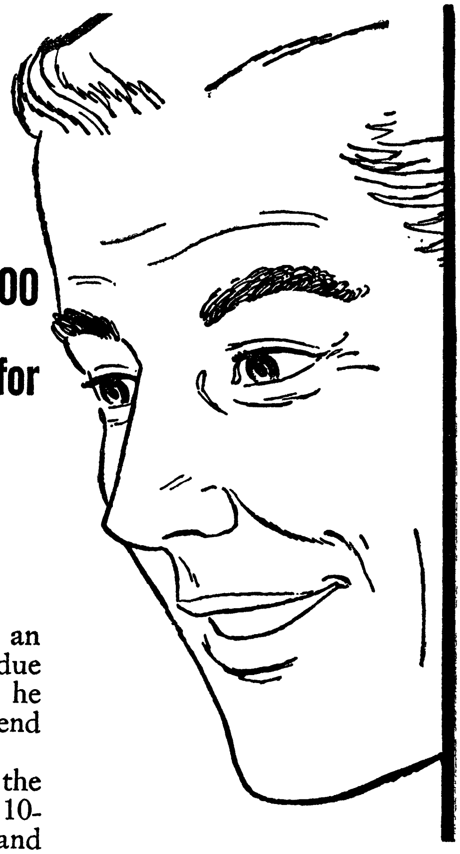
AZELLE B. WALTCHER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.
ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.
INDIANA, May 4, 1957.
IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.
KANSAS
KENTUCKY, Berea College, Berea, April 20, 1957.
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.
METROPOLITAN NEW YORK, April 27, 1957.
MICHIGAN, Wayne State University, Detroit, March 23, 1957.
MINNESOTA, Carleton College, Northfield, May 11, 1957.
MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.
NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.
NEW JERSEY
NORTHEASTERN
NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA
ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.
SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.
SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.
SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.
TEXAS, University of Houston, Houston, April, 1957.
UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.
WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.



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TRAMMEL ROTORS IN REGULAR POLYGONS

MICHAEL GOLDBERG, Bureau of Ordnance, U.S. Navy

1. Introduction. A rotor in a polygon (plane or spherical) is defined as a closed convex curve which remains tangent to all the sides of the polygon during a complete rotation of the curve. All the plane rotors were obtained and described analytically by Meissner [1] by means of a Fourier series expansion. Extensions to spherical rotors were shown by the author [3, 4].

In "*Circular-arc Rotors in Regular Polygons*" [2], the author derived kinematically a rotor bounded entirely by arcs of circles for each regular polygon. In "*Rotors in Spherical Polygons*" [3], he obtained an analogous rotor for each regular spherical polygon. For odd polygons, this rotor had two axes of symmetry; for even polygons ($n>4$), only one axis of symmetry.

In the present paper, a new series of circular-arc rotors is derived in the plane characterized by the possession of all the higher orders of symmetry. The corresponding rotors in spherical polygons are derived in a similar manner. However, the bounding arcs of spherical rotors are not all circular. In the plane, the generation is accomplished by the motion of vertices or other points of the rotor along straight lines. On the sphere, rotor points move along great circles. Hence, to distinguish the circular-arc rotors and the corresponding spherical rotors from the more general ones, I choose to call them *trammel rotors*.

The trammel rotors in the square include the Reuleaux polygons which are rotors bounded by arcs whose radii are equal to the edge of the square. The trammel rotors in the triangle include rotors whose radii are all equal to the height of the triangle. These were investigated by Fujiwara [7]. However, for the pentagon, there are no trammel rotors which have the same radii; at least two different radii are needed. Among those which have the height of the pentagon as one radius are the regular rotor described by Fujiwara [7, p. 245], shown in Figure 3, the biaxial rotor described by the author [2] and the family described here.

2. Construction of the (n-1)-lobed regular trammel rotor in the n-gon. Given a regular n-gon, inscribe a regular $(n-1)$-gon in the following manner. See Figure 1. Take the midpoint B of a side of the given n-gon as a vertex of the $(n-1)$-gon. Draw lines from this point inclined at the angle $\pi/(n-1)$ with the side. The intersections A and C of these drawn lines with the adjacent sides of the n-gon will be taken as the adjacent vertices of the $(n-1)$-gon. With the sides AB and BC as a beginning, complete the $(n-1)$-gon.

Now rotate the $(n-1)$-gon counter-clockwise within the n-gon while the vertices B and C are constrained to move along the sides of the n-gon. After a rotation of $\alpha=2\pi/(n-1)-2\pi/n=2\pi/n(n-1)$, the next vertex D will touch a side of the n-gon. The relative positions of the polygons at mid-position of the motion are shown in Figure 2. The envelopes of the positions of the fixed sides of the n-gon on the moving plane of the $(n-1)$-gon are $n-2$ arcs of circles by Bobillier's theorem [2, pp. 393–394]. The centers of these circles will lie on a

circle through B and C and the intermediate vertex of the n-gon. The totality of these points will be equally spaced on the circle. Also, these centers lie on diagonals of the n-gon. Hence, the radii of the circular arcs are the distances of these diagonals from the sides of the n-gon parallel to them.

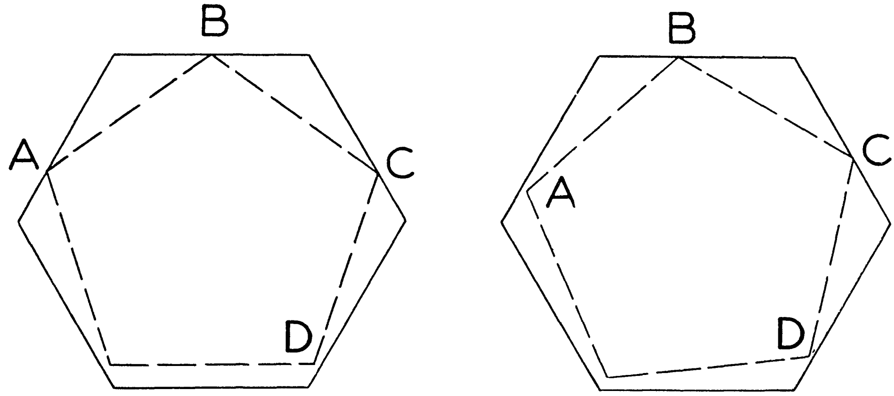

FIG. 1 and FIG. 2. Vertices of regular $(n-1)$-lobed rotor in n-gon.

The rotation of the $(n-1)$-gon is now continued but with the vertices C and D moving along the sides of the n-gon. Each time a new vertex touches a side of the n-gon, it continues by moving along this side. The contour of the rotor is thus molded by the sides of the n-gon. The complete contour is formed by a rotation of $[(n-1)/2]\alpha$ where $[(n-1)/2]$ is the integral part of $(n-1)/2$. Further rotation will retrace the same contour. The deflection angle of the contour at a cusp (or vertex) of the rotor is also α.

3. Construction of the $(n+1)$-lobed regular trammel rotor in the n-gon. Given a regular n-gon, inscribe a regular $(n+1)$-gon as the $(n-1)$-gon was inscribed in Section 2. Draw lines from the midpoint B of a side making angles $\pi/(n+1)$ with the side. The intersections A and C with the adjacent sides of the n-gon will be taken as adjacent vertices of the $(n+1)$-gon. With the sides AB and BC as a beginning, complete the $(n+1)$-gon.

Now rotate the $(n+1)$-gon within the n-gon while the vertices B and C are constrained to move along the sides of the n-gon. After a rotation of $\alpha=2\pi/n-2\pi/(n+1)=2\pi/n(n+1)$, a new vertex will touch a side of the n-gon. Continue the rotation as in Section 2 by keeping two vertices moving along sides of the n-gon. The contour of the rotor is thus molded by the sides of the n-gon. The radii are the same as in Section 2, and the total measure of the arcs of each radius is the same.

4. The regular trammel rotors. A completed regular four-lobed rotor in the regular pentagon is shown in Figure 3. The circle of one set of centers is shown as well as the radii of the sectors. The normals at the points of contact intersect in the instantaneous center of rotation.

The regular $(n-1)$-lobed and $(n+1)$-lobed rotors for $n=3$, 4, 5, and 6 that

are generated by the methods of Sections 2 and 3 are shown in Figure 4 for comparison. These should be compared also with the previously published set of trammel rotors [2, Fig. 4]. The duals of these rotors are another set of regular trammel rotors. The rotors in the even polygons are self-dual.

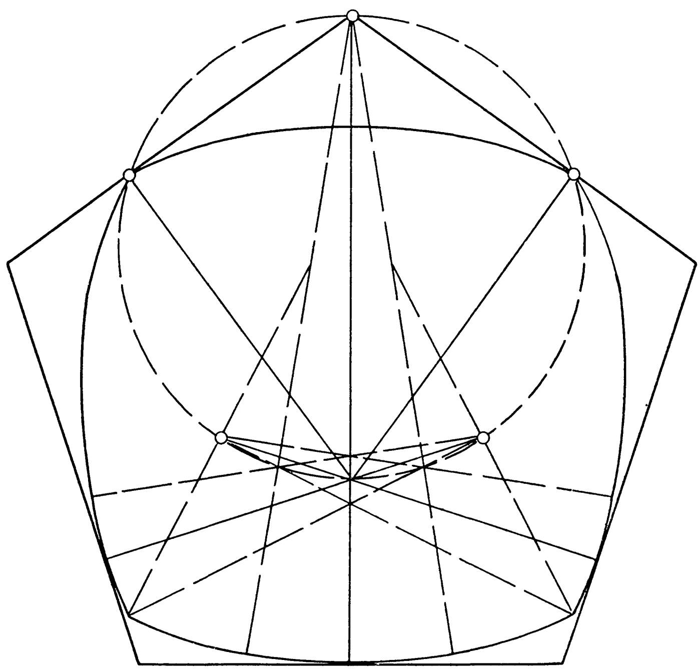

FIG. 3. Regular four-lobed rotor in regular pentagon.

5. Other trammel rotors. Every rotor in a regular n-gon is described by the polar tangential equation, due to Meissner [1],

$$p(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \sin k\theta + \sum_{k=1}^{\infty} b_k \cos k\theta, \tag{1}$$

where

$$a_k = b_k = 0 \quad \text{for} \quad k \not\equiv \pm 1 \pmod{n}.$$

The weighted mean of the equations for two rotors in an n-gon will give another equation satisfying the same condition on the coefficients. Therefore, the new equation will also describe a rotor in the same n-gon. The radius of curvature ρ is given by the equation

$$\rho = p(\theta) + p''(\theta) \geqq 0. \tag{2}$$

For convexity, ρ cannot be negative. This condition imposes a limiting relation among the constants a_k and b_k.

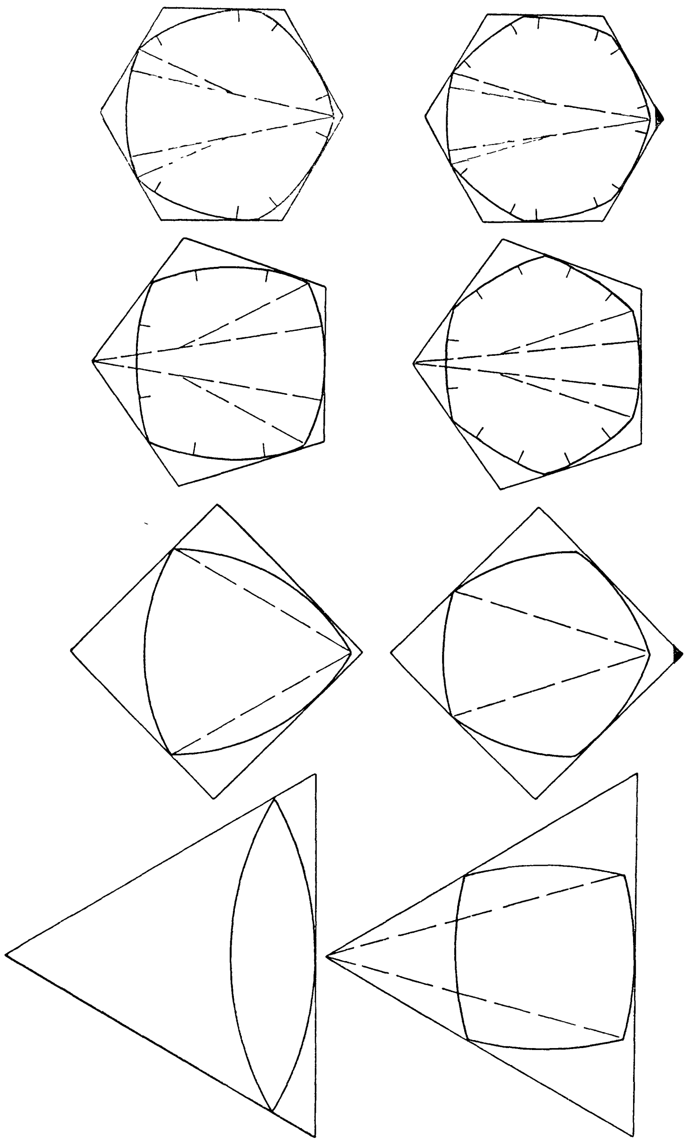

Fig. 4. Regular trammel rotors.

Since a rotor in a kn-gon is also a rotor in an n-gon, the foregoing sections show, for each regular polygon, the constructions for an infinity of different trammel rotors. They are not, however, all the possible rotors. New ones are described by linear combinations of the equations for those previously derived.

For example, if two trammel rotors in a polygon are described by the equations

$$p_1 = f_1(\theta) \quad \text{and} \quad p_2 = f_2(\theta),$$

then new trammel rotors in the same polygon are their weighted means given analytically by the equation

$$p_3 = \frac{uf_1(\theta) + vf_2(\theta)}{u + v},$$

where u and v are any real numbers. This equation includes non-convex as well as convex curves. Of course, only the convex curves may be used. If one of the given rotors is a circle, then the new rotors are similar to the parallel rotors. The rotor similar to the convex internal parallel rotors are the *dual rotors*.

These new rotors are obtained geometrically as follows. Each arc of a new rotor has a center which is the weighted mean of the two centers of the corresponding arcs of the two given rotors and whose radius is the weighted mean of the corresponding arcs of the given rotors.

The foregoing may be summarized in the following theorems.

THEOREM 1. *A regular $(n-1)$-lobed rotor in a regular n-gon can be made of $n-1$ equal parts, each composed of a circular arc of measure $\alpha = 2\pi/n(n-1)$ and radius r_1 equal to the height of the n-gon, flanked on each side by successively tangent circular arcs of measure α and radii $r_2, r_3, \cdots, r_k$, where $k = [(n-1)/2]$ and $r_2, r_3, \cdots, r_k$ are the distances of the diagonals of the n-gon from the parallel sides.*

THEOREM 2. *A regular $(n+1)$-lobed rotor in a regular n-gon can be made of $n+1$ equal parts, each composed of a circular arc of measure $\alpha = 2\pi/n(n+1)$ and radius r_1 equal to the height of the n-gon, flanked on each side by successively tangent circular arcs of measure α and radii $r_2, r_3, \cdots, r_k$, where $k = [(n-1)/2]$, and $r_2, r_3, \cdots, r_k$ are the distances of the diagonals of the n-gon from the parallel sides.*

THEOREM 3. *Parallel curves of the rotors of Theorems 1 and 2 are also rotors in regular n-gons. The cusps of the rotors in Theorems 1 and 2 are replaced by circular arcs of measure α.*

THEOREM 4. *A convex linear combination of the trammel rotors of Theorems 1, 2, and 3 in an n-gon is also a trammel rotor in the n-gon.*

The rotors described in the foregoing paragraphs still do not exhaust the possible trammel rotors. For example, the rotors of four and five arcs for the equilateral triangle, described by Fujiwara [7], are not derivable by finite combinations in this way. However, both trammel rotors and non-trammel rotors may conceivably be represented asymptotically by a linear combination of an infinite series of $(n-1)$-lobed and $(n+1)$-lobed regular trammel rotors. It would be interesting to find a minimal set of trammel rotors from which each of the others could be obtained by a finite number of linear combinations.

6. On rotors of minimum area. All the rotors in a given polygon have the same perimeter. However, the areas vary. In each case, the maximum area is attained by the circle. The minimum area and the shapes that attain it are not generally known. In fact, only for the equilateral triangle and the square have minimum rotors been determined. The minimum rotor for the triangle is bounded by two circular arcs whose radius is equal to the height of the triangle. The minimum rotor for the square is bounded by three equal circular arcs whose radius is equal to the height of the square.

Fujiwara, who used an analytical method for deriving the minimum rotors for the triangle and the square [6, 7], stated that his method was not successful for the pentagon [7, p. 246]. However, he submitted the regular trammel rotor (described in Section 2 and pictured in Figure 3) as a possible contender for this distinction.

A family of four-lobed trammel rotors in the pentagon is constructable by re-arranging the arcs of the Fujiwara rotor. If e is the edge of the pentagon, then the radii of the arcs are given by the equations

$$r_1 = e(1 + 2\cos 72°) = \text{height of pentagon},$$

$$r_2 = e \sin 72°.$$

The bounding arcs of a quadrant of a rotor of the family are given by arcs of the following radii and angles

$$(r_1,\ 18° - \beta),\quad (r_2,\ 2\beta),\quad (0,\ 36°),\quad (r_2,\ 36° - 2\beta),\quad (r_1,\ \beta),$$

where $0 \leqq \beta \leqq 9°$. When $\beta = 9°$, the area is minimized and the Fujiwara rotor is obtained. When $\beta = 0$, the four-arc rotor of biaxial symmetry is obtained. These rotors may be derived as dual rotors of the mean of two biaxial four-arc rotors placed at an angle to each other. When the angle is 90°, the Fujiwara rotor is obtained.

The area A of a closed convex curve is given by the equation

$$A = \frac{1}{2}\int_0^{2\pi} \rho p\, d\theta. \tag{3}$$

The determination of the rotor of minimum area may be formulated as the determination of the coefficients a_k and b_k which satisfy equations (1) and (2) and which minimize A in equation (3).

An examination of the possibilities has led the author to venture the following conjecture.

Among all the possible rotors in a given n-gon, the rotor of least area is the regular trammel rotor of $n-1$ lobes.

Rotors may be ranked by the quantity $K = A/L^2$ where A is the area and L is the perimeter. Then K for several rotors in the pentagon are listed below.

Description of rotor in pentagon	K
1. Regular trammel rotor (Fujiwara rotor)	.07603
2. Trammel rotor with biaxial symmetry	.07679
3. Basic rotor $p=a(15+\cos 4\theta)$ [4]	.07692
4. Dual rotor of 2	.07724
5. Circle	.07958

7. Regular trammel rotors in spherical polygons. The same procedure for generating trammel rotors in the plane can be used for generating the regular

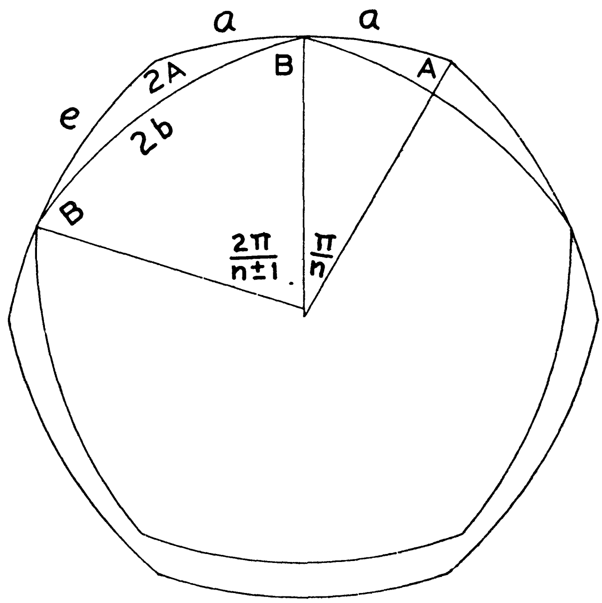

FIG. 5. Vertices of regular rotor in spherical polygon.

trammel rotors in spherical polygons. See Figure 5. If the side of the given n-gon is $2a$, and the side of the inscribed $(n\pm 1)$-gon is $2b$, the relation between them is obtained by eliminating B, C and e from the following equations.

$$\begin{aligned}
\sin A \cos a &= \cos \pi/n,\\
\sin B \cos b &= \cos \pi/(n \pm 1),\\
\sin e \sin 2A &= \sin 2b \cos B,\\
\cos e &= \cos a \cos 2b + \sin a \sin 2b \sin B.
\end{aligned}$$

If two consecutive vertices of the inscribed polygon are moved along adjacent sides of the fixed n-gon until another vertex touches the n-gon, the envelopes of the other sides of the n-gon on the surface of the moving inscribed polygon will be part of the boundary of the generated rotor. Similar motions will complete the generation of the rotor boundary. The 90° parallels will give a set of

regular trammel rotors external to the polar polygons.

The boundary curves are not plane and, therefore, they are not circles. However, they do have the property, shared by the circle in the plane, of being the locus of a vertex of a moving constant angle whose sides pass through two fixed points.

Other spherical trammel rotors, possessing only one and two axes of symmetry, were described in an earlier paper by the author [3].

A law of combination for trammel rotors on the sphere, similar to the law of combination described in Section 5, has not yet been obtained. However, a law of combination for the basic rotors, which resemble the regular trammel rotors, has been derived by the author [4].

8. Trammel rotors in polyhedra. Surfaces of constant width may be considered as rotors in a cube. Trammel rotors in a polyhedron may be taken as those generated by restraining the motion of each of three of its points to a plane for each finite interval of the motion. The surfaces generated are spherical and toroidal surfaces. One obvious family of such rotors in a cube is the family of surfaces of revolution obtained by rotating a symmetrical trammel rotor in a square about its axis of symmetry. Other trammel rotors are also possible. Meissner described one [8] which has the four vertices of a regular tetrahedron, four spherical surfaces opposite them and three toroidal strips along three of the edges of the tetrahedron. Still others may be based on any tetrahedron and will have, in general, eight spherical surfaces, (two centered on each vertex), and six toroidal surfaces (one corresponding to each edge of the tetrahedron). As in the case of plane rotors, new rotors may be obtained as combinations of known rotors.

Rotors in the regular tetrahedron and octahedron exist, but it is not known if any of these are trammel rotors.

References

1. E. Meissner, Über die Anwendung von Fourier-Reihen auf einige Aufgaben der Geometrie und Kinematik, Vierteljschr. Naturforsch. Ges. Zürich, vol. 54, 1909, pp. 309–329.

2. M. Goldberg, Circular-arc rotors in regular polygons, this MONTHLY, vol. 55, 1948, pp. 393–402.

3. ———, Rotors in spherical polygons, J. Math. Physics, vol. 30, 1952, pp. 235–244.

4. ———, Basic rotors in spherical polygons, J. Math. Phys., vol. 34, 1956, pp. 322–327.

5. W. Blaschke, Konvexe Bereiche gegebener Breite und kleinsten Inhalts, Math. Ann., vol. 76, 1914, pp. 504–513.

6. M. Fujiwara and S. Kakeya, On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle, Tôhoku Math. J., vol. 11, 1917, pp. 92–110.

7. M. Fujiwara, Über die innen-umdrehbare Kurve eines Vieleckes, Sci. Rep., Tôhoku Univ., Ser. 1, vol. 8, 1919, pp. 221–246.

8. E. Meissner, Drei Gipsmodelle von Flächen konstanter Breite, Z. Math. Phys., vol. 60, 1912, pp. 92–94.

9. W. Wunderlich, Über eine Klasse zwangläufiger höherer Elementenpaare, Z. Angew. Math. Mech., vol. 19, 1939, pp. 177–181. (This paper describes examples of non-convex trammel rotors in star polygons.)

A METHOD FOR DERIVING NUMERICAL DIFFERENTIATION FORMULAS

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The problem of polynomial interpolation usually involves the approximation of a function $y=y(x)$ by a suitably chosen nth degree polynomial $P_n(x)$ such that $y_i \equiv y(x_i)=P_n(x_i)$ for $i=0, 1, \cdots, n$. It is well known that a unique polynomial of degree n passes through $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$, where $x_0<x_1< \cdots <x_n$, though it may be derived in different ways and hence may appear in different forms (*e.g.*, the interpolation formulas of Newton, Lagrange, *etc.*). We can obtain an approximation to $y(x_k)$, if $x_k \neq x_i$, by computing $P_n(x_k)$. If $x_0<x_k<x_n$ this process is called interpolation; otherwise extrapolation.

To find approximations to the derivatives of $y(x)$ we usually differentiate the interpolating polynomial $P_n(x)$. However, in this discussion we shall be interested only in values of the derivatives at the mesh points x_i and in this particular case there is no need to find $P_n(x)$. Instead we propose to find formulas of the form

$$y_i^{(k)} = h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j + O(h^s), \tag{1}$$

where $y_i^{(k)}$ denotes the kth derivative of $y(x)$ evaluated at the point x_i and $O(h^s)$ indicates the truncation error. Thus, in effect, we can express the derivatives at the mesh points as linear combinations of the known ordinates y_i. For examples of such formulas see Milne [1], Hildebrand [2], and Nielsen [3].

The purpose of this paper is to describe an algorithm for finding the coefficients A_{ij}^{ks} for a formula of type (1) when the order of the derivative k and the quantity s are specified. We shall use an "undetermined coefficients" method which requires that $y(x)$ possess a continuous $(k+s)$th derivative in the region of consideration.

Consider the Taylor expansion of y_j about the point x_i

$$y_j = \sum_{t=0}^{k+s-1} (j-i)^t \frac{h^t}{t!} y_i^{(t)} + \frac{(j-i)^{k+s} h^{k+s}}{(k+s)!} y^{(k+s)}(\xi_{ij}), \tag{2}$$

where ξ_{ij} lies in (x_i, x_j). If we write the expansion (2) for $j=0, 1, 2, \cdots, n$ and then form a linear combination of these ordinates (where the coefficients are denoted by A_{ij}^{ks}) we obtain

$$\begin{aligned}\sum_{j=0}^{n} A_{ij}^{ks} y_j &= y_i \sum_{j=0}^{n} A_{ij}^{ks} + h y_i^{(1)} \sum_{j=0}^{n} (j-i) A_{ij}^{ks} + \cdots \\ &\quad + \frac{h^{k+s-1}}{(k+s-1)!} y_i^{(k+s-1)} \sum_{j=0}^{n} (j-i)^{k+s-1} A_{ij}^{ks} \\ &\quad + \frac{h^{k+s}}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}), \end{aligned}$$

$$= \sum_{m=0}^{k+s-1} \frac{h^m}{m!} y_i^{(m)} B_{im}^{ks} + \frac{h^{k+s}}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}), \tag{3}$$

where

$$B_{im}^{ks} = \sum_{j=0}^{n} (j-i)^m A_{ij}^{ks}. \tag{4}$$

Dividing (3) by h^k gives

$$h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j = \sum_{m=0}^{k+s-1} \frac{h^{m-k}}{m!} y_i^{(m)} B_{im}^{ks} + E_i^{ks}, \tag{5}$$

where

$$E_i^{ks} = \frac{h^s}{(k+s)!} \sum_{j=0}^{n} (j-i)^{k+s} A_{ij}^{ks} y^{(k+s)}(\xi_{ij}), \tag{6}$$

and we determine the coefficients A_{ij}^{ks} so as to make

$$B_{im}^{ks} = \delta_{mk} k! \qquad (m = 0, 1, \cdots, k+s-1). \tag{7}$$

Using (7) in (5) we obtain

$$y_i^{(k)} = h^{-k} \sum_{j=0}^{n} A_{ij}^{ks} y_j - E_i^{ks}, \tag{8}$$

where (6) shows (8) to be a formula of type (1).

Let us use (4) to write the system of linear equations (7) in the form

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ (-i) & (1-i) & (2-i) & \cdot & (n-i) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-i)^k & (1-i)^k & (2-i)^k & \cdot & (n-i)^k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-i)^t & (1-i)^t & (2-i)^t & \cdot & (n-i)^t \end{bmatrix} \begin{bmatrix} A_{i0}^{ks} \\ A_{i1}^{ks} \\ A_{i2}^{ks} \\ \cdot \\ \cdot \\ A_{in}^{ks} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ k! \\ \cdot \\ 0 \end{bmatrix}, \tag{7'}$$

where $t = k+s-1$.

There are $k+s$ equations in the $n+1$ variables A_{ij}^{ks} and in general we must have

$$n+1 \geqq k+s. \tag{9}$$

if the equations are to be consistent. The inequality (9) gives us a clear insight into the relationship that exists among the quantities n, k, and s. For instance, if we are given k and s we automatically have a lower bound on the number of ordinates y_i necessary for a formula of type (1). On the other hand, if we have a fixed value of n, and k is given, we have an upper bound on the accuracy of the formula of type (1) which can be derived.

In the usual case where $n+1=k+s$ we have a unique solution to the system (7′) since the coefficient matrix is square and non-singular. The non-singularity follows directly if we observe that the determinant has the form of the determinant of Vandermonde

$$\begin{vmatrix} 1 & 1 & 1 & \cdot & 1 \\ z_0 & z_1 & z_2 & \cdot & z_n \\ z_0^2 & z_1^2 & z_2^2 & \cdot & z_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_0^n & z_1^n & z_2^n & \cdot & z_n^n \end{vmatrix} = \prod_{j,k=0,j>k}^{n} (z_j - z_k)$$

and, in (7′), $z_j - z_k \neq 0$ for $j \neq k$.

The writer has found occasion to choose $n+1>k+s$, *i.e.*, more ordinates were used than absolutely necessary to obtain a differentiation formula for a specified k and s. In this case it is easily seen that the system (7′) has rank $k+s$ and hence the solutions A_{ij}^{ks} are functions of $(n+1)-(k+s)$ arbitrary parameters.

To illustrate the algorithm, let us find formulas for the second derivative which have a truncation error $O(h^3)$. Here $k=2$ and $s=3$. First we set $n+1=5$ (*i.e.*, we use 5 mesh points) and then find the unique set of coefficients for the cases $i=0, 1, 2, 3, 4$. The algorithm merely involves solving (7′) for these five values of i. For instance, when $i=0$ we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 16 & 81 & 256 \end{bmatrix} \begin{bmatrix} A_{00}^{23} \\ A_{01}^{23} \\ A_{02}^{23} \\ A_{03}^{23} \\ A_{04}^{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

with the solutions

$$A_{00}^{23} = \frac{35}{12}, \quad A_{01}^{23} = -\frac{104}{12}, \quad A_{02}^{23} = \frac{114}{12}, \quad A_{03}^{23} = -\frac{56}{12}, \quad A_{04}^{23} = \frac{11}{12}.$$

In like manner the other four sets of coefficients can be found and we have the following five formulas:

$$y_0^{(2)} = \frac{1}{12h^2}[35y_0 - 104y_1 + 114y_2 - 56y_3 + 11y_4],$$

$$y_1^{(2)} = \frac{1}{12h^2}[11y_0 - 20y_1 + 6y_2 + 4y_3 - y_4],$$

$$y_2^{(2)} = \frac{1}{12h^2}[-y_0 + 16y_1 - 30y_2 + 16y_3 - y_4],$$

$$y_3^{(2)} = \frac{1}{12h^2}[-y_0 + 4y_1 + 6y_2 - 20y_3 + 11y_4],$$

and

$$y_4^{(2)} = \frac{1}{12h^2}[11y_0 - 56y_1 + 114y_2 - 104y_3 + 35y_4].$$

The errors in these formulas are given by (6). For example, the error in $y_2^{(2)}$ becomes

$$E_2^{23} = \frac{h^5}{90}[2y^{(5)}(\xi_{20}) - y^{(5)}(\xi_{21}) + y^{(5)}(\xi_{23}) - 2y^{(5)}(\xi_{24})],$$

and

$$|E_2^{23}| \leqq \frac{h^5}{15}M,$$

where M is the maximum absolute value of $y^{(5)}(x)$ in the interval (x_0, x_4).

Notice that the coefficient matrix on the left in equation (7′) is a function of the combination $k+s$. Thus we use the same matrix for finding a formula for $y_i^{(k-r)}$ with truncation error $O(h^{s+r})$ that we use for finding a formula for $y_i^{(k)}$ with truncation error $O(h^s)$. The only difference in the two cases is the vector on the right.

References

1. W. E. Milne, Numerical Calculus, Princeton, 1949, pp. 96–99.
2. F. B. Hildebrand, Introduction to Numerical Analysis, New York, 1956, p. 82.
3. K. L. Nielsen, Methods in Numerical Analysis, New York, 1956, pp. 342–347.

A CURIOUS TRIGONOMETRIC IDENTITY

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Recently I discovered, to my surprise, that

$$|\sin(x+iy)| = |\sin x + \sin iy|$$

for real values of x and y. This is easily verified. In the first place, by the addition formula, we have

$$\begin{aligned}\sin(x+iy) &= \sin x \cos iy + \cos x \sin iy\\ &= \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

From this, it follows that

$$\begin{aligned}|\sin(x+iy)|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y\\ &= \sin^2 x\,(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y\\ &= \sin^2 x + \sinh^2 y = |\sin x + i\sinh y|^2\\ &= |\sin x + \sin iy|^2,\end{aligned}$$

which yields the desired identity.

In a similar way, or using the above identity, we find also that

$$|\sinh(x+iy)| = |\sinh x + \sinh iy|.$$

We shall now show that z, $\sin z$, and $\sinh z$ are essentially the only functions with this property.*

THEOREM. *If $f(z)$ is regular for $|z|<r$, and satisfies the functional equation*

$$|f(x+iy)| = |f(x)+f(iy)|$$

for real values of x and y, then

$$f(z) = Az, \qquad f(z) = A\sin bz, \quad \text{or} \quad f(z) = A\sinh bz,$$

where A and b are constants, and b is real.

Proof. From the preceding, it is clear that these functions do satisfy the functional equation. It remains to show that there is no other solution.

Suppose that $f(z)$ is regular for $|z|<r$, and satisfies the functional equation. Putting $z=0$, we find that $f(0)=0$. Hence we may put

* After this result was presented to the Northern California Section of the Association on January 14, 1956, Professor G. Szegö called my attention to a related paper by Einar Hille, *A Pythagorean functional equation*, Ann. of Math., vol. 24, 1922, pp. 175–180. Hille draws the same conclusion for a different functional equation, namely, $|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$. It is not clear how to modify Hille's method to apply to the functional equation considered here. On the other hand, the calculations of the present paper need only minor modifications to yield Hille's result.

$$f(z) = \sum_{n=p}^{\infty} c_n z^n \qquad \text{for } |z| < r,$$

where $p \geqq 1$. Unless $f(z)$ vanishes identically, we may suppose that $c_p \neq 0$. It will clearly be sufficient to consider normalized solutions, with $c_p = 1$.

We see that

$$\begin{aligned} |f(x+iy)|^2 &= \sum_k c_k(x+iy)^k \cdot \sum_l \bar{c}_l(x-iy)^l \\ &= \sum_{k,l} c_k\bar{c}_l(x+iy)^k(x-iy)^l \end{aligned}$$

and

$$\begin{aligned} |f(x)+f(iy)|^2 &= \sum_k c_k[x^k+(iy)^k] \cdot \sum_l \bar{c}_l[x^l+(-iy)^l] \\ &= \sum_{k,l} c_k\bar{c}_l[x^k+(iy)^k][x^l+(-iy)^l]. \end{aligned}$$

If the given functional equation is to be satisfied, then the terms in these two series of degree m in x and y must agree. This yields the recursion formula

$$\sum_{k+l=m} c_k\bar{c}_l(x+iy)^k(x-iy)^l = \sum_{k+l=m} c_k\bar{c}_l[x^k+(iy)^k][x^l+(-iy)^l]$$

for the coefficients of $f(z)$.

The first case of this recursion formula to be considered is $m = 2p$. This yields

$$(x+iy)^p(x-iy)^p = [x^p+(iy)^p][x^p+(-iy)^p],$$

or

$$(x^2+y^2)^p = \begin{cases} x^{2p}+y^{2p} & \text{if } p = 2q+1, \\ (x^p+y^p)^2 & \text{if } p = 4q, \\ (x^p-y^p)^2 & \text{if } p = 4q+2. \end{cases}$$

It is seen that this is an identity only if $p=1$. Thus the expansion of $f(z)$ must necessarily start with the first power of z, so that

$$f(z) = z + c_2z^2 + c_3z^3 + \cdots.$$

Since the terms in the recursion formula occur in conjugate pairs, it may be rewritten in the form

$$\Re \sum_{1 \leqq l \leqq m/2}{}' c_{m-l}\bar{c}_l\{(x+iy)^{m-l}(x-iy)^l - [x^{m-l}+(iy)^{m-l}][x^l+(-iy)^l]\} = 0,$$

where the prime on the summation sign indicates that for even m, the term with $l=m/2$ must be multiplied by $1/2$. Putting $m=3$, we find that

$$\Re\{c_2(x-iy)[(x+iy)^2-(x^2-y^2)]\} = 0$$

or

$$\Re\{c_2(x - iy)\cdot 2ixy\} = 0.$$

Thus both the real and imaginary parts of c_2 must vanish, so that $c_2=0$. Next, putting $m=4$, we find that

$$\Re\{c_3(x - iy)[(x + iy)^3 - (x^3 - iy^3)]\} = 0$$

or

$$\Re\{c_3(x^2 + y^2)\cdot 3ixy\} = 0.$$

Thus the imaginary part of c_3 must vanish, or c_3 must be real. Finally, putting $m=n+1$, where $n>3$, we find that

$$\Re\{c_n(x - iy)[(x + iy)^n - (x^n + i^n y^n)] + \cdots\} = 0,$$

where the omitted terms involve earlier coefficients only. From this we have

$$\Re\{c_n(x - iy)[nx^{n-1}iy + (1/2)n(n - 1)x^{n-2}(iy)^2 + \cdots] + \cdots\} = 0.$$

and hence

$$\Re\{c_n[inx^n y - (1/2)n(n - 3)x^{n-1}y^2 + \cdots] + \cdots\} = 0.$$

This shows that both the real and imaginary parts of c_n are determined uniquely in terms of earlier coefficients for $n>3$, so that c_n itself is so determined. Summarizing, we have

$$f(z) = z + c_3z^3 + c_4z^4 + \cdots,$$

where c_3 is real, and the remaining coefficients are uniquely determined in terms of c_3.

On the other hand, the known solutions

$$z, \qquad (1/b)\sin bz, \qquad (1/b)\sinh bz$$

provide solutions of this form with $c_3=0$, $c_3=-b^2/6$, and $c_3=b^2/6$, that is, solutions with arbitrary real c_3. Since the remaining coefficients are uniquely determined in terms of c_3, there can be no other normalized solutions. Every solution is found from one of these by multiplying by a constant factor.

AN ACKNOWLEDGEMENT

F. A. VALENTINE

Theorem 1 in my paper "*The motion of a particle constrained to move on a rough convex curve*," this MONTHLY, vol. 63, 1956, pp. 16–20, is essentially contained on page 221 of "*Dynamics of a particle*" by Tait and Steele, 5th Edition, Macmillan, 1882.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

EXPRESSION OF IRRATIONALS OF ANY DEGREE AS REGULAR CONTINUED FRACTIONS WITH INTEGRAL COMPONENTS

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By means of Thiele's Theorem [1], $f(x+h)$ can be expressed as a continued fraction in terms of h, $f(x)$, and the reciprocal derivatives of $f(x)$, thus

$$f(x+h) = f(x) + \frac{h}{rf(x)+}\,\frac{h}{2rrf(x)+}\,\frac{h}{3rr_2f(x)+}\cdots, \tag{1}$$

where

$$rf(x) = \frac{1}{f'(x)}, \qquad r_2f(x) = f(x) + 2rrf(x),$$

and in general

$$r_nf(x) = r_{n-2}f(x) + nrr_{n-1}f(x).$$

The purpose of this note is to show that (1) can be applied to the problem of expressing the mth root of the integral number N as a continued fraction with a simple and regular development.

We will write N in the form $\alpha^m+\beta$, where α, β, and m are all integral, and apply (1) with $f(x)=x^{1/m}$, finally giving h and x the values β and α^m respectively.

We have

$$a_0 = x^{1/m},$$

$$a_1 = rx^{1/m} = mx^{(m-1)/m},$$

$$a_2 = 2rrx^{1/m} = \frac{2x^{1/m}}{m-1}, \qquad r_2x^{1/m} = \frac{m+1}{m-1}x^{1/m},$$

$$a_3 = 3rr_2x^{1/m} = \frac{3m(m-1)}{m+1}x^{(m-1)/m}, \qquad r_3x^{1/m} = \frac{2m(2m-1)}{m+1}x^{(m-1)/m},$$

$$a_4 = 4rr_3x^{1/m} = \frac{2(m+1)}{(m-1)(2m-1)}x^{1/m}, \qquad r_4x^{1/m} = \frac{(m+1)(2m+1)}{(m-1)(2m-1)}x^{1/m},$$

$$a_5 = 5rr_4x^{1/m} = \frac{5m(m-1)(2m-1)}{(m+1)(2m+1)}x^{(m-1)/m},$$

$$r_5x^{1/m} = \frac{3m(2m-1)(3m-1)}{(m+1)(2m+1)}x^{(m-1)/m}, \text{ etc.}$$

The expressions for the r's suggest the general relations.

$$r_{2n-1}x^{1/m} = \frac{nm(2m-1)(3m-1)\cdots(nm-1)}{(m+1)(2m+1)(3m+1)\cdots\{(n-1)m+1\}}x^{(m-1)/m},$$

and

$$r_{2n}x^{1/m} = \frac{(m+1)(2m+1)(3m+1)\cdots(nm+1)}{(m-1)(2m-1)(3m-1)\cdots(nm-1)}x^{1/m}.$$

These can be proved by induction.

We can now write the continued fraction for $f(x+h)=N^{1/m}=(\alpha^m+\beta)^{1/m}$ in the form

$$(2)\quad \alpha+\frac{\beta}{m\alpha^{m-1}+}\;\frac{\beta}{\dfrac{2}{m-1}\alpha+}\;\frac{\beta}{\dfrac{3m(m-1)}{m+1}\alpha^{m-1}+}\;\frac{\beta}{\dfrac{2(m+1)}{(m-1)(2m-1)}\alpha+}\cdots.$$

To transform this into a more convenient expression we note that if b_r, a_r and b_{r+1} in the continued fraction

$$a_0+\frac{b_1}{a_1+}\;\frac{b_2}{a_2+}\;\frac{b_3}{a_3+}\cdots\frac{b_r}{a_r+}\cdots$$

are all multiplied by the same number k, the value of the continued fraction is unchanged [2]. Also if $b_{r+1}k$, a_{r+1} and b_{r+2} in the new fraction are all divided by k, the value of the continued fraction is still unchanged. By a repetition of this process using different k's we obtain

$$a_0+\frac{b_1k_1}{a_1k_1+}\;\frac{b_2k_2}{\dfrac{k_2}{k_1}a_2+}\;\frac{b_3k_3}{\dfrac{k_1k_3}{k_2}a_3+}\;\frac{b_4k_4}{\dfrac{k_2k_4}{k_1k_3}a_4+}\cdots$$

as equivalent to the original fraction.

By applying this process to (2) with $k_1=1$, $k_2=m-1$, $k_3=m+1$, $k_4=2m-1$, $k_5=2m+1$, *etc.*, we obtain

$$(3)\quad N^{1/m}=\alpha+\frac{\beta}{m\alpha^{m-1}+}\;\frac{\beta(m-1)}{2\alpha+}\;\frac{\beta(m+1)}{3m\alpha^{m-1}+}\;\frac{\beta(2m-1)}{2\alpha+}\;\frac{\beta(2m+1)}{5m\alpha^{m-1}+}\cdots,$$

wherein the numerators are β times the successive terms of the series

$$1,\quad m-1,\quad m+1,\quad 2m-1,\quad 2m+1,\quad 3m-1,\quad 3m+1,\quad etc.,$$

and the denominators are alternately $m\alpha^{m-1}$ times the successive terms of the series 1, 3, 5, 7, $\cdots$ and 2α.

If m is not integral but rational and of the form m/l we write the required $N^{l/m}$ in the form $(\alpha^{m/l}+\beta)^{l/m}$ choosing α so that it is an lth power, *e.g.*, if $36^{3/5}$

were required we could choose α to be 8 so that $36=8^{5/3}+4$, and then form the continued fraction for $N^{l/m}$ thus

$$(4) \qquad N^{l/m}=\alpha+\frac{\beta l}{m\alpha^{(m-l)/l}+}\,\frac{\beta(m-l)}{2\alpha+}\,\frac{\beta(m+l)}{3m\alpha^{(m-l)/l}+}\cdots.$$

Some numerical examples are as follows.

$$2^{1/3}=1+\frac{1}{3+}\,\frac{2}{2+}\,\frac{4}{9+}\,\frac{5}{2+}\,\frac{7}{15+}\,\frac{8}{2+}\,\frac{10}{21+}\,\frac{11}{2+}\cdots,$$

$$2^{2/3}=1+\frac{2}{3+}\,\frac{1}{2+}\,\frac{5}{9+}\,\frac{4}{2+}\,\frac{8}{15+}\,\frac{7}{2+}\,\frac{11}{21+}\,\frac{10}{2+}\cdots,$$

$$131^{5/7}=32+\frac{15}{28+}\,\frac{6}{64+}\,\frac{36}{84+}\,\frac{27}{64+}\,\frac{57}{140+}\cdots.$$

References

1. L. M. Milne-Thomson, Calculus of Finite Differences, London, Macmillan 1933.
2. S. Barnard and J. M. Child, Advanced Algebra, London, Macmillan, 1937, p. 197.

ON A PROPERTY OF COMPLEX POWER SERIES

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Let $f(z)$ be a complex-valued function of the complex variable $z=x+iy$, defined for $|z|\leqq 1$. We define the set H as the set of functions f such that the Lebesgue-integral

$$\|f\|^2=\iint_K |f(z)|^2dxdy$$

exists, the integral being extended over the area of the real unit circle K, $x^2+y^2\leqq 1$ with functions f and g defining the same element of H if $f=g$ except on a set of plane measure zero. For two elements $f\in H$ and $g\in H$ we define the scalar-product

$$(f,g)=\iint_K f(z)\overline{g(z)}dxdy.$$

It is obvious that H is a Hilbert-space. If f is regular for $|z|\leqq 1$ then $f\in H$.

Let ν, μ be non-negative integers. Then we have

$$(z^\nu,z^\mu)=\iint_K z^\nu\bar{z}^\mu dxdy=\int_0^{2\pi}\int_0^1 r^{\nu+\mu+1}e^{i(\nu-\mu)\phi}drd\phi=\begin{cases}0 & (\nu\neq\mu)\\ \dfrac{\pi}{\nu+1} & (\nu=\mu).\end{cases}$$

Therefore, the set of functions

$$f_\nu(z) = \sqrt{\frac{\nu+1}{\pi}}\, z^\nu \qquad (\nu = 0, 1, \cdots)$$

is an orthonormal set in the sense that $(f_\nu, f_\mu) = \delta_{\nu\mu}$.

As an immediate consequence we see that if $f \in H$ and $f = \sum_0^\infty a_\nu z^\nu$, in the sense that $\|f - \sum_{\nu=0}^n a_\nu z^\nu\| \to 0$, then

$$a_\nu = \frac{\nu+1}{\pi}(f(z), z^\nu) = \frac{\nu+1}{\pi}\iint_K f(z)\bar{z}^\nu dxdy.$$

If $f = \sum_0^\infty a_\nu z^\nu$ is regular for $|z| \leqq 1$, this formula reduces to Cauchy's formula. In order to see this we transform the last integral as follows

$$\iint_K f(z)\bar{z}^\nu dxdy = \int_0^1 \int_0^{2\pi} f(z) r^{\nu+1} e^{-i\nu\phi} d\phi dr = \int_0^1 r^{2\nu+1} \int_0^{2\pi} f(z) r^{-\nu} e^{-i\nu\phi} d\phi dr.$$

Here we have

$$\int_0^{2\pi} f(z) r^{-\nu} e^{-i\nu\phi} d\phi = \int_0^{2\pi} f(z) z^{-\nu} d\phi = -i \int f(z) z^{-(\nu+1)} dz,$$

the integral being taken along the circle $|z| = r$. Now, by Cauchy's theorem, this integral is independent of r, $r \leq 1$. Accordingly, the path of integration may be taken to be any closed rectifiable curve C contained in $|z| \leqq 1$. So we get

$$a_\nu = -\frac{i(\nu+1)}{\pi}\int_C f(z) z^{-(\nu+1)} dz \int_0^1 r^{2\nu+1} dr = \frac{1}{2\pi i}\int_C f(z) z^{-(\nu+1)} dz.$$

ANOTHER NOTE ON HERMITE POLYNOMIALS

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In a recent note (this MONTHLY, vol. 62, 1955, 646–647) L. Carlitz gave a new proof of the fact that the Hermite polynomials, defined as

$$H_m(x) = \sum_s (-1)^s (2x)^{m-2s} \frac{m!}{s!(m-2s)!},$$

satisfy the identities

$$H_m(x) H_n(x) = \sum_r 2^r r! \binom{m}{r}\binom{n}{r} H_{m+n-2r}(x) \qquad (m, n = 0, 1, \cdots).$$

We give here a very brief and elementary verification of these identities.

Writing out both sides (for given m, n) as explicit polynomials in x and comparing the coefficients of x^{m+n-2v} on either side (for a given non-negative integer $v \leq \frac{1}{2}(m+n)$), we see at once that the identities are equivalent to

$$\sum_{s+t=v}(-1)^{s+t}2^{m+n-2s-2t}\frac{m!n!}{s!t!(m-2s)!(n-2t)!}$$
$$=\sum_{r+u=v}2^r r!\binom{m}{r}\binom{n}{r}(-1)^u 2^{m+n-2r-2u}\frac{(m+n-2r)!}{u!(m+n-2r-2u)!},$$

i.e.

$$\sum_s \frac{(m+n-2v)!}{s!(v-s)!(m-2s)!(n-2v+2s)!}=\sum_r\frac{(-2)^r(m+n-2r)!}{r!(v-r)!(m-r)!(n-r)!},$$

which we may in turn rewrite in the binomial form

$$\sum_s\binom{v}{s}\binom{m+n-2v}{m-2s}=\sum_r(-2)^r\binom{v}{r}\binom{m+n-2r}{m-r} \tag{1}$$
$$(m,n=0,1,\cdots;v=0,1,\cdots,[(m+n)/2]).$$

But (even for arbitrary complex n, v) this is just what we get on comparing coefficients of x^m in the obvious relation

$$(1+x^2)^v(1+x)^{m+n-2v}=\left\{1-\frac{2x}{(1+x)^2}\right\}^v(1+x)^{m+n}. \tag{2}$$

The special cases of (1), (2) corresponding to the substitution $n=2v-m$ were noted by E. Netto in his *Lehrbuch der Combinatorik* (Leipzig, 1927); in particular, (1) becomes

$$\sum_r(-2)^r\binom{v}{r}\binom{2v-2r}{m-r}=\binom{v}{m/2},$$

where the sum is of course to be taken over all integral values of r for which

$$\binom{v}{r}\binom{2v-2r}{m-r}$$

is non-zero, and where the binomial coefficient on the right is to be interpreted as zero if m is odd. As T. S. Nanjundiah (this MONTHLY, vol. 61, 1954, 700–702) has shown, this in turn includes a formula of E. Grosswald (this MONTHLY, vol. 60, 1953, 179–181).

Finally, it is perhaps worth noting also the rather similar relation

$$(1+x^3)^v(1+x)^{m+n-3v}=\left\{1-\frac{3x}{(1+x)^2}\right\}^v(1+x)^{m+n},$$

whence

$$\sum_s \binom{v}{s}\binom{m+n-3v}{m-3s} = \sum_r (-3)^r \binom{v}{r}\binom{m+n-2r}{m-r},$$

and in particular

$$\sum_r (-3)^r \binom{v}{r}\binom{3v-2r}{m-r} = \binom{v}{m/3}.$$

ON A GENERALIZATION OF THE GEOMETRIC SERIES

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In a previous paper, (this MONTHLY, vol. 56, p. 325), R. Stalley sums a generalized geometric series obtaining the following results:

$$(1)\quad K_n(x) = \sum_{s=1}^{\infty} s^n x^s = (1-x)^{-n-1}\sum_{r=1}^{n}\left[\sum_{m=1}^{r}(-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n\right]x^r$$

where $|x|<1$. In this note $K_n(x)$ will be obtained in a different form by an immediate application of a theorem of Montmort [1]. Also, it will be shown how to express the sum in terms of the generalized Bernoulli numbers or Stirling numbers of the first kind.

Montmort's theorem states that

$$(2)\qquad \sum_{r=1}^{\infty} a_r x^r = \left(\frac{x}{1-x}\right)a_1 + \left(\frac{x}{1-x}\right)^2 \Delta a_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 a_1 + \cdots,$$

where it is assumed that the series $\sum_{r=1}^{\infty} a_r x^r$ converges for $|x|<1$, and that also $|x|<|1-x|$. If $a_r = r^n$, then $\Delta^m a_1 = 0$ for $m \geqq n+1$. For this case, $\sum_{r=1}^{\infty} a_r x^r$ will be given in closed form, and there will be no need for the restriction $|x|<|1-x|$. Consequently,

$$(3)\qquad K_n(x) = \sum_{s=1}^{\infty} s^n x^s = (1-x)^{-n-1}\sum_{r=0}^{n} x^{r+1}(1-x)^{n-r}\Delta^r 1^n.$$

Another way of obtaining (3) would be to express s^n in the following form [2]:

$$(4)\qquad s^n = 1 + \binom{s-1}{1}\Delta 1^n + \binom{s-1}{2}\Delta^2 1^n + \cdots + \Delta^{s-1}1^n.$$

Comparing (1) and (3), one obtains the identity

$$(5)\quad \sum_{s=0}^{n} x^{s+1}(1-x)^{n-s}\Delta^s 1^n = \sum_{r=1}^{n}\sum_{m=1}^{r}(-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n x^r.$$

A direct proof of (5) is rather long but will lead to interesting results. It would

be quicker to show that each side of the equation satisfies the recurrence relation

$$K_{n+1}(x) = xK_n'(x), \tag{6}$$

which was the basis for obtaining (1). This is easily done using the properties of the difference operator Δ [4].

To prove (5) directly we use the relation

$$\Delta^s 1^n = \sum_{m=0}^{s} (-1)^m \binom{s}{m} (s - m + 1)^n, \tag{7}$$

expand $(1-x)^{n-s}$ by the binomial theorem, and then after interchanging the order of summation and equating coefficients of like powers of x, we obtain the identity

$$\begin{aligned}(-1)^t \sum_{s=0}^{t-1} \sum_{m=0}^{s} (-1)^{s+m+1} \binom{n-s}{t-s-1} \binom{s}{m} (s - m + 1)^n \\ = \sum_{m=1}^{t} (-1)^{m+1} \binom{n+1}{m-1} (t - m + 1)^n.\end{aligned} \tag{8}$$

By letting $s-m+1=a$, and $t-m+1=b$, and interchanging order of summations, (8) is transformed into

$$\sum_{b=1}^{t} (-1)^b \binom{n+1}{t-b} b^n = \sum_{a=1}^{t} (-1)^a a^n \sum_{s=a}^{t} \binom{n-s+1}{t-s} \binom{s-1}{a-1}. \tag{9}$$

Consequently, it follows that

$$\begin{aligned}\binom{n+1}{t-b} &= \sum_{s=b}^{t} \binom{n-s+1}{t-s} \binom{s-1}{b-1} \\ &= \sum_{p=0}^{t-b} \binom{n-p-b+1}{t-p-b} \binom{p+b-1}{p}.\end{aligned} \tag{10}$$

Equation (10) is a known binomial identity [3]. We have thus shown that (10) follows from (1) and (3), and that also (3) and (10) imply (1).

To express $K_n(x)$ in terms of generalized Bernoulli numbers or Stirling numbers of the first kind we use the following relations [4], [5]:

$$\begin{aligned}\Delta^r 0^n + \Delta^{r+1} 0^n &= \Delta^r 1^n, \\ \Delta^r 0^n &= \frac{n!}{(n-r)!} B_{n-r}^{(-r)}, \\ S_n^m &= \binom{n-1}{m-1} B_{n-m}^{(n)}.\end{aligned} \tag{11}$$

References

1. G. Chrystal, Textbook of Algebra, Chelsea, 1952, vol. 2, p. 406.
2. H. S. Hall and S. R. Knight, Higher Algebra, Macmillan, 1932, p. 324.
3. W. Feller, An Introduction to Probability Theory and Applications, John Wiley, 1950, p. 48.
4. L. M. Milne-Thompson, The Calculus of Finite Differences, Macmillan, 1951, pp. 52, 134–135.
5. C. Jordan, Calculus of Finite Differences, Chelsea, 1947, p. 146.

ON THE SKEW QUADRANGLE*

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1. LEMMA. *Given two vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$, and two points M and M' which divide $\overrightarrow{AB}$ and $\overrightarrow{B'A'}$ in the same ratio, let π' and π be planes through M and M' perpendicular to $A'B'$ and AB. If O' is any point in π', then a necessary and sufficient condition that a point O lies in π is that*

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'} = \overrightarrow{O'B}\cdot\overrightarrow{OB'}.$$

Proof. If we set $MA/MB = M'B'/M'A' = k$, $(k\neq 1)$, the point M being the barycenter of points A and B with coefficients -1 and k, we obtain

$$\overrightarrow{O'M}\cdot(k-1) = \overrightarrow{O'B}\cdot k - \overrightarrow{O'A},$$

whence

$$(1)\qquad \begin{aligned}\overrightarrow{O'M}\cdot\overrightarrow{A'B'}(k-1) &= (\overrightarrow{O'B}k - \overrightarrow{O'A})\cdot(\overrightarrow{OB'} - \overrightarrow{OA'})\\ &= \overrightarrow{O'B}\cdot\overrightarrow{OB'}k - \overrightarrow{O'A}\cdot\overrightarrow{OB'} - \overrightarrow{O'B}\cdot\overrightarrow{OA'}k + \overrightarrow{O'A}\cdot\overrightarrow{OA'}.\end{aligned}$$

Symmetrically,

$$(2)\qquad \overrightarrow{OM'}\cdot\overrightarrow{AB}(k-1) = \overrightarrow{OA'}\cdot\overrightarrow{O'B}k - \overrightarrow{OB'}\cdot\overrightarrow{O'B} - \overrightarrow{OA'}\cdot\overrightarrow{O'A}k + \overrightarrow{OB'}\cdot\overrightarrow{O'A}$$

On adding the relations (1), (2) term by term, one obtains, after division by $k-1$,

$$(3)\qquad \overrightarrow{O'M}\cdot\overrightarrow{A'B'} + \overrightarrow{OM'}\cdot\overrightarrow{AB} = \overrightarrow{O'B}\cdot\overrightarrow{OB'} - \overrightarrow{O'A}\cdot\overrightarrow{OA'}$$

We recall that the scalar product of two vectors is zero if, and only if, either at least one of the vectors is null, or, the two vectors are perpendicular. If the point O' of π' is at M, the vector $\overrightarrow{O'M}$ is null; if it is distinct from M, the vectors $\overrightarrow{O'M}$ and $\overrightarrow{A'B'}$ are perpendicular, and $\overrightarrow{O'M}\cdot\overrightarrow{A'B'}=0$. In order that the point O

* Translated by E. A. Nordhaus.

lie in the plane π, it is necessary and sufficient that $\overrightarrow{OM'}\cdot\overrightarrow{AB}=0$, so that by (3)

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'} = \overrightarrow{O'B}\cdot\overrightarrow{OB'}.$$

COROLLARY. *Given two skew quadrangles $ABCD$, $A'B'C'D'$ with the points $M, N, P, Q, M', N', P', Q'$ taken on $AB, BC, CD, DA, A'B', B'C', C'D', D'A'$ so that*

$$\frac{MA}{MB} = \frac{M'B'}{M'A'} = k_1, \quad \frac{NB}{NC} = \frac{N'C'}{N'B'} = k_2, \quad \frac{PC}{PA} = \frac{P'D'}{P'D'} = k_3, \quad \frac{QA}{QD} = \frac{Q'D'}{Q'A'} = k_4,$$

if the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ are concurrent, then the planes through M', N', P', Q' perpendicular to AB, BC, CD, DA are also concurrent.

Proof. Let O' be the intersection of the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ and O the intersection of the planes through M', N', P' perpendicular to AB, BC, CD. From preceding remarks, we obtain

$$\overrightarrow{O'A}\cdot\overrightarrow{OA'} = \overrightarrow{O'B}\cdot\overrightarrow{OB'}, \qquad \overrightarrow{O'B}\cdot\overrightarrow{OB'} = \overrightarrow{O'C}\cdot\overrightarrow{OC'}, \qquad \overrightarrow{O'C}\cdot\overrightarrow{OC'} = \overrightarrow{O'D}\cdot\overrightarrow{OD'},$$

and, since $\overrightarrow{O'D}\cdot\overrightarrow{OD'}=\overrightarrow{O'A}\cdot\overrightarrow{OA'}$, the point O belongs to the plane through Q' perpendicular to DA.

2. A Particular Case (a) $k_1=k_2=k_3=k_4=k$.

LEMMA. *In order that the planes through the points M, N, P, Q (which divide the sides AB, BC, CD, DA in the ratio k) perpendicular to the corresponding sides of another skew quadrangle $A'B'C'D'$ shall be concurrent, it is necessary and sufficient that*

$$\begin{aligned} &(\overline{A'A}^2 + \overline{B'B}^2 + \overline{C'C}^2 + \overline{D'D}^2)(1 + k) \\ (4) \qquad &= \overline{A'B}^2k + \overline{B'A}^2 + \overline{B'C}^2k + \overline{C'B}^2 + \overline{C'D}^2k + \overline{D'C}^2 + \overline{D'A}^2k + \overline{A'D}^2. \end{aligned}$$

Proof. Supposing that the planes through M, N, P, Q perpendicular to $A'B', B'C', C'D', D'A'$ are concurrent at point O', one obtains

$$\overline{MA'}^2 - \overline{MB'}^2 = \overline{O'A'}^2 - \overline{O'B'}^2, \qquad \overline{NB'}^2 - \overline{NC'}^2 = \overline{O'B'}^2 - \overline{O'C'}^2$$
$$\overline{PC'}^2 - \overline{PD'}^2 = \overline{O'C'}^2 - \overline{O'D'}^2, \qquad \overline{QD'}^2 - \overline{QA'}^2 = \overline{O'D'}^2 - \overline{O'A'}^2,$$

which yields, on adding these relations term by term,

$$(5) \qquad \overline{MA'}^2 - \overline{MB'}^2 + \overline{NB'}^2 - \overline{NC'}^2 + \overline{PC'}^2 - \overline{PD'}^2 + \overline{QD'}^2 - \overline{QA'}^2 = 0.$$

Since $MA/MB=k$, M is the barycenter of A and B with coefficients 1 and

$-k$, and consequently

$$\overline{A'A}^2 - \overline{A'B}^2k = \overline{A'M}^2(1-k) + \overline{MA}^2 - \overline{MB}^2k,$$
$$\overline{B'A}^2 - \overline{B'B}^2k = \overline{B'M}^2(1-k) + \overline{MA}^2 - \overline{MB}^2k,$$

whence

$$\overline{MA'}^2 - \overline{MB'}^2 = \frac{1}{1-k}(\overline{A'A}^2 - \overline{A'B}^2k - \overline{B'A}^2 + \overline{B'B}^2k), \tag{6}$$

with analogous relations holding for $\overline{NB'}^2-\overline{NC'}^2$, $\overline{PC'}^2-\overline{PD'}^2$, $\overline{QD'}^2-\overline{QA'}^2$, which, when added to (6) give, using (5), the relation (4), and conversely.

COROLLARY. *If the planes through points M, N, P, Q (which divide the edges AB, BC, CD, DA of a skew quadrangle $ABCD$ in the ratio k) perpendicular to the corresponding edges of another quadrangle $A'B'C'D'$ are concurrent, then the planes through the points M', N', P', Q' (which divide the edges $B'A'$, $C'B'$, $D'C'$, $A'D'$ in the same ratio k) perpendicular to the corresponding sides of the quadrangle $ABCD$ are also concurrent* [1].

Proof. By the preceding lemma, in order that the planes through M', N', P', Q' and perpendicular to AB, BC, CD, DA be concurrent, it is necessary and sufficient that

$$(\overline{AA'}^2 + \overline{BB'}^2 + \overline{CC'}^2 + \overline{DD'}^2)(k+1) = \overline{AB'}^2 + \overline{BA'}^2k + \overline{BC'}^2 + \overline{CB'}^2k + \overline{CD'}^2 + \overline{DC'}^2\cdot k + \overline{DA'}^2 + \overline{AD'}^2\cdot k.$$

This is the relation (4).

(b) Provided $k=-1$, the points M, N, P, Q, M', N', P', Q' are the midpoints of the corresponding edges of the quadrangles $ABCD$, $A'B'C'D'$, which are called *orthologic quadrangles relative to the midpoints of corresponding edges* [2], we may define them in the following manner:

In order that two skew quadrangles $ABCD$ and $A'B'C'D'$ shall be orthologic relative to the midpoints of corresponding edges, it is necessary and sufficient that

$$\overline{AB'}^2 - \overline{BA'}^2 + \overline{BC'}^2 - \overline{CB'}^2 + \overline{CD'}^2 - \overline{DC'}^2 + \overline{DA'}^2 - \overline{AD'}^2 = 0.$$

COROLLARY. *In order that two skew quadrangles $ABCD$ and $A'B'C'D'$ shall be orthologic relative to the midpoints of corresponding edges, it is necessary and sufficient that*

$$\overline{B'D'}\cdot\overline{A_1C_1} = \overline{A'C'}\cdot\overline{B_1D_1} \quad \text{or} \quad \overline{BD}\cdot\overline{A_1'C_1'} = \overline{AC}\cdot\overline{B_1'D_1'},$$

where A_1 and C_1, B_1 and D_1, A_1' and C_1', B_1' and D_1' designate the orthogonal projections of A and C, B and D, A' and C', B' and D' on $B'D'$, $A'C'$, BD, AC.

Proof. If m' denotes the midpoint of $B'D'$, we obtain first

$$\overline{AB'}^2 - \overline{AD'}^2 = 2\overline{B'D'}\cdot\overline{m'A_1}, \qquad \overline{CD'}^2 - \overline{CB'}^2 = 2\overline{D'B'}\cdot\overline{m'C_1},$$

and then, by analogy,

$$\overline{AB'}^2 - \overline{AD'}^2 + \overline{CD'}^2 - \overline{CB'}^2 = 2\overline{B'D'}\cdot\overline{C_1A_1},$$
$$\overline{BC'}^2 - \overline{BA'}^2 + \overline{DA'}^2 + \overline{DC'}^2 = 2\overline{A'C'}\cdot\overline{B_1D_1},$$

so that

$$\overline{B'D'}\cdot\overline{A_1C_1} = \overline{A'C'}\cdot\overline{B_1D_1}.$$

3. In the discussion which has preceded, the corresponding edges of the quadrangles $A'B'C'D'$ and $ABCD$ may be replaced by the opposite edges.* In the particular case in which $k_1=k_2=k_3=k_4=k=-1$, the skew quadrangles $ABCD$ and $A'B'C'D'$ are, at the same time, orthologic relative to the midpoints of corresponding edges and relative to the midpoints of opposite edges.

References

1. R. Blanchard, Mathesis, vol. 56, p. 352, question 3355.
2. V. Thébault, Parmi les belles figures de la géométrie dans l'espace, p. 3, Librairie Vuibert, Paris, 1955.

AN IDENTITY FOR THE SUM OF MULTINOMIAL COEFFICIENTS

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1. Summary. Let r, n be positive integers with $r \geqq n$, and let $\{k_i\}$, $i=1, \cdots, n$, be a proper n-fold partition of r. A simple formula (2.6) is given for the sum of multinomial coefficients of degree r over all proper n-fold partitions of r. A second proof is obtained by the explicit construction of two probability models (Theorems 3.1 and 3.2) for the same "collector's problems."†

2. Sum of Multinomial Coefficients. If r and n are positive integers ($r \geqq n$), an *n-fold partition* $\sigma^*_{n,r}$ of r is an ordered set of nonnegative integers k_i ($i=1, \cdots, n$) such that $\sum_{i=1}^n k_i = r$. A *proper n-fold partition* $\sigma_{n,r}$ of r is a $\sigma^*_{n,r}$ with the further restriction: $k_i \geqq 1$ ($i=1, \cdots, n$). For given r and n, let $\mathfrak{S}^*_{n,r}$ be the set of all n-fold partitions of r, and $\mathfrak{S}_{n,r}$ be the set of all proper n-fold partitions of r.

By a *multinomial coefficient of degree r* is meant any coefficient in the expansion of the following power series in the indeterminates $x_1, \cdots, x_n$:

$$(2.1) \qquad (x_1 + \cdots + x_n)^r = \sum_{\sigma^*_{n,r} \in \mathfrak{S}^*_{r,n}} \frac{r!}{\prod_{i=1}^n k_i!} x_1^{k_1} \cdots x_n^{k_n}.$$

* Thus, the edges $C'D'$, $D'A'$, $A'B'$, $B'C'$ of the quadrangle $A'B'C'D'$ are associated with the edges AB, BC, CD, DA and *vice versa.*

† See W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, John Wiley, 1950, pp. 63–64.

The problem under consideration is to find the sum

$$\sum_{\sigma_{n,r}\in\mathfrak{S}_{n,r}} \frac{r!}{\prod_{i=1}^{n} k_i!}. \tag{2.2}$$

LEMMA 2.1. *For* $0 \leqq q \leqq n-1$,

$$S_{n,q} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^q = \begin{cases} (-1)^{n+1} & \text{if } q = 0, \\ 0 & \text{if } 1 \leqq q \leqq n-1. \end{cases} \tag{2.3}$$

*Proof.** From the identity

$$(n-i)^q = n(n-i)^{q-1} - i(n-i)^{q-1}, \tag{2.4}$$

it follows that the quantity $S_{n,q}$ defined by (2.3) satisfies the equation

$$S_{n,q} = nS_{n,q-1} + nS_{n-1,q-1}. \tag{2.5}$$

Since $S_{n,0} = (-1)^{n+1}$ by definition, we have immediately $S_{n,q} = 0$ for $1 \leqq q \leqq n-1$.

THEOREM 2.2. *For* $r \geqq n$,

$$\sum_{\sigma_{n,r}\in\mathfrak{S}_{n,r}} \frac{r!}{\prod_{i=1}^{n} k_i!} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^r. \tag{2.6}$$

Proof. We first proceed by induction on n. For $n=1$ and arbitrary r, (2.6) becomes

$$r!/r! = 1 = 1^r = \sum_{i=0}^{0} (-1)^i \binom{1}{i} (1-i)^r. \tag{2.7}$$

Assume the theorem true for all r', n' such that $0 \leqq r' \leqq r$, $0 \leqq n' \leqq n$ with $r' \geqq n'$. Consider the left side of (2.6) for r and $n+1$, where $r \geqq n+1$. Then, since $\sum_{i=1}^{n} k_i \geqq n$ by hypothesis, it follows that $1 \leqq k_{n+1} \leqq r-n \leqq r - \sum_{i=1}^{n} k_i$ and $r' = r - k_{n+1} \leqq r$. We have

$$\begin{aligned} \sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} &= \sum_{k_{n+1}=1}^{r-n} \sum_{\sigma_{n,r-k_{n+1}}\in\mathfrak{S}_{n,r-k_{n+1}}} \frac{r!}{\prod_{i=1}^{n} k_i!\,k_{n+1}!} \\ &= \sum_{k_{n+1}=1}^{r-n} \binom{r}{k_{n+1}} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^{r-k_{n+1}}. \end{aligned} \tag{2.8}$$

By Lemma 2.1, the inner sum is zero for $k_{n+1} = r-n+1, \cdots, r-1$ so that we may extend the upper limit of summation of k_{n+1} to $r-1$ without changing the

* We are indebted to the referee for this simpler proof. Our original proof was longer and based on differentiation of power series. Also, *ibid.*, p. 77.

value of the entire sum. Adding and subtracting terms corresponding to $k_{n+1} = 0, r$ and interchanging the order of summation, we get for (2.8)

$$
\begin{aligned}
&\sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} \\
(2.9)\qquad &= \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\sum_{k_{n+1}=0}^{r}\binom{r}{k_{n+1}}(n-i)^{r-k_{n+1}} \\
&\quad - \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^r + (-1)^n \\
&= \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n+1-i)^r - \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^r + (-1)^n.
\end{aligned}
$$

Changing $i = j-1$ in the second term and using the identity

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i},$$

we get

$$\sum_{\sigma_{n+1,r}\in\mathfrak{S}_{n+1,r}} \frac{r!}{\prod_{i=1}^{n+1} k_i!} = \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}(n+1-i)^r. \tag{2.10}$$

For induction on r, we have to show that (2.6) holds for $r=1$; and if it holds for all r', n' such that $0 \leqq r' \leqq r$, $0 \leqq n' \leqq n$ with $r' \geqq n'$, then it also holds for $r+1$ and n. The case $r=1$ is trivial. For the rest of the proof, we proceed thus: for $n=1$, the theorem holds for *arbitrary* r as the computation (2.7) shows. For this fixed n, we replace r by $r+1$ and then repeat the above induction on n; so the theorem is proved.

3. Probability Models. We now offer a second proof for Theorem 2.2 by constructing two different models for the following probability problem: Random sampling with replacement is made from a population of n objects, we seek the distribution of the random variable X = the number of drawings it takes to get objects of all kinds. Let $p_r = \Pr\{X=r\}$. Clearly, $p_r \equiv 0$ for $r<n$.

THEOREM 3.1.

$$p_r = \frac{1}{n^r}\cdot\sum_{i=1}^{n}(-1)^{i-1}i\binom{n}{i}(n-i)^{r-1}. \tag{3.1}$$

Proof. Define the sample space $\Omega_{n,r}$ to be the set of all (ordered) r-tuples, each coordinate of which may take on the values $1, \cdots, n$ (a set of labels for the n objects). $\Omega_{n,r}$ has therefore n^r points. Define a sequence of events A_i

($i=1, \cdots, n$) with A_i = the event that object i is not drawn in $r-1$ trials. Then,

$$p_r = \frac{1}{n} \cdot \Pr\{\text{exactly one of the } A_i\text{'s occurs}\} = \frac{1}{n} \cdot P_{[1]}. \tag{3.2}$$

By a well-known formula,*

$$P_{[1]} = S_1 - 2S_2 + 3S_3 - \cdots \pm nS_n \tag{3.3}$$

where

$$S_1 = \sum_i p_i = \sum_i \Pr\{A_i\}, \qquad S_2 = \sum_{i<j} p_{i,j} = \sum_{i<j} \Pr\{A_iA_j\}, \cdots. \tag{3.4}$$

In our problem,

$$S_i = \binom{n}{i}\left(\frac{n-i}{n}\right)^{r-1} \qquad (i = 1, \cdots, n),$$

so that (3.1) immediately follows from (3.2) after computing $P_{[1]}$.

THEOREM 3.2.

$$p_r = \frac{1}{n^{r-1}} \cdot \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.5}$$

Proof. Let $\Omega_{n,r}$ be defined as in Theorem 3.1, each point of which has probability $1/n^r$ by hypothesis. Consider the set of all points or sequences in $\Omega_{n,r}$ with one of the labels $1, \cdots, n$ appearing for the first time in the sequence at the last position. For this *fixed* label, calculate the number of sequences with $n-1$ remaining labels in $r-1$ positions. This number is exactly

$$A = \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.6}$$

Since any one of the labels can be placed at the last positions, we find

$$p_r = \frac{A}{n^r} \cdot n = \frac{1}{n^{r-1}} \sum_{\sigma_{n-1,r-1} \in \mathfrak{S}_{n-1,r-1}} \frac{(r-1)!}{\prod_{i=1}^{n-1} k_i!}. \tag{3.7}$$

Equating (3.1) and (3.5), we see that Theorem 2.2 follows after simple substitutions.

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* Cf. *ibid.*, p. 64.

4. Further Extensions. The boundary condition: $k_i \geqq 1 (i=1, \cdots, n)$ on $\mathfrak{S}_{n,r}$ may be generalized to other cases:

(a) $k_i \geqq a$ for $1 \leqq a \leqq r$; (b) $k_i \geqq a_i$ for $1 \leqq a_i \leqq r$;

(c) $b_i \geqq k_i \geqq a_i$ for $1 \leqq a_i \leqq b_i \leqq r$; (d) $\sum_{j=1}^{i} k_j \geqq a_i$ for $1 \leqq a_i \leqq r$;

$(i = 1, \cdots, n)$.

The formulas for the sum of multinomial coefficients corresponding to these more general boundary conditions on k_i remain to be worked out. In each case, a meaningful collector's problem can be defined.

ON SYLVESTER'S LAW OF NULLITY

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When one extends Sylvester's Law of Nullity* from square matrices over a field of rectangular ones,† the question arises which are the most general conditions. The answer is as follows.

Sylvester's Law of Nullity. *The nullity of AB is less than or equal to the sum of the nullities of A and B. It is greater than or equal to the nullity of A; it is greater than or equal to that of B provided A does not have more columns than it has rows.*

Proof. Let $R(M)$ and $N(M)$ denote the rank and the nullity of the matrix M, and let A and B be $m \times n$ and $n \times p$ matrices respectively.

(a) Following standard procedure,*,† choose non-singular $m \times m$ and $n \times n$ matrices P and Q such that the $m \times n$ matrix $PAQ = A^*$ is diagonal, having the entries of an identity matrix of rank $r = R(A)$ in the left upper corner and zeros elsewhere. Set $B^* = Q^{-1}B$, then A^*, B^*, $A^*B^* = PAB$ are equivalent to, and therefore have the ranks and nullities of A, B, and AB, respectively. The first r rows of A^*B^* are those of B^*, and the remaining $m-r$ rows consist of zeros; hence A^*B^*, which lacks $n-r$ possibly independent rows of B^*, has a rank of at least $R(B^*) - (n - R(A^*))$. Therefore $N(A^*B^*) = m - R(A^*B^*) \leqq m - R(A^*) + n - R(B^*) = N(A^*) + N(B^*)$.

(b) The null space of A is clearly a subspace of that of AB, hence $N(A) \leqq N(AB)$.

(c) Since $R(AB) \leqq R(B)$, we have $N(B) \leqq m - R(B) \leqq N(AB)$ provided $n \leqq m$. We may have $N(AB) < N(B)$ if $n > m$, as is shown by the example

$$A = [1\ \ 0], \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad AB = A, \qquad N(AB) = 0, \qquad N(B) = 1.$$

* See, for instance, M. Bôcher, Introduction to higher algebra, New York, 1907, pp. 78 and 80.

† See G. Birkhoff and S. Mac Lane, A survey of modern algebra, revised edition, New York, 1953, p. 235.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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ON THE DIOPHANTINE EQUATION $x^2+y^2+z^2+2xyz=1$

A. OPPENHEIM, University of Malaya, Singapore

In his note "A Diophantine equation characterizing the law of cosines" (this MONTHLY, vol. 62, 1955, pp. 251–252) Barnett gives the most general rational solution of the cubic equation

$$x^2 + y^2 + z^2 + 2xyz = 1, \tag{1}$$

in the form

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad y = \frac{c^2 + a^2 - b^2}{2ca}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}, \tag{2}$$

where a, b, c are any integers.

Now (1) has an infinity of rational integral solutions, *e.g.*

$$x = 16u^5 - 20u^3 + 5u, \quad y = 3u - 4u^3, \quad z = u^2 - 1, \tag{3}$$

where u is any integer. However, it is not obvious what values of a, b, c produce this solution. It is of some interest then to give the general solution of (1) in rational integers.

THEOREM. *Apart from the trivial solutions* $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, *all integral solutions of the Diophantine equation*

$$x^2 + y^2 + z^2 + 2xyz = 1,$$

are given by the following rule: let p, q, r be any integers with greatest common divisor unity such that one of them is equal to the sum of the other two; let $u \geqq 1$ be any integer: then, writing ch for cosh,

$$x = \pm ch(p\theta), \quad y = \pm ch(q\theta), \quad z = \pm ch(r\theta), \tag{4}$$

where $\theta = \log(u + (u^2 - 1)^{1/2})$, are integers which satisfy the Diophantine equation provided the ambiguous signs are chosen to have product -1.

That x, y, z given by (4) do satisfy (1) is clear since

$$ch^2\,\alpha + ch^2\,\beta + ch^2\,\gamma - 2\,ch\,\alpha\,ch\,\beta\,ch\,\gamma = 1,$$

for any numbers α, β, γ such that $\alpha+\beta+\gamma=0$.

Again, for any integer $n \geqq 1$, $ch(n\theta)$ is an integral polynomial of degree n in $ch\,\theta$. In the present case $ch\,\theta = u$, an integer, so that x, y, z given by (4) are rational integers which satisfy (1). (It may be noted that the trivial solutions are also included in these polynomials for the value $u=0$.)

It remains to show that every non-trivial solution in integers of (1) can be put into the form (4). For this purpose note that (1) is an integral quadratic in each of x, y, z with leading coefficients which divide the coefficients of xyz. Hence, from any integral triad (x, y, z) satisfying (1), can be derived associated integral solutions (x', y, z), (x, y', z), (x, y, z') by the rule

$$x + x' = -2yz, \qquad (xx' = y^2 + z^2 - 1) \textit{ etc.}$$

Thus any admissible triad gives rise to a class or chain (finite or infinite) of admissible triads: any two members of the chain are derivable one from the other by repeated use of association.

Now if $x = \pm ch\ \theta$, $y = \pm ch\ \phi$, $z = \pm ch\ \psi$ where $\theta + \phi + \psi = 0$, then clearly

$$x' = \pm 2ch\,\phi\, ch\,\psi \mp ch\,\theta = \pm ch(\psi - \phi)$$

(the product of the ambiguous signs being -1, it follows that $xx' > 0$) so that the associate solution (x', y, z) has the same form as (x, y, z). Thus if one member of a chain has the form (4), every member has this form.

Next suppose that there is an integral solution (x, y, z) such that (without loss of generality) $|x| \geqq |y| \geqq |z| \geqq 1$.

If $|z| > 1$, *then the associate solution* (x', y, z) *is such that*

$$|x| > |y| > |x'|.$$

For consider the quadratic in t, $f(t) \equiv t^2 + y^2 + z^2 + 2tyz - 1 = 0$ for which $f(x) = 0$, $f(x') = 0$, $f(0) = y^2 + z^2 - 1 > 0$. Plainly

$$\begin{aligned} f(-y \operatorname{sgn} z) &= 2y^2 + |z|^2 - 2y^2|z| - 1 \\ &= -(|z| - 1)(2y^2 - 1 - |z|) < 0, \end{aligned}$$

since $|y| \geqq |z| > 1$. Hence $|x| > |y| > |x'|$.

This descent can be carried on so long as $\min(|x|, |y|, |z|) > 1$, which can hold for only a finite number of steps. Thus a solution must be reached in the chain for which $|z| = 1$ (or $|x| = 1$ or $|y| = 1$). If $z = 1$, then $x = u$, $y = -u$: if $z = -1$, then $x = u$, $y = u$, where u is any integer not zero. But each of these solutions has the form of the theorem with $p = 1$, $q = -1$, $r = 0$.

It follows that all integral solutions of (1) apart from the trivial ones are given by (4).

It may be noted that $(\pm 1, 0, 0)$ (and its permutations), $(-1, -1, -1)$, $(-1, 1, 1)$, *etc.* give rise only to finite chains. Any integer u such that $|u| \geqq 2$ gives rise to the basic solutions $(u, -u, 1)$, $(u, u, -1)$ (and permutations), each of which generates an infinite chain.

The method of descent used here was first employed by Markoff [**3**] for the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$ for which the only basic solutions are $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$. Hurwitz [**2**] later applied it to the case of n variables,

$$x_1^2 + x_2^2 + \cdots + x_n^2 = tx_1x_2 \cdots x_n,$$

the main result being roughly that if $|t|$ is too large any basic solution must have one of $x_1, \cdots, x_n$ zero.

Reference may also be made to Mills [4] and to Barnes [1] who considered respectively the equations

$$x^2 + y^2 + ax + ay + 1 = txy,$$
$$x^2 + y^2 + c = txy.$$

It is plain that the method applies equally to an equation such as

$$2x_1^2 - 3x_2^2 + 4x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_1 + 3x_2 + 5$$
$$= 60ax_2x_3x_4 + 12bx_1x_2x_3 + 30cx_1x_2x_4 + 20dx_1x_3x_4 + 60tx_1x_2x_3x_4$$

which has the property that it is a quadratic with integral coefficients in each of the variables such that if one of the roots in x_1, say, is integral, so also is the other. It can be shown that if $|t|$ is too large then any basic solution must have at least one of $x_1, \cdots, x_4$ zero.

Added in proof: I regret that I have overlooked the valuable paper of L. J. Mordell, *J. London Math. Soc.*, vol. 28, 1953, pp. 500–510.

References

1. E. S. Barnes, J. London Math. Soc., vol. 28, 1953, pp. 242–244.
2. A. Hurwitz, Mathematische Werke, vol. II, pp. 412–421.
3. A. Markoff, Math. Annalen, vol. 17, 1880, pp. 379–399.
4. W. H. Mills, Pacific J. Math., vol. 3, 1953, pp. 209–220.

A LOW ENERGY PROOF OF THE RECIPROCITY LAW

D. H. LEHMER, University of California

Most textbooks in the theory of numbers give a proof of the Law of Quadratic Reciprocity based on the celebrated Lemma of Gauss which he used in the so-called third proof of the theorem. An examination of these proofs shows at least in most cases, that in the deduction of the theorem from the lemma, certain by-products are obtained that are not really needed for the conclusion. These extra results, which are usually properties of the greatest integer function, are of some interest in themselves and stimulate the reader's interest in related problems. However, one may take the point of view that the reader is interested in reaching the conclusion without having to waste energy in passing over summits from which views may be had of the surrounding terrain. The proof of the Reciprocity Law given below is intended to be one of minimum energy in which the possibly unwilling reader is led along a path of constant gradient. The attempt is made to translate the reciprocal relationship stated in the theorem into the more elementary notions of geometrical symmetry. In

lattice point below the diagonal.

To consider the lattice points above the diagonal we have only to view our rectangle from its side AJ, interchanging axes and p and q, and to repeat the above reasoning. Thus we have proved that there are $\mu_1+\mu_2$ lattice points in H. Statement (4) now follows from (6) and the theorem is proved.

INTERVAL-ADDITIVE PROPOSITIONS

L. R. FORD, Illinois Institute of Technology

The present paper results from an attempt to unify the proofs of a large number of theorems connected with closed intervals. The basic properties of continuous functions in an interval, for example, or the facts about sets of points, are sometimes treated by appeals to the Heine-Borel theorem, sometimes by repeated divisions into subintervals, or by other devices. The concept to be presently introduced is believed to be generally rather easy to apply and to be comprehensible to a student at a fairly early level of mathematical attainment.

1. Interval-Additive propositions. A statement P concerning intervals will be called *interval-additive* if whenever P is true for each of two overlapping intervals (with common interior points) it is also true for the interval obtained by combining them; that is, their union.

As an example, the proposition that an interval is more than one unit long is interval-additive, whereas the proposition that it is less than one unit long, is not. The proposition that for a given number M a function satisfies $f(x)<M$ is interval-additive, but the proposition that $\int f(x)dx<M$, the integral being taken over the interval, is not. The proposition that a function is integrable, or continuous, or positive or of bounded variation, or strongly monotonic in an interval is interval-additive.

2. Propositions true at a point. We shall be concerned with a closed base interval $a\leqq x\leqq b$, which we shall call I. We shall say that a proposition P is true at a point x_0 of this interval, if it is true in some subinterval I_0 of I enclosing x_0. Here x_0 is an interior point of I_0 in general, but the subinterval may be closed or open. However, if x_0 is an end point a or b, which cannot be interior to a subinterval, we say that P is true there, if it is true in a subinterval terminating at, and including the end point.

We wish to proceed from truth at each point of an interval to truth in the whole interval. Our fundamental theorem is the following.

THEOREM. *If P is an interval-additive proposition which is true at each point of a closed interval $a\leqq x\leqq b$, then P is true in the whole interval.*

By hypothesis P is true in a subinterval ac, including the end point a. Consider all such subintervals in which P is true, and let u be the least upper bound of c. It is given that P is true at u.

If $u<b$, an interval about u combines with some ac_0 to produce an interval

ac extending to the right of u in which P is true. This is impossible, since u was defined as the least upper bound of c.

So $u=b$, and P is true in an interval ac which is either the closed interval I, or I with the end point b excluded, or a subinterval with c as near b as we like. In the latter cases a subinterval $a'b$ (including b) combines with ac to establish the truth of P in the whole closed interval.

The usefulness of this theorem arises from the fact that interval-additivity is a rather widespread phenomenon and that it is frequently easy to detect the additive character of a proposition. It is often so nearly obvious that it need merely be mentioned.

3. Functions continuous in a closed interval. Let $f(x)$ be continuous at all points of a closed interval I, $a\leqq x\leqq b$. We shall use the preceding theorem to derive certain of its properties.

About each x_0 and for any given $\epsilon_0>0$ there is a subinterval I_0 within which $|f(x)-f(x_0)|<\epsilon_0$, with the convention previously made for $x_0=a$ and $x_0=b$.

(a) We observe that $f(x)$ is *bounded* in each I_0, and boundedness is obviously an interval-additive property. Hence, $f(x)$ is bounded in I.

Here we have not used the fact that ϵ_0 can be made arbitrarily small. It suffices for boundedness in I that we have boundedness at each point; in other words, that $f(x)$ have a finite oscillation at each x_0.

(b) In I then the values of $f(x)$ have a least upper bound M and a greatest lower bound m. We show that these values are actually taken on by the function.

Suppose, on the contrary, that $f(x)$ is nowhere equal to M in I. Then ϵ_0 can be chosen so that in the resulting I_0 we have $f(x)<M_0<M$ for some M_0; for example, $M_0=\frac{1}{2}[M+f(x_0)]$. Now the statement that $f(x)$ is less than a number less than M in an interval is clearly interval-additive. Hence we have, for some M', $f(x)<M'<M$ in I, and the least upper bound is less than M. This contradiction establishes the result. That $f(x)=m$ somewhere in I is proved similarly.

An alternative proof consists in observing that if the theorem is not true the function $1/[M-f(x)]$ is continuous in I and hence bounded, $1/[M-f(x)]<K$. Then $f(x)<M-1/K$, which is impossible.

(c) The function $f(x)$ takes on in I each value between M and m.

Suppose, on the contrary, that $f(x)$ is nowhere equal to an intermediate value n. Then $f(x)-n$ is nowhere zero. About each x_0 we can construct I_0 throughout which $f(x)-n$ is positive or is negative. Now, to say that a function retains the same sign throughout an interval is an interval-additive proposition. It follows that $f(x)-n$ has the same sign throughout I. But since $f(x)-n$ has the values $M-n$ and $m-n$, which differ in sign, this situation is impossible, and the theorem is true.

(d) A property of the function that is useful in proving the existence of the integral is the following: Given $\epsilon>0$, it is possible to divide I into a finite number of abutting subintervals in each of which the oscillation of $f(x)$ is less than ϵ.

That it is possible so to divide an interval is an interval-additive statement.

For if each of two overlapping intervals can be so divided and if we put in the points of division of both intervals we shall have the combined interval divided into subintervals of the desired kind. Also, the proposition is true at each point, for by taking $\epsilon_0 < \epsilon/2$ the oscillation in I_0 is less than ϵ, and I_0 requires no further division. The proposition then holds for I.

The uniform continuity of the continuous function follows from the preceding result. Take δ to be the length of the shortest of the subintervals into which I is divided. Then for any x_0 in I we have $|f(x) - f(x_0)| < 2\epsilon$ for $|x - x_0| < \delta$.

4. Sets of points. Let S be a set of points lying on a line. To say that an interval on the line contains no point of S is interval-additive, to say that it contains one point of S, or exactly n points $(n > 0)$, is not.

The following corollary of our main theorem is useful here in the statement of results. Its proof is immediate.

THEOREM. *If P is an interval-additive proposition which is not true for a closed interval I, then there is at least one point ξ of I such that P is not true for any subinterval of I which contains ξ.*

The statement that an interval contains a finite number of points of S is interval-additive. Hence, if I contains an infinite number of points of S we have the Bolzano-Weierstrass theorem that I contains at least one *limit point* ξ. In every neighborhood of ξ there are infinitely many points of S.

Also interval-additive, is the statement that an interval contains a finite or denumerably infinite number of points of S. Hence, if there is a non-denumerable set of points of S in I there is a so-called *point of condensation*. In every neighborhood of the point there is a non-denumerable infinity of points of S.

5. Generalizations. Extensions of these ideas to two or more dimensions will occur to the reader at once. A proposition P will be *region-additive* for example, if, whenever P is true for each of two overlapping regions, it is true also for the union of these regions. In a closed bounded base region R we can say that P is *true at a point*, if it is true in some subregion R_0 enclosing the point, where some suitable convention is made to cover the case of points on the boundary.

We can then prove a theorem analogous to our basic theorem and use it to treat a variety of propositions. Properties of functions in two or more variables akin to those discussed in Section 3 and theorems about point sets in R are thus readily established.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1251. *Proposed by J. S. Lew, Princeton University*

"Did your teacher give you that problem?" I asked. "It looks rather tedious."

"No," said Willie, "I made it up. It's a polynomial with my age as a root. That is, x stands for my age last birthday."

"Well, then," I remarked, "It shouldn't be so hard to work out—integer coefficients, integral root. Suppose I try $x=7 \cdots$. No, that gives 77."

"Do I look only seven years old?" demanded Willie.

"Well, let me try a larger integer $\cdots$. No, that gives 85, not zero."

"Oh, stop kidding!" said Willie, looking over my shoulder. "You know I'm older than that!"

How old is Willie?

E 1252. *Proposed by Jerome Rothstein, Signal Corps Engineering Laboratories, Fort Monmouth, N. J.*

Prove that the difference between two positive integral powers of the same integer is exactly divisible by six unless the integer gives the remainder two on division by three and one power is odd while the other is even. Show that in this exceptional situation the sum of the two powers is exactly divisible by six.

E 1253. *Proposed by T. R. Jenkins, University of Idaho, and by Nathaniel Macon and Abraham Spitzbart, Flight Propulsion Laboratory, General Electric Co.*

For n and p positive integers with $n>p$, and x and y arbitrary, show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+ky)^p = 0.$$

Find the value of the sum for $n=p$.

E 1254. *Proposed by Robin Robinson, Dartmouth College*

Prove that if two conics intersect in four distinct points, these points are concyclic if and only if the axes of the two conics are parallel or perpendicular.

E 1255. *Proposed by J. P. Ballantine, University of Washington*

A constant gravitational field operates in the direction of the negative y-axis.

A particle starts from point (a, b) with a given initial velocity, travels along a straight line to the point (x, y), thence without loss of velocity along another straight line to point (c, d). The point (x, y) is so chosen that the total time is a minimum. Prove or disprove that the particle reaches the point (x, y) when the time is precisely half gone.

SOLUTIONS

De Polignac's Theorem

E 1221 [1956, 420]. *Proposed by S. W. Golomb, University of Oslo, Norway*

If n is an integer greater than 1, then 2^n-1 is never a perfect square, cube, or higher power.

Solution by D. C. B. Marsh, Colorado School of Mines. For $n>1$, $2^n-1\equiv 3 \pmod 4$; so, if we were able to write $2^n-1=x^m$, $m>1$, x and m would both have to be odd. But this would imply the impossible relation

$$2^n = x^m + 1 = (x+1)[(x^m+1)/(x+1)],$$

where the final factor is integral, odd, and greater than 1.

Also solved by D. S. Adorno, H. W. Becker, H. F. Bennett, David Bloom, Julian Braun, Leonard Carlitz, P. L. Chessin, M. P. Drazin, L. R. Ford, Jr., George Franzisko, Marshall Freimer and A. L. Tritter (jointly), A. M. Glicksman, I. L. Goldhirsh, R. K. Guy, Virginia S. Hanly, A. R. Hyde, Lawrence Isenecker, Ronald Jacobowitz, J. B. Johnston, M. S. Klamkin and D. J. Newman (jointly), Andrew Kraus, Sidney Kravitz, Joe Lipman, Marshall Luban, E. W. Marchand, Patricia G. Mâssé, J. B. Muskat, E. A. Nordhaus, Margaret Olmsted, Anatol Rapaport, Azriel Rosenfeld, C. M. Sandwick, Sr., J. P. Scholz, J. R. Smart, Mirko Stojaković, D. R. Sudborough, A. V. Sylwester, Lincoln Teng, R. M. Warten, R. E. Wyllys, F. H. Young, and the proposer.

Becker called attention to Dickson's *History of the Theory of Numbers*, vol. II, p. 753, where it is pointed out that in 1887 C. de Polignac proved that $a^n-2^k=\pm 1$ is impossible unless $a=3$, $n=1$ or 2. Chessin remarked that the given problem is covered by the solution to E 444 [1941, 482]. Drazin established the stronger theorem: *the equation $p^n-c^m=1$ has no solution in integers c, m, n, p, with p a positive prime and $m>1$, $n>1$, other than that given by $3^2-2^3=1$.*

The Regular Heptagon

E 1222 [1956, 421]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If we designate by C_1, C_2, C_3 the sides of the regular convex heptagon and of the two regular star heptagons inscribed in a circle of radius R, then

$$C_1^2 + C_2^2 + C_3^2 = 7R^2.$$

I. *Solution by P. L. Chessin, Westinghouse Electric Corporation.* We generalize. Let $A_0, A_1, \cdots, A_{n-1}$ be the consecutive vertices of a regular n-gon inscribed in a circle of radius R and center O, and let $C_k=A_0A_k$, $1\leqq k\leqq n-1$.

THEOREM. $\sum_{k=1}^{n} C_k^2=2R^2[m-\sin m\pi/n \cos (m+1)\pi/n \csc \pi/n]$, $1\leqq m\leqq n-1$.

In triangle OA_0A_k apply the law of cosines to obtain $C_k^2=2R^2(1-\cos 2k\pi/n)$. Then

$$\sum_{k=1}^{m} C_k^2 = 2R^2\left[m - \sum_{k=1}^{m}\cos 2k\pi/n\right].$$

But, by a well known result,

$$\sum_{k=1}^{m} \cos k\phi = \sin m\phi/2 \cos (m+1)\phi/2 \csc \phi/2.$$

COROLLARY 1.

$$\sum_{k=1}^{n-1} C_k^2 = 2nR^2.$$

COROLLARY 2. *If n is even,*

$$\sum_{k=1}^{n/2-1} C_k^2 = (n-2)R^2.$$

COROLLARY 3. *If n is odd,*

$$\sum_{k=1}^{(n-1)/2} C_k^2 = nR^2.$$

The given problem is established by Corollary 3 with $n=7$.

II. *Solution by W. B. Carver, Cornell University.* Let $t=e^{2\pi i/7}$. Then t^s, $s=0, 1, \cdots, 6$, are the seven seventh roots of unity, t^{7-s} is the conjugate of t^s, and

$$1+t+t^2+t^3+t^4+t^5+t^6=0.$$

The vertices of the regular heptagon correspond to the complex numbers Rt^s, and we have

$$C_1^2 = R^2(t-1)(t^6-1), \qquad C_2^2 = R^2(t^2-1)(t^5-1), \qquad C_3^2 = R^2(t^3-1)(t^4-1),$$

and

$$C_1^2+C_2^2+C_3^2 = R^2(3t^7-t-t^2-t^3-t^4-t^5-t^6+3) = (3+1+3)R^2 = 7R^2.$$

III. *Solution by Norman Anning, Alhambra, Calif.* Put unit masses at the vertices of the regular heptagon. Taking moments of inertia with respect to any vertex and applying the familiar parallel-axis theorem, we have

$$2(C_1^2)+2(C_2^2)+2(C_3^2) = 7R^2+7R_2^2.$$

Also solved by Leon Bankoff, A. D. Bradley, Leonard Carlitz, Edward Fleisher, Michael Goldberg, Cornelius Groenewoud, R. K. Guy, A. R. Hyde, V. F. Ivanoff, M. S. Klamkin and D. J. Newman (jointly), J. D. E. Konhauser, Sidney Kravitz, Josef Langr, D. C. B. Marsh, Margaret

Olmsted, D. K. Pease, L. A. Ringenberg, C. M. Sandwick, Sr., B. M. Stewart, Chih-yi Wang, F. H. Young, and the proposer.

Bankoff supplied the following items of interest relative to the figure of the problem. In the triangle ABC whose sides are $a\equiv C_1$, $b\equiv C_2$, $c\equiv C_3$, we have:

(1) $bc=a(b+c)$, $ac=b(c-a)$, $ab=c(b-a)$

(2) $\cos A\cos B\cos C=-1/8$, $\sin A\sin B\sin C=\sqrt{7}/8$

(3) $\sin^2 A=(3a-c)/4a$, $\sin^2 B=(3b-a)/4b$, $\sin^2 C=(3c+b)/4c$

(4) $\cos^2 A+\cos^2 B+\cos^2 C=5/4$, $\sin^2 A+\sin^2 B+\sin^2 C=7/4$

(5) $\cot V=\cot A+\cot B+\cot C=\sqrt{7}$,

where V is the Brocard angle of triangle ABC

(6) $\cos^2 B\cos^2 C+\cos^2 C\cos^2 A+\cos^2 A\cos^2 B=3/8$

(7) $h_a=h_b+h_c$,

where h_i are the altitudes of triangle ABC

(8) $h_a^2+h_b^2+h_c^2=(a^2+b^2+c^2)/2$

(9) $b^2/a^2+c^2/b^2+a^2/c^2=5$

(10) $OH=OI_a=R\sqrt{2}$, $r_a=R/2$, $I_aH=R$

References: *Mathesis*: 1913—204; 1938—169; 1950—344; 1955—329; 1956—106 and 149. *Journal de mathématiques élémentaires*, 69° année, p. 25 (1944).

Editorial Note. Solutions II and III can be extended readily to yield Corollary 3 of Solution I. An easy additional corollary of Solution I is: *In a regular n-gon the sum of the squares of all sides and diagonals is n^2R^2.* An alternative proof of this, using the theory of mean position, may be found in M'Clelland, *A Treatise on the Geometry of the Circle* (1891), p. 103.

Limit of a Curious Sequence

E 1223 [1956, 421]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Starting with two arbitrary non-negative numbers a_1 and a_2, build a sequence a_k, $k=1, 2, 3, \cdots$, by taking alternately the arithmetic and the geometric means of the two preceding numbers, for example,

$$0,\ 1,\ 1/2,\ \sqrt{2}/2,\ (1+\sqrt{2})/4,\ \cdots.$$

Evaluate $\lim a_k$ as $k\to\infty$.

Solution by the proposer. If $a_1=a_2$, then, trivially, $\lim a_k=a_1$. Suppose $a_1<a_2$. We make use of the following elementary facts: (a) the arithmetic mean of $\cot 2\alpha$ and $\csc 2\alpha$ is $(\cot\alpha)/2$, (b) the geometric mean of $\csc 2\alpha$ and $(\cot\alpha)/2$ is $(\csc\alpha)/2$. Consider the sequence

$$c\cot\theta,\quad c\csc\theta,\quad (c/2)\cot(\theta/2),\quad (c/2)\csc(\theta/2),\quad (c/4)\cot(\theta/4),\ \cdots,$$

which already fits the desired pattern of alternating arithmetic and geometric means. This can be identified with the desired sequence by choosing c and θ such that $a_1=c\cot\theta$ and $a_2=c\csc\theta$, whence $\theta=\cos^{-1}(a_1/a_2)$ and $c=(a_2^2-a_1^2)^{1/2}$. Since

$$\lim_{n\to\infty}2^{-n}\cot(\theta/2^n)=\lim_{n\to\infty}2^{-n}\csc(\theta/2^n)=1/\theta,$$

the desired limit is

$$\lim a_k = c/\theta = (a_2^2 - a_1^2)^{1/2}/\cos^{-1}(a_1/a_2).$$

For $a_1 > a_2$ the above argument needs but little alteration. If $a_2 = 0$, then $\lim a_k = 0$. If $a_2 > 0$, we can make use of the facts: (a) the arithmetic mean of coth 2α and csch 2α is $(\coth \alpha)/2$, (b) the geometric mean of csch 2α and $(\coth \alpha)/2$ is $(\operatorname{csch} \alpha)/2$, and set

$$a_1 = c \coth u, \qquad a_2 = c \operatorname{csch} u,$$

so that

$$\lim a_k = (a_1^2 - a_2^2)^{1/2}/\cosh^{-1}(a_1/a_2).$$

Also solved by N. J. Fine, M. S. Klamkin, H. L. Krall, D. C. B. Marsh, K. W. Morris, and Chih-yi Wang.

Morris pointed out that the problem appears, although in somewhat different form, in Knopp, *Theory and Application of Infinite Series*, 1928, Ex. 8, parts (c) and (d), p. 41, and Ex. 91, p. 228. Klamkin credited the result to Borchardt, and called attention to Bromwich, *Introduction to the Theory of Infinite Series*, Problem 8, p. 23, and Gibson, *Advanced Calculus*, Problem 3, p. 50.

A Fixed Point Theorem

E 1224 [1956, 421]. *Proposed by G. K. Wenceslas, Providence, R. I.*

Let G be a finite group of rigid motions (or more generally affine transformations) of a Euclidean space. Then there is a point of the space which is left fixed by all the transformations of G.

Solution by N. J. Fine, University of Pennsylvania. Let the group elements T_i be given by $T_i(x) = A_i x + b_i$ $(i = 1, \cdots, n)$, where the A_i are linear. Let c be an arbitrary vector, $x = (1/n) \sum_{j=1}^{n} T_j(c)$. Then

$$\begin{aligned} T_i(x) &= A_i x + b_i = A_i\left[(1/n) \sum_{j=1}^{n} T_j(c)\right] + b_i \\ &= (1/n) \sum_{j=1}^{n} [A_i T_j(c) + b_i] = (1/n) \sum_{j=1}^{n} T_i[T_j(c)] \\ &= (1/n) \sum_{k=1}^{n} T_k(c) = x. \end{aligned}$$

Hence x is invariant under the entire group.

Also solved by the proposer.

Editorial Note. One cannot help but notice the following property. Let c and d be any two points and let c_i and d_i be the maps of c and d, respectively, under the transformation T_i. Then, if there is only one fixed point of G, the centroids of the two sets of points $c_1, \cdots, c_n$ and $d_1, \cdots, d_n$ coincide. Examples, like the octic group of the symmetries of a square, are abundant.

Laying a Rug

E 1225 [1956, 421]. *Proposed by L. R. Ford, Illinois Institute of Technology*

Find necessary and sufficient conditions that it be possible to lay an $a\times b$ rug on an $A\times B$ floor. (Take $a\geqq b$, $A\geqq B$.)

Solution by W. B. Carver, Cornell University. The conditions $a\geqq b>0$ and $A\geqq B>0$ are imposed by the problem itself and are understood to hold throughout the discussion. There are two cases to be considered: the trivial case when the rug can be placed with its length parallel to the length of the room, and the case when the length of the rug is greater than the length of the floor so that it must be placed diagonally.

In the first case the necessary and sufficient conditions are obviously

$$A \geqq a \text{ and } B \geqq b. \tag{1}$$

In the second case we have $a>A$. In Figure 1 the radius of the circle is

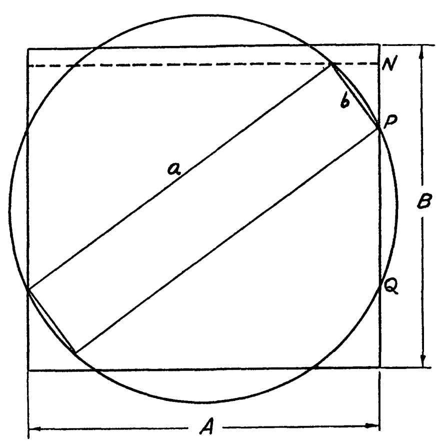

FIG. 1

$(a^2+b^2)^{1/2}/2$, and hence distance $PQ=(a^2+b^2-A^2)^{1/2}$. Letting $PN=h$, we must have

$$B \geqq (a^2+b^2-A^2)^{1/2}+2h. \tag{2}$$

From similar triangles we have

$$h/b = [A-(b^2-h^2)^{1/2}]/a,$$

and this gives

$$h = [abA - b^2(a^2+b^2-A^2)^{1/2}]/(a^2+b^2).$$

(The larger value of h: $h=[abA+b^2(a^2+b^2-A^2)^{1/2}]/(a^2+b^2)$, satisfies the equation $h/b=[A+(b^2-h^2)^{1/2}]/a$, which has no significance in the problem.) Putting the value of h in (2) we have

$$B \geqq [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2). \tag{3}$$

Hence the necessary and sufficient conditions in this case are

$$a > A \text{ and } B \geqq [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2). \tag{4}$$

From conditions (4) certain other necessary (but not sufficient) conditions for this second case may be derived, namely, the rather obvious conditions $A^2+B^2 > a^2+b^2$ and $B>b$, and the not so obvious condition $a+b\leqq A\sqrt{2}$.

The necessary and sufficient conditions required in the problem are *either* (1) *or* (4), in the sense that if the rug can be placed on the floor either (1) or (4) must hold, and if either (1) or (4) are satisfied the rug can be placed on the floor.

The inequality conditions (1) and (4) may be represented graphically by means of a rectangular Cartesian system, (a, b, B), in space (see Figure 2).

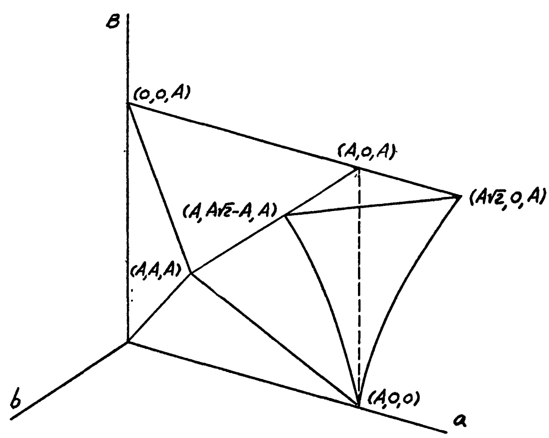

FIG. 2

One plots the points $(0, 0, A)$, $(A, 0, A)$, $(A, 0, 0)$, (A, A, A), $(A\sqrt{2}, 0, A)$, $(A, A\sqrt{2}-A, A)$. Conditions (1) are satisfied by all points in a region which is a quadrangular pyramid with a square base in the plane $b=0$ and vertex at (A, A, A); and conditions (4) by points in the pseudo-tetrahedron bounded by the three planes $a=A$, $B=A$, $b=0$, and a portion of the curved surface

$$B = [2abA + (a^2 - b^2)(a^2 + b^2 - A^2)^{1/2}]/(a^2 + b^2).$$

This triangular piece of curved surface is bounded by the line $a+b=A\sqrt{2}$ in the plane $B=A$, the hyperbola $a^2-B^2=A^2$ in the plane $b=0$, and the curve $(b^2+A^2)B=3A^2b-b^3$ in the plane $a=A$. All points on the boundaries of the regions belong to the regions except points in the planes $a=0$ and $b=0$, and the

points of the plane $a=A$ bounding the second region.

Also solved by H. F. Bennett, D. A. Breault, Michael Goldberg, R. K. Guy, D. C. B. Marsh, C. S. Ogilvy, C. D. Olds, M. J. Pascual, and Azriel Rosenfeld.

A number of the above solutions gave necessary but not sufficient conditions. Guy obtained the necessary and sufficient conditions in the form

$$B \geqq b \text{ and either } A \geqq 0 \text{ or } 3\pi/2 - 2\sin^{-1} a/(a^2+b^2)^{1/2} \leqq \sin^{-1} A/(a^2+b^2)^{1/2} + \sin^{-1} B/(a^2+b^2)^{1/2}.$$

As an example of the second case he gave $a=410$, $b=41$, $A=409$, $B=130$.

Editorial Note. For allied problems see Problem 416 (algebra) [1920, 327], Problem 3036 [1925, 47], and Problem No. 563, *National Mathematics Magazine*, 1945, p. 259.

The necessary condition $a+b \leqq A\sqrt{2}$ is apparent in Figure 2; a purely algebraic derivation of the condition constitutes a nice problem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4723. *Proposed by Joachim Lambek, McGill University*

Given a non-negative integer n and a prime p, obtain an expression for the number of binomal coefficients $\binom{n}{r}$ which are not divisible by p. (This generalizes problem I, 7 of the Putnam Competition Examination for 1956, this MONTHLY, vol. 64, 1957, p. 24.)

4724. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Show that the necessary and sufficient condition that

$$\limsup_{x\to 1} (x^{n_1} - x^{n_2} + x^{n_3} - \cdots) = 1$$

is that to every c and ϵ there should exist j and k, $k \geqq j$, such that $n_{2k+2} > cn_{2j+1}$ and

$$(n_{2j+2} - n_{2j+1}) + \cdots + (n_{2k+2} - n_{2k+1}) > (1-\epsilon)(n_{2k+2} - n_{2j+1}).$$

4725. *Proposed by Olga Taussky, National Bureau of Standards*

A square $n \times n$ matrix $A=(a_{ik})$ with complex numbers as elements is called

normal if $AA^*=A^*A$, where A^* is the transposed and complex conjugate of A. A matrix A is called nilpotent if for some integer $r\geqq 1$ the matrix A^r is the zero matrix 0. Prove (by rational methods only) that a normal and nilpotent matrix is the zero matrix.

4726. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the parallels to the asymptotes of a conic (C), drawn through an arbitrary point P of its plane, intersect (C) in P_1 and P_2, if the perpendiculars to PP_1 and PP_2 at P_1 and P_2 intersect in a point O, and if the polar of P with respect to (C) intersects the conic in M_1' and M_2', then the perpendicular bisector of segment $M_1'M_2'$ passes through O.

4727. *Proposed by G. R. MacLane, the Rice Institute*

Find a function $f(x)$, upper semi-continuous and non-negative on $[0, \infty)$, bounded on each finite interval $(0, T)$, such that $\int_0^\infty f(x)dx=\infty$ and $\sum_{n=1}^\infty f(nh) < \infty$ for every $h>0$. (Cf. problem 4670 [1956, 47, 190]; solution in this number.)

SOLUTIONS

A Sum of Fractional Parts

4667 [1955, 734]. *Proposed by J. L. Ullman, University of Michigan*

Let λ_i be an infinite sequence of positive numbers such that $\sum\lambda_i=1$. Prove that

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}((N\lambda_i)) = 0,$$

where $((x))$ means the fractional part of x.

Solution by D. S. Greenstein, University of Pennsylvania. Since $0\leqq((x))<1$ and $((x))\leqq x$, we have

$$\frac{((N\lambda_i))}{N}<\frac{1}{N}, \qquad 0\leqq\frac{((N\lambda_i))}{N}\leqq\frac{N\lambda_i}{N}=\lambda_i.$$

Given $\epsilon>0$, choose k such that $\sum_{i=k+1}^\infty\lambda_i<\epsilon/2$. Then

$$\left|\frac{1}{N}\sum_{i=1}^{\infty}((N\lambda_i))\right|<\frac{K}{N}+\frac{\epsilon}{2}<\epsilon$$

for sufficiently large N. Therefore the limit is zero as asserted.

Note that no use has been made of the hypothesis $\sum\lambda_i=1$. Hence the conclusion holds if $\sum\lambda_i$ is merely convergent.

Also solved by Leonard Carlitz, A. E. Danese, R. O. Davies, Leopold Flatto, J. B. Kelly, P. G. Kirmser, A. E. Livingston, O. Mourmaki, Chih-yi Wang, J. G. Wendel, and the proposer.

Special Summation

4668 [1956, 46]. *Proposed by D. C. Russell, University of London, England*

Construct the matrix A of a regular method of summation which sums $\sum_{n=0}^{\infty} z^n$ to $(1-z)^{-1}$ at each of the points $z=-1, -2, -3, \cdots$, but which sums the series at no other point in $|z|>1$.

Solution by Albert Wilansky, Lehigh University. Let $v_n^{(m)}$ $(n=0, 1, 2, \cdots, m, \cdots; m=1, 2, 3, \cdots)$ be defined by the equations

$$\sum_{n=0}^{\infty} v_n^{(m)} z^n = \left(\sum_n u_n^{(1)} z^n\right)\left(\sum_n u_n^{(2)} z^n\right)\cdots\left(\sum_{n=0}^{\infty} u_n^{(m)} z^n\right), \tag{1}$$

$$\sum_{n=0}^{\infty} u_n^{(m)} z^n = \frac{m}{m+1} + \frac{z}{m+1}. \tag{2}$$

Let A be the matrix whose mth row is the sequence starting with m zeros and then the sequence $v^{(m)}$, *e.g.*, the second row is

$$(0, 0, v^{(2)}) = (0, 0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}, 0, 0, 0, \cdots).$$

A is regular since all its row-sums are one, as is seen by setting $z=1$ in (1) and (2), its elements are non-negative, and its columns terminate in zeros. We now show that if $|z|>1$, A sums the sequence $\{z^n\}$ if and only if z is a negative integer, and to the sum 0 in each case. Since $\sum_{k=0}^{m} z^k=(1-z^{m+1})/(1-z)$, this yields the result.

Let $\{\tau_n\}$ be the transform of $\sum_{n=0}^{\infty} z^n$ by the matrix A. Then

$$\tau_m = \frac{1}{1-z} - \frac{z}{1-z} t_m,$$

where

$$t_m = \sum_{n=0}^{\infty} v_n^{(m)} z^{n+m} = z^m \prod_{r=1}^{m} \frac{r+z}{r+1},$$

as we saw above. If z is a negative integer, $t_m=0$ for sufficiently large m. It remains only to show that $\{t_m\}$ is divergent if $|z|>1$, z not a negative integer. It will be sufficient to show that it is unbounded. The product term is, in absolute value, not less than

$$C \prod_{r=p+1}^{m+1} \left(1 - \frac{w}{r}\right)$$

where

$$w = |z-1|, \qquad C = \left| \prod_{r=1}^{p} \frac{r+z}{r+1} \right| \neq 0,$$

and $p=[w]+1$. This in turn is not less than

$$C\prod_{r=p+1}^{m+1}\left(1-\frac{p}{r}\right)=C\Big/\binom{m+1}{p}\sim Km^{-p}$$

as $m\to\infty$, $K\neq 0$. Thus $|t_m|$ is asymptotically not less than $K|z|^m m^{-p}\to\infty$.

Also solved by Chih-yi Wang and the proposer.

Rectangular Pentagon

4669 [1956, 47]. *Proposed by Michael Goldberg, Washington, D. C.*

1. What is the relation connecting the lengths of the sides of a rectangular skew (spatial) pentagon?

2. Among all possible rectangular skew pentagons, what are the lengths of the sides of the pentagon in which the ratio of the longest to the shortest side is a minimum?

Solution (*part* 1) *by the proposer.* Let the lengths of the sides be a, b, c, d, e in that order. Take the plane P of c, d as the xy-plane. If the projection of b on P is x and the projection of e on P is y, while the end of b is below P a distance z_1, and the end of e is above P a distance z_2, then the following five equations are clearly true:

$$z_1^2=b^2-x^2,\qquad a^2+e^2=(d-x)^2+z_1^2+c^2,$$
$$z_2^2=e^2-y^2,\qquad a^2+b^2=(c-y)^2+z_2^2+d^2,$$

$$a^2=(d-x)^2+(c-y)^2+(z_1+z_2)^2.$$

Hence a necessary condition is obtained by the elimination of x, y, z_1, z_2, which gives

$$\{(s^2-a^2-e^2)^2-b^2d^2\}\{(s^2-a^2-b^2)^2-c^2e^2\}=c^2d^2(s^2-c^2-d^2)^2$$

in which we have put $2s^2=a^2+b^2+c^2+d^2+e^2$. Note, as a check, that the relation is unaltered by cyclic permutation of the sides.

Editorial Note. The condition is not sufficient because of squaring during the elimination; *e.g.*, there is no pentagon with sides 1, 1, 1, 1, $\sqrt{6}$. No best almost-equilateral pentagon has been suggested, but E. D. Schell notes that if space is four-dimensional an equilateral rectangular pentagon is possible. For example, consider

$$(0,0,0,1),\quad (0,0,0,0),\quad (1,0,0,0),$$
$$(1,\sqrt{3/8},\sqrt{3/8},1/2),\quad (1/2,(2+\sqrt{5})/2\sqrt{6},(2-\sqrt{5})/2\sqrt{6},1).$$

Divergent integral and series

4670 [1956; 47, 190]. *Proposed by K. L. Chung, Syracuse University*

If $f(x)$ is continuous and non-negative in $[0,\infty)$, and $\int_0^\infty f(x)dx=\infty$, then there exists an $h>0$ such that $\sum_{n=1}^\infty f(nh)=\infty$.

I. *Solution by A. R. Hyde, West Hartford, Conn.* Let $k>1$ and

$$M(b)=\frac{1}{k}\int_0^b f(x)dx.$$

$M>0$ for some finite value of b, since $f(x)\geqq 0$ and $M\to\infty$ as $b\to\infty$. Then, if $N\Delta x=b$, we have

$$\lim_{\Delta x\to 0,N\to\infty}\sum_{n=1}^{N} f(n\Delta x)\Delta x = kM,$$

i.e., for some finite positive $\Delta x=h$ and $N=b/h$,

$$\sum_{n=1}^{b/h} f(nh)h > M, \qquad h\lim_{b\to\infty}\sum_{n=1}^{b/h} f(nh) \geqq \lim_{b\to\infty} M,$$

whence the desired result follows.

II. *Solution by N. J. Fine, University of Pennsylvania.* Define the extended real-valued function $G(x)=\sum_{n=1}^{\infty} f(nx)$. It is easy to see that G is lower semi-continuous, *i.e.*,

$$\liminf_{x\to a} G(x) \geqq G(a)$$

for every a. Hence the set $A_N=\{x\mid G(x)\leqq N\}$ is closed.

Now suppose that A_N contains an interval (a, b), $a<b$. Then

$$\int_a^b G(x)dx < \infty.$$

But

$$\int_a^b G(x)dx = \sum_{n=1}^{\infty}\int_a^b f(nx)dx = \sum_{n=1}^{\infty}\frac{1}{n}\int_{na}^{nb} f(x)dx = \int_0^{\infty}\phi(x)f(x)dx,$$

where

$$\phi(x) = \sum_{na<x<nb}\frac{1}{n} = \sum_{(x/b)<n<(x/a)}\frac{1}{n}\cdot$$

As $x\to\infty$, $\phi(x)\to\log(b/a)$, whence $\int_a^b G(x)dx=\infty$. This contradiction shows that A_N contains no interval and, being closed, is nowhere dense. Therefore $A=\{x\mid G(x)<\infty\}=\bigcup_{N=1}^{\infty} A_N$ is of first category, and its complement in $(0, \infty)$ is of second category, hence everywhere dense in $(0, \infty)$.

This has as a corollary Problem 4605 [1955, 738]. For, if S is any unbounded open set, we can construct a non-negative continuous function f such that $\int_0^\infty f(x)dx=\infty$ and $f(x)=0$ for $x\notin S$. For this function, $G(x)=\infty$ implies that $nx\notin S$ for infinitely many n.

Also solved by A. E. Danese, A. S. Davis, G. Lorentz, G. R. MacLane, O. E. Stanaitis, and G. N. Wollan.

A Generalization of Taylor's Expansion

4671 [1956; 47, 191]. *Proposed by J.P. Ballantine, University of Washington, Seattle*

Show that, if $f(x)$ is any function possessing $2n+1$ continuous derivatives,

$$\sum_{r=0}^{n} \binom{2n-r}{n} \frac{a^r}{r!} [(-D)^r f(a) - D^r f(0)] = (-1)^n \frac{D^{2n+1}f(\xi)}{(2n+1)!} a^{2n+1},$$

where the first parenthesis under the summation is a binomial coefficient, and ξ lies between 0 and a.

Editorial Note. C. A. Dyche points out that the proposed formula is a special case ($m=n$, $x=0$) of A Generalization of Taylor's Expansion, by Hummel and Seebeck (this MONTHLY, vol. 56, 1949, pp. 243–247). Andress extended the theory to more than two points by means of contour integration (this MONTHLY, vol. 60, 1953, pp. 394–396).

F. B. Hildebrand refers to his treatise, *Introduction to Numerical Analysis*, equations (6.14.6) and (6.14.7). The present result is equivalent to Obrechkoff's quadrature formula with an error term.

Independent solutions were submitted by A. E. Danese, C. A. Dyche, Berthold Schweizer, and Chih-yi Wang.

Integers Uniquely Expressible in Terms of a Sequence

4672 [1956, 47]. *Proposed by D. J. Newman, A VCO Research Division, Lawrence, Mass.*

Find a sequence of positive integers such that almost all (in the usual sense of limit density) positive integers are uniquely expressible as a sum of two of them.

Solution by Paul Erdös. Technion Mathematics Department, Haifa, Israel. Let S_1 be the set of integers $\sum_k a_k 10^{2k}$, $0 \leqq a_k \leqq 9$, and S_2 be the set $\sum_k b_k 10^{2k+1}$, $0 \leqq b_k \leqq 9$. Clearly every integer can be written uniquely in the form $x+y$ where $x \in S_1$ and $y \in S_2$. The numbers x_1+x_2, x_1 and x_2 in S_1, are of the form $\sum a_k\ 10^{2k} + \sum b_k\ 10^{2k+1}$ where $0 \leqq a_k \leqq 9$, $0 \leqq b_k \leqq 1$, whence their number up to x is $o(x)$. The same holds for the numbers of the form y_1+y_2, with y_1, $y_2 \in S_2$. Thus the numbers $S_1 \cup S_2$ satisfy the requirements of the problem.

Also solved by the proposer.

A Diophantine Equation

4674 [1956, 126]. *Proposed by R. Venkatachalam Iyer, Karamana, Trivandrum, India*

The general solution in rational numbers of the equation

$$x^2 + y^2 + z^2 + 2xyz = 1, \tag{1}$$

is given by I. A. Barnett as

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad y = \frac{c^2 + a^2 - b^2}{2ca}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

(See: *A Diophantine Equation Characterizing the Law of Cosines,* this MONTHLY, vol. 62, 1955, p. 251.) Find the general solution of (1) in integers.

Solution by D. C. B. Marsh, Colorado School of Mines. (1) may be written in the form

$$(x + yz)^2 = (y^2 - 1)(z^2 - 1),$$

which yields solutions if and only if

$$y^2 - 1 = pq^2 \quad \text{and} \quad z^2 - 1 = pr^2,$$

for p, q, r positive integers and p square-free. For p fixed and >1, the Pell equation $s^2 - pt^2 = 1$ has infinitely many solutions. If the least positive integral solution is (s_1, t_1) then the general integral solution is (s_v, t_v) where

$$s_v + t_v\sqrt{p} = \pm (s_1 + t_1\sqrt{p})^v.$$

Hence y and z may be taken as s_m and s_n (both positive or both negative) for any integral m and n, whereupon $x = -s_{m\pm n}$, since

$$x = - s_m s_n \pm pqr = - s_m s_n \pm pt_m t_n.$$

Also solved by Norman Anning, Leonard Carlitz, Walter Penney, Chih-yi Wang, and the proposer.

Editorial Note. Anning traces the problem back to Francis van Schotten the Younger, about 1660. See note, this MONTHLY, 1926, p. 212, also *Enciclopedia della Matematiche Elementari,* vol. I, part 2, pp. 330, 377–8.

Carlitz remarks that a slightly more general equation was solved by P. Bachmann, *Arch. Math. und Physik* (3), vol. 24, 1916, p. 89. See also a paper by L. J. Mordell, *J. London Math. Soc.,* vol. 28, 1953, pp. 500–510.

Infinite Product

4675 [1956, 126]. *Proposed by D. H. Lehmer, University of California at Berkeley*

Show that

$$\left(\frac{3\cdot 3}{1\cdot 5}\right)^4\left(\frac{15\cdot 15}{13\cdot 17}\right)^4\left(\frac{27\cdot 27}{25\cdot 29}\right)^4\left(\frac{39\cdot 39}{37\cdot 41}\right)^4 \cdots = 12.$$

Solution by D. C. Russell, University College of North Staffordshire, England. We will employ three well known results concerning the gamma function. (See Whittaker and Watson, *Modern Analysis,* 4th ed., pp. 238–240.)

$$(1) \qquad \prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \cdots (n - a_k)}{(n - b_1)(n - b_2) \cdots (n - b_k)} = \prod_{m=1}^{k} \frac{\Gamma(1 - b_m)}{\Gamma(1 - a_m)},$$

provided that $a_1 + \cdots + a_k = b_1 + \cdots + b_k$,

(2) $$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right)=(2\pi)^{(n-1)/2}n^{(1-2nx)/2}\Gamma(nx),$$

(3) $$\Gamma(x)\cdot\Gamma(1-x)=\pi/\sin\pi x.$$

By (1), the desired product, P, without the fourth power, reduces to

$$P=\Gamma(1/12)\Gamma(5/12)\{\Gamma(1/4)\}^{-2}.$$

(2) and (3) give

$$\Gamma(1/12)\Gamma(5/12)\Gamma(3/4)=2\pi\cdot 3^{1/4}\cdot\Gamma(1/4),\qquad \Gamma(1/4)\Gamma(3/4)=\pi\cdot 2^{1/2},$$

whence $P=2^{1/2}\cdot 3^{1/4}$ which is equivalent to the required result.

Also solved by Leonard Carlitz, A. E. Danese, N. J. Fine, Emil Grosswald, J. R. Hatcher, Peter Henrici, Edgar Karst, M. S. Klamkin, A. E. Landry, A. E. Livingston, Kovina Milosevich, Hermann Schmidt, M. R. Spiegel, Ernst Trost, Chih-yi Wang, J. V. Whittaker, and David Zeitlin.

Arithmetic and Harmonic Means

4676 [1956, 126]. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, White Plains, N. Y.*

Let $f(x)$ be a real valued measurable function defined on a measurable set A with measure μ such that

$$0<a\leqq f(x)\leqq b<\infty,$$

almost everywhere on A. Let $\bar{f}$ be the average value of f and $\tilde{f}$ be the reciprocal of the average value of the reciprocal of f. Find least upper and greatest lower bounds for $\bar{f}/\tilde{f}$.

Solution by J. Horváth, University of the Andes, Bogota, Colombia. We have

$$1\leqq\frac{\bar{f}}{\tilde{f}}=\frac{1}{\mu^2}\left(\int_A f dx\right)\left(\int_A f^{-1}dx\right)\leqq\frac{1}{4}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2.$$

The bound on the left is attained when f is constant, the bound on the right when $f(x)=a$ on a set B with measure $\mu/2$ and $f(x)=b$ on $A-B$.

The left-hand inequality is the well known one between the arithmetic and the harmonic means (Hardy-Littlewood-Polya, *Inequalities*, p. 144). The one on the right follows from the inequality (*op. cit.*, p. 166, ex. 230)

$$\int_A f_1^2dx\cdot\int_A f_2^2dx\leqq\frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\int_A f_1f_2dx\right)^2,$$

where $0<m_1\leqq f_1(x)\leqq M_1$, $0<m_2\leqq f_2(x)\leqq M_2$. It suffices in fact to put

$$f=f_1^2,\quad 1/f=f_2^2,\ m_1=\sqrt{a},\quad M_1=\sqrt{b},\quad m_2=\sqrt{1/b},\quad M_2=\sqrt{1/a}.$$

Also solved by Neill McShane, Chih-yi Wang, Albert Wilansky, and the proposer.

Convergent Series

4677 [1956, 126]. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

For what values of θ does the series

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\theta}$$

converge?

Solution by Leonard Carlitz, Duke University. Put $\theta = \alpha\pi$, where we may assume that $0<\alpha<2$. Also put $\alpha = 2^{-n_1}+2^{-n_2}+2^{-n_3}+\cdots$ $(0 \leqq n_1 < n_2 < \cdots)$ and $k_r = n_{r+1} - n_r (r=1, 2, \cdots)$.

Clearly we may assume that $\alpha \neq m/2^t$ where m is an integer. We shall prove that the series

(1) $$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\alpha\pi}$$

converges if and only if, as $r \to \infty$,

(a) $$2^{k_r}/n_r \to 0,$$

and the following series converges

(b) $$\sum_{r=1}^{\infty} 2^{k_r}/n_r^2.$$

Proof. The necessity of (*a*) is obvious, for otherwise the n_rth term of (1) does not approach zero. In the next place it follows from the identity

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x,$$

that

(2) $$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n \sin 2^n\alpha\pi} &= \sum_{n=1}^{N} \frac{1}{n} (\cot 2^{n-1}\alpha\pi - \cot 2^n\alpha\pi) \\ &= \cot \alpha\pi - \sum_{n=1}^{N-1} \frac{\cot 2^n\alpha\pi}{n(n+1)} - \frac{1}{N} \cot 2^N\alpha\pi. \end{aligned}$$

It follows from (a) that as $N \to \infty$, $(\cot 2^N\alpha\pi)/N \to 0$. As for

(3) $$\sum_{n=1}^{N-1} \frac{\cot 2^n\alpha\pi}{n(n+1)},$$

note first that the only negative terms are those for which $n = n_r - 1$; since the series

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n_r-1}\alpha\pi}{n_r(n_r - 1)}$$

is evidently convergent, we may ignore such terms in (3). In other words, if

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n}\alpha\pi}{n(n + 1)} \tag{4}$$

converges, it converges absolutely. Consequently the convergence of (4) implies the convergence of

$$\sum_{r=1}^{\infty} \frac{\cot 2^{n_r}\alpha\pi}{n_r(n_r + 1)}. \tag{5}$$

Since the fractional part of $2^{n_r}\alpha$ is equal to $2^{n_r-n_{r+1}}+2^{n_r-n_{r+2}}+ \cdots$, it is clear that (5) converges if and only if

$$\sum_{r=1}^{\infty} \frac{2^{k_r}}{n_r(n_r + 1)}$$

converges; this is equivalent to (b).

Conversely when (a) and (b) hold, it is clear from (2) that it is only necessary to prove the convergence of (4). But

$$\left|\sum_{n=1}^{N} \frac{\cot 2^{n}\alpha\pi}{n(n + 1)}\right| \leqq \sum_{n=1}^{N} \frac{|\csc 2^{n}\alpha\pi|}{n(n + 1)},$$

and $\sin 2^n\alpha\pi$ is negative only for $n=n_r$. Then the convergence of

$$\sum_{r=1}^{\infty} \frac{\csc 2^{n_r}\alpha\pi}{n_r(n_r + 1)}$$

is a consequence of (b), while the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n(n + 1) \csc 2^{n}\alpha\pi}$$

is easily proved by summation by parts as in (2). This completes the proof.

Remark. The series (1) is certainly convergent if the differences k_r are bounded; in particular, (1) converges for rational α (except $m/2^t$). An example of divergence is furnished by $\sum 2^{-r^2}$; the condition (a) is not satisfied.

For $n_r=[r \log_2 r]$ both (a) and (b) are satisfied; for

$$n_r = [r(\log_2 r + \log_2 \log_2 r)].$$

neither (a) nor (b) holds; while (a) is satisfied but (b) is not if

$$n_r = [r(\log_2 r + \log_2 \log_2 r - \log_2 \log_2 \log_2 r)].$$

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction to Mathematical Logic, vol. I. By Alonzo Church, Princeton University Press, 1956. ix+376 pages. $7.50.

The main body of this book is devoted to a precise and detailed exposition of the propositional calculus and the functional calculi of first and second order. Although intended primarily as a text for a beginning course, the book is capable of serving in other ways. Its use as a standard reference is assured. Perhaps its greatest value will be as a text (in conjunction with vol. II) for those often neglected students who are neither beginners nor experts. Exercises, which are given at the end of most of the sections, increase greatly the value of the book. Both the number and range of the exercises are large. Some of the exercises are routine, but many are extensions or variations of the theory, and a few are unsolved problems.

In broad outline, the book follows the author's 1944 monograph, *Introduction to Mathematical Logic, Part I.* Revisions have been made and much new material has been added. (The monograph has 118 pages and no exercises.) Some of the more important changes and additions are given below.

New is a sixty-eight page introduction, in which the author brings together a number of topics, such as *Names, The Logistic Method, Syntax, Semantics,* which another writer might prefer to scatter throughout the book or include in appendices. Although the author's treatment of this difficult material is uniformly excellent, its massing at the beginning of the book makes it take on the nature of an obstacle. The principal formulation of the propositional calculus is different, and leads quickly to an elegant Kalmár-type proof of deductive completeness. For the pure functional calculus of first order, Henkin's proof of completeness is given in addition to Gödel's. Henkin's method is used again in a completeness proof for the second order calculus. A new section, *Postulate Theory*, discusses systems with non-logical axioms.

The book is not intended to be comprehensive (*e.g.*, modal logic is omitted), but its wide scope is indicated by the presence of forty-one formulations of propositional calculi and thirty-five formulations of functional calculi.

An extensive bibliography is given in the form of footnotes. Two interesting sections are devoted to historical notes. An excellent numbering scheme makes an individual theorem, section or exercise easy to find.

The author has taken great pains to combine precision with clarity, and has succeeded notably. Because the book presupposes "some substantial mathematical background," it is not universally suitable as a text for a first course.

Only a few typographical errors were noted. One worth mentioning occurs in the formula at the top of page 97. (The formula should agree with the preceding one.)

This book will play an important part in the training of mathematical logicians.

ANGELO MARGARIS
Oberlin College

Numerical Analysis. By Z. Kopal. New York, John Wiley and Sons Inc., 1955, xiv+556 pages. $12.00.

This book is intended to be a research handbook as well as a text in numerical analysis as applied to functions of a single real variable. Although elementary calculus and "some algebra" are stated to be the only really necessary prerequisites, nevertheless, it would appear that considerably more mathematical background would be needed in order to understand some of the more advanced chapters.

The book will probably find wider use as a research handbook than as a text for a course in numerical analysis. It would only be possible to cover a fraction of the material in a single semester, and in a two-semester course in numerical analysis, one would almost certainly wish to include a considerable number of topics which are not treated in the book such as algebraic and transcendental equations involving a single variable, systems of equations, matrix problems, and partial differential equations.

The book is written from a point of view considerably removed from that of the user of high speed computing machines and lying somewhere between applied mathematics and hand computation. Some material is included which appears to fall in the area of classical analysis rather than numerical analysis. For instance, four pages are devoted to the standard proof of the Picard iteration process for solving ordinary differential equations and ten pages are used to treat the method of Frobenius. Since these topics are not treated specifically from the point of view of numerical computation, it would seem that only the results could be given, together with a reference to a book on ordinary differential equations.

The introductory chapter provides an interesting history of number systems and of numerical analysis as well as motivations for the study of the subject. Chapter II contains descriptions of the usual methods of polynomial interpolation, including the "throwback" method of Cromie which uses modified differences, and describes an interesting application to curve fitting.

The next chapter contains descriptions of the standard methods for solving ordinary differential equations. In addition, some new material on "successive extrapolation" is developed and applied to equations of the form $y''+f(x)y=g(x)$. The use of adjoint systems to estimate the propagated errors of various numerical procedures is also discussed.

Chapter V which is entitled "Boundary Value Problems" is primarily concerned with characteristic value problems associated with linear ordinary differential equations. By the use of finite difference methods the problem is reduced to that of finding the characteristic values of a matrix. The use of higher order difference equations and also various "extrapolations to zero grid size" are also considered. Chapter VI treats these problems by variational methods including the Rayleigh-Ritz method, Schwarz's method, a "collocation" method which is related to Lagrange's method of interpolation, and a least squares method.

Chapter VII deals with a number of methods for performing numerical quadrature. Methods which use unequally spaced intervals as well as those using equally spaced intervals are discussed. The concluding chapter deals with integral equations and integro-differential equations. The following subjects are treated in the five appendices: operational approach to finite difference formulas; trigonometric interpolation and Tchebysheff polynomials; coefficients for "mechanical" quadrature formulas; and algebraic equations and systems of linear equations.

In general, there seems to be too much attention devoted to some of the topics with the result that the scope is much narrower than one would expect from a book of this length. The book is very clearly written in a pleasing style and the arguments are easy to follow. The chapters on variational methods for solving characteristic value problems and on integral equations are particularly interesting.

At the end of each chapter a fair number of examples are provided together with supplementary notes and some research problems. The book is not entirely devoid of misprints; for instance on page 20, line 27, $a_1, a_2, \cdots, a_n$, should be $a_0, a_1, a_2, \cdots, a_n$ and on page 21 formula (II-B-4) should have $p_n^I(a_j)$ in the denominator.

Although this book is not recommended as a text to be used in a course in numerical analysis, it contains a great deal of interesting material and should prove very useful to those who work in the field of numerical analysis.

David Young
The Ramo-Wooldridge Corporation
Los Angeles, California

Introduction to Numerical Analysis. By F. B. Hildebrand. New York, McGraw-Hill Book Co., Inc., 1956, x+511 pages. $8.50.

In the preface, the author states that the book is intended to provide an introductory treatment of the fundamental processes of numerical analysis which is compatible with the expansion of the field brought about by the development of high speed computing machines. However, he indicates that he intends to take into account the fact that very large amounts of computation will continue to be effected by desk calculators (and by hand or slide rule) and that familiarity with computation on a desk calculator is a desirable preliminary to computation on a high speed computer. He expects that it would be pos-

sible to provide a survey of a substantial portion of the text in a single semester, and that a more thorough coverage could be provided in two semesters.

It is evident that the book will be much more valuable to the user of desk computers than to the user of high speed computers. A better background for the field of modern computation would be provided by a book which covered only the most frequently used methods of hand computation and which at the same time was written from the standpoint of high speed computing. Until such a book appears, however, the present volume will probably find extensive use as a text for courses in numerical analysis. It is very clearly written in sufficient detail so that a good student should be able to learn the material with little or no assistance. At the end of each chapter a large number of illustrative numerical problems are given. These should be of immense value to the instructor, since one of the drawbacks of other available texts in numerical analysis has been the lack of enough good problems.

The introductory chapter contains a discussion of various types of errors, including statistical errors, and analyzes the growth of such errors. Some mathematical preliminaries are also provided. The next four chapters, covering 140 pages, contain a thorough discussion of interpolation, numerical quadrature, and numerical differentiation. Use is made of divided differences, methods based on Lagrange's interpolation formula, finite differences, and difference operators.

The next chapter contains a description of the standard methods for solving ordinary differential equations together with an analysis of so-called "parasitic" solutions and of propagated errors. One section is devoted to two-point boundary value problems and another section describes methods for solving characteristic value problems.

Chapter VII deals with least squares polynomial approximation with special reference to Legendre, Laguerre, Hermite, and Chebyshev approximations. Factorial power functions and summation formulas are also discussed. The chapter is concluded by a description of several techniques for smoothing empirical data. Chapter VIII treats Gaussian quadrature and other related quadrature methods including Hermite, Legendre-Gauss, Laguerre-Gauss, Hermite-Gauss, Chebyshev-Gauss, Jacobi-Gauss, Radan, and Chebyshev quadrature. Chapter IX considers approximations of various types including Fourier approximation for both continuous and discrete ranges, exponential approximation, Chebyshev interpolation, and approximation by continued fractions.

The concluding chapter, which is practically independent of the preceding chapters, summarizes a number of methods for the numerical solution of sets of linear algebraic equations, non-linear algebraic or transcendental equations, and non-linear algebraic equations in particular. There is a bibliography with 276 references and an appendix with a directory of methods.

In general, it seems that too many methods are covered and that there is not enough discrimination nor enough discussion of the relative merits of the various methods. In particular, far too much space is devoted to interpolation and to finite differences. It would certainly seem that a knowledge of only a fraction

of the methods presented would enable one to handle in a satisfactory way all except the most unusual interpolation problems. Except for the section on error formulas, the entire chapter on operator methods might well have been eliminated. Similarly a large number of methods are presented for solving initial value problems for ordinary differential equations and yet only a single section is devoted to the important topic of two-point boundary value problems.

The treatment of methods for solving systems of linear algebraic equations could perhaps have been more detailed, if the length of some of the earlier chapters on interpolation had been reduced.

The material in Chapters VII, VIII, and IX on least squares polynomial approximation, Gaussian quadrature, and approximations, respectively, is very well presented, and does not appear to be readily accessible in as palatable a form elsewhere.

By concentrating only on those methods which are most useful for actual computation, an instructor should be able to present a very satisfactory one semester course in numerical analysis using this book, which is probably the best textbook available today for an elementary course in numerical analysis. The book should be part of the personal library of anyone working in the computing field.

DAVID YOUNG
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BRIEF MENTION

Publications of potential interest, but which are more properly reviewed in other periodicals, are described briefly below.

Sphere Grid Kit. Sphere Grid Sales, 3829 Davis Place, N.W., Washington 7, D.C. $2.95.

The Sphere Grid is a drafting underlay about 5 inches in diameter of the type discussed in the article *On Spherical Drawing and Computation* by Milton Felstein which appeared in this MONTHLY, vol. 62, 1955, a reprint of which accompanies the Sphere Grid.

Offerings and Enrollments in Science and Mathematics in Public High Schools. By Kenneth E. Brown. Office of Education Pamphlet No. 118. 1956. 24 pages. For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D.C. 15 cents.

This low cost pamphlet is of considerable interest to persons concerned with the teaching of science and mathematics in the public high schools. This study places emphasis on the *number of pupils* enrolled in schools offering certain courses. It should help disperse the complacency of certain smaller schools that have been lulled into the acceptance of their eviscerated offerings by statistics concerning the *number of schools* offering given courses. For example, about ninety percent of the students in the United States are enrolled in high schools

which offer intermediate algebra, trigonometry, or solid geometry, even though only two-thirds of the high schools offer such courses. Similarly, ninety-four percent of the students in the United States are enrolled in high schools offering either physics or chemistry or both, although only seventy-five percent of the high schools themselves are in this situation. The statistics used in preparing this booklet are from the fall of 1954.

Abacs or Nomograms. By A. Giet, New York, Philosophical Library, 1956. ix +225 pages. $12.00.

A translation by Helen Phippen and J. W. Head of the 1953 publication *Abaques ou Nomogrammes* by A. Giet. It presents nomograms and alignment charts "in an elementary way intended for practical engineers rather than mathematicians." However, many of the actual charts and examples are of sufficient mathematical interest to merit examination by teachers of mathematics.

A Manual of Engineering Geometry and Graphics. By Hollie W. Shupe and Paul E. Machovina. New York, McGraw-Hill Book Co., 1956. vii+347 pages. $5.25.

Chapters on surfaces, map projections, vectors, alignment charts, nomography, and graphic calculus may be of interest to students of mathematics as well as those of engineering geometry for whom the text is primarily intended. The inclusion of a bibliography of visual aids is of interest.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

AAAS CONFERENCE ON MATHEMATICS INSTRUCTION

Recognizing the concern about mathematics instruction and the interest of scientists, as well as mathematicians, in the teaching of mathematics, the AAAS called a Conference on Mathematics Instruction in Washington in October, 1956. The conference was made possible by a grant from the Carnegie Corporation of New York. A report of this conference appeared in *Science*, vol. 124, 1956, p. 1219.

CONFERENCE ON HIGH-SPEED COMPUTERS

The 1957 Conference on High-Speed Computers will be held at Louisiana State University, Baton Rouge, March 5–8, 1957. This conference is open to businessmen, office

managers, accountants, engineers, chemists, physicists, economists, statisticians, and other potential users from all sections of the country. Topics scheduled for discussion by nationally recognized speakers include office procedures, statistical operations and numerical methods designed for the adaptation of problems to machine solution. Several manufacturers of computing equipment will be represented through exhibits or demonstrations of computers in operation.

Inquiries concerning the conference may be directed to C. W. Barnett, Research Programmer, Office of the Comptroller, Louisiana State University, or to Dr. J. W. Brouillette, Director, General Extension Division, Louisiana State University, Baton Rouge 3, Louisiana.

PERSONAL ITEMS

Dean W. L. Duren, Jr., of the University of Virginia represented the Association at a convocation held on November 4, 1956, at Reed College at which the honorary degree of Doctor of Laws was conferred on Dr. F. L. Griffin. Dean Duren gave the principal address, "Mathematics and a Liberal Education." Dr. Griffin has also been honored by the creation of a "Frank Loxley Griffin professorship of mathematics" at Reed College.

Professor W. M. Whyburn of the University of North Carolina was the representative of the Association at the inauguration of President W. T. Gibbs of North Carolina Agricultural and Technical College on November 9, 1956.

Bradley University: Miss Rosamond J. Jones, St. Mary's Seminary Junior College, has been appointed Assistant Professor; Assistant Professor H. E. Sandstrom has been promoted to Associate Professor.

Georgia Institute of Technology: Assistant Professor E. R. Immel, University of Wisconsin, Dr. R. D. Johnson, Jr., University of Virginia, Dr. J. W. Walker, University of North Carolina, and Assistant Professor J. W. Wray, University of Idaho, have been appointed Assistant Professors; Mr. J. W. Jayne and Mr. A. J. Kainen have been appointed Instructors; Assistant Professor B. M. Drucker has been promoted to Associate Professor; Associate Professor A. L. Starrett has been promoted to Professor.

Michigan State University, Department of Statistics: Professor Leo Katz has been named Head of the Department; Professor W. D. Baten, Associate Professors K. J. Arnold, Ingram Olkin, Visiting Associate Professor Gopinath Kallianpur, on leave from the Indian Statistical Institute, Calcutta, Assistant Professors J. F. Hannan, C. H. Kraft A. G. Laurent, Visiting Assistant Professor Morris Skibinsky, on leave from Purdue University, Instructors C. H. Proctor and John Van Dyke are staff members; Professor R. A. Fisher will spend the Fall Quarter, 1957, in the Department as Visiting Distinguished Professor.

Mount Allison University: Mr. J. A. Flemming and Mr. Edgar Sparkes have been appointed Lecturers; Professor W. S. H. Crawford, Head of the Department of Mathematics, has been appointed Dean of Science.

New York University, Department of Mathematics, announces the following transfers: Associate Professor H. E. Wahlert and Assistant Professor John Schoonmaker of Washington Square College to the School of Commerce.

New York University, Institute of Mathematical Sciences: Associate Professor Herbert Greenberg, Carnegie Institute of Technology, has been appointed Associate Professor of Mathematics; Dr. George Witham, Lecturer, University of Manchester, has been appointed Associate Professor of Fluid Dynamics and Applied Mathematics; Dr. Jurgen Moser and Dr. Jerome Berkowitz have been appointed Assistant Professors; Dr. Gian-Carlo Rota, Yale University, has been appointed Research Assistant; Dr. M. Kneser, University of Heidelberg, is a Fulbright Scholar; Dr. Lars Hormander is in residence for the first term of the academic year 1956–57.

Ohio University: Assistant Professors R. K. Butner, W. T. Fishback, and S. J.

Jasper have been promoted to Associate Professors; Dr. W. E. Baxter, University of Pennsylvania, has been appointed Assistant Professor; Miss B. L. Bernhardt, Mrs. N. D. Johnson, Mr. R. N. King, and Miss E. O. Uhl have been appointed Instructors.

Tulane University: Dr. G. B. Preston, Lecturer, Royal Military College of Science, England, has been appointed Visiting Assistant Professor; Dr. H. H. Corson, III, and Assistant Professor V. R. Hancock, Virginia Polytechnic Institute, have been appointed Instructors; Dr. Heinz Renggli, an assistant for higher mathematics at the Swiss Federal Institute of Technology, and Dr. Gert Sabidussi, University of Minnesota, have been appointed Research Instructors; Dr. Anatole Beck, Williams College, has been appointed ONR Research Associate.

University of Colorado: Assistant Professor Arne Magnus, University of Nebraska, and Dr. R. W. McKelvey, a post-doctoral fellow at the University of Maryland, Institute for Fluid Dynamics, have been appointed Assistant Professors; Assistant Professor A. Zirakzadeh, University of Teheran, Iran, has been appointed Instructor; Dr. W. E. Briggs has been promoted to Assistant Professor.

University of Southern California: Dr. Henry Dye, State University of Iowa, has been appointed Associate Professor; Dr. A. V. Balakrishnan, an engineer for the Radio Corporation of America, and Dr. Heinz Cordes, an assistant at the University of Göttingen, have been appointed Assistant Professors; Dr. Douglas Anderson, an associate mathematician at the RAND Corporation, and Dr. John Maybee, University of Minnesota, have been appointed Instructors; Associate Professor A. L. Whiteman has been promoted to Professor.

Vanderbilt University: Professor J. A. Hyden has retired as Head of the Department of Mathematics and is continuing as Professor of Mathematics; Professor E. B. Shanks has been appointed Head of the Department.

Wayne State University: Professor A. L. Nelson has retired as Chairman of the Department and is continuing as Professor of Mathematics; Professor Wallace Givens has been appointed Chairman of the Department.

Wellesley College: Professor Helen G. Russell is on sabbatical leave for the year 1956–57 at Harvard University; Professor R. N. Johanson, Boston University, has been appointed Lecturer for the year.

West Virginia University: Mr. W. L. Anderson, Miss Jean M. Coover, Miss Lois V. Heflin, Mr. W. W. Hokman, Mr. R. P. L. Lu, Mr. B. H. Youell, Jr., and Mr. Harry McClung have been appointed Instructors.

Western Washington College of Education: Dr. H. G. H. Bartram, University of Oregon, has been appointed Instructor; Assistant Professor J. L. Hildebrand has been promoted to Associate Professor.

Mr. Eugene Albert of Brooklyn College is employed as a mathematician at General Electric Company, Schenectady, New York.

Associate Professor R. D. Anderson, University of Pennsylvania, has been appointed Professor at Louisiana State University.

Lt. W. R. Ballard of the Air Force Institute of Technology has been promoted to Assistant Professor.

Associate Professor D. H. Ballou, Middlebury College, has been promoted to Professor.

Mr. R. A. Barron, West Virginia University, has been appointed Assistant Professor at the University of Rhode Island.

Associate Professor S. Louise Beasley of Lindenwood College has been promoted to Professor.

Dr. S. R. Bodner, a senior scientist at AVCO Manufacturing Corporation, Stratford, Connecticut, has been appointed Assistant Professor of Engineering at Brown University.

Assistant Professor Fred Brafman, Wayne State University, has been appointed Assistant Professor at Southern Illinois University.

Mr. J. P. Brannen, an instructor at Sweeny High School, Texas, has been appointed Instructor at Sam Houston State Teachers College.

Dr. N. A. Brigham, a mathematician at the Applied Physics Laboratory, Johns Hopkins University, has a position as a senior scientist at AVCO Manufacturing Corporation, Advanced Development Division, Lawrence, Massachusetts.

Professor Leonard Bristow, Wisconsin State College, has been appointed Professor at the University of Santa Clara.

Assistant Professor C. E. Burgess, University of Utah, has been promoted to Associate Professor.

Dr. J. R. Byrne, Portland State College, has been appointed Assistant Professor at San Jose State College.

Mr. A. A. Caporaso of New York University has accepted a position as a mathematician with the International Business Machines Corporation, New York City.

Mr. M. M. Chirico, a scientist for the Westinghouse Electric Corporation, Atomic Power Division, Pittsburgh, Pennsylvania, has a position as an engineer at the American Machine and Foundry Corporation, Greenwich, Connecticut.

Dr. C. J. Clark of the Continental Oil Company has accepted a position as research scientist with Lockheed Aircraft Corporation, Palo Alto, California.

Miss Virginia Clover, University of Arizona, is employed as an aeronautical research engineer by the National Advisory Committee on Aeronautics, Ames Aeronautical Laboratory, Moffet Field, California.

Mr. W. R. Cooper, a teacher at Northeastern Academy, New York City, has been appointed Assistant Professor at Knoxville College.

Miss Helen E. Core, Northwestern Michigan College, is on leave of absence for the year 1956–57 and is at the University of Hawaii as an instructor.

Mr. R. J. Cormier, Teaching Assistant, University of Tennessee, has been appointed Assistant Professor at Northern Illinois State College.

Mr. C. G. Cullen, Graduate Assistant, University of New Hampshire, has been appointed Instructor at Worcester Polytechnic Institute.

Mr. G. J. Duffy, a research technician at Argonne National Laboratory, Lemont, Illinois, has been appointed an assistant mathematician in the Physics Division of the Laboratory.

Professor H. S. Everett of the University of Chicago has retired.

Miss Constance Foley, West Virginia University, has a position at the University of New Hampshire.

Mr. W. Y. Gateley, Clarkson College of Technology, has been appointed Assistant Professor at Colorado College.

Dr. R. D. Glauz, University of California, Los Alamos Laboratory, New Mexico, has accepted a position as a specialist with the General Electric Company, Cincinnati, Ohio.

Dr. A. J. Goldman, Princeton University, has a position as a mathematician at the National Bureau of Standards, Washington, D.C.

Miss Louisa S. Grinstein, Teaching Fellow, University of Michigan, has been appointed Instructor at Hunter College.

Mr. W. J. Hardell, Michigan State University, is employed as a mathematician at Remington-Rand UNIVAC, St. Paul, Minnesota.

Mr. D. F. Hayes, a student at St. Mary's University, has a position as an engineer with Boeing Airplane Company, Seattle, Washington.

Mr. W. A. Heinly, a statistician in the Office of Special Contracts, University of Pittsburgh, Washington, D.C., is employed as an engineer at the Monroe Calculating Machine Company, Morris Plains, New Jersey.

Associate Professor J. G. Herriot, Stanford University, has been promoted to Professor.

Mr. R. E. Hill, a student at the University of California, has a position as a computer for the Shell Development Company, Emeryville, California.

Mr. David Horwitz, an associate engineer for the Armour Research Foundation, Chicago, Illinois, has a position as a materials engineer at the Englander Company, Chicago, Illinois.

Assistant Professor L. Aileen Hostinsky, Pennsylvania State University, has been promoted to Associate Professor.

Mr. Walter James, a research engineer for Automatic Control Company, St. Paul, Minnesota, has been appointed an associate scientist for the St. Anthony Falls Hydraulics Laboratory, University of Minnesota.

Mr. Douglas Jones, a student at the University of Oklahoma, has a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Associate Professor J. R. F. Kent of Harpur College has been promoted to Professor.

Dr. Leo Lapidus, Michigan State University, has accepted a position as a senior research engineer at Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mr. Martin Lipschutz, Fairleigh Dickinson University, has been awarded a research grant for the academic year.

Mr. J. C. McCully, University of Rhode Island, has been appointed Assistant Professor at Western Michigan College.

Dr. Knox Millsaps, Massachusetts Institute of Technology, has accepted a position as chief scientist with the Holloman Air Development Center, New Mexico.

Professor Stella R. Mizell, Chowan College, has been appointed Instructor at Simpson College.

Mr. Dewey Moore, Naval Ordnance Laboratory, has accepted a position as senior engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. J. T. Morse has been appointed Instructor at Boston University.

Mr. H. A. Mortensen, a student at Utah State Agricultural College, has been appointed Chairman of the Department of Mathematics of Box Elder High School, Brigham City, Utah.

Assistant Professor T. J. Pignani, University of North Carolina, has been appointed Assistant Professor at the University of Kentucky.

Professor R. J. Pitts, Fort Valley State College, has been appointed Assistant Professor at Los Angeles State College.

Dr. R. G. Pohrer of the Chemical Corps, U.S. Army, has been appointed to the staff of the Mathematics Division of the Air Force Office of Scientific Research, Washington, D.C.

Dr. Anthony Ralston, Massachusetts Institute of Technology, has been appointed a member of the technical staff of Bell Telephone Laboratories, Whippany, New Jersey.

Mr. J. G. Renno, Jr., University of Wisconsin, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. B. E. Rhoades of Lafayette College has been promoted to Assistant Professor.

Mr. S. T. Rio, Pacific University, has been promoted to Assistant Professor.

Mrs. Jane Ingersoll Robertson, University of Maryland, has been appointed Instructor at the University of Illinois.

Mr. D. A. Rux, University of Kansas, has been appointed Professor and Chairman of the Department of Mathematics of Wisconsin State College, Oshkosh.

Associate Professor Judson Sanderson, Jr., U.S. Air Force Institute of Technology, has been appointed Associate Professor at the University of Redlands.

Mr. E. T. Sheffield, a physicst at the U.S. Naval Radiological Defense Laboratory, San Francisco, California, has been appointed Instructor at California State Polytechnic College.

Professor Jack Silber has returned to Roosevelt University after spending the spring and summer semester as Consultant to the Assistant for Operations Analysis, U.S. Air Force.

Mr. W. B. Simmons, Jr., an associate engineer at Lockheed Aircraft Corporation, Burbank, California, is now employed as an engineer at the American Bosch Arma Corporation, Garden City, New York.

Sister Mary Corona, Cardinal Stritch College, has been promoted to Assistant Professor.

Dr. C. V. L. Smith, a scientific liaison officer for the Office of Naval Research, London Branch, is now Chief of the Computation Laboratory, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. R. J. Smith has been appointed Instructor at Queen's University, Kingston, Ontario, Canada.

Mr. R. F. Steinhart, New Jersey State Teachers College, Montclair, has a position with the International Business Machines Corporation, Newark, New Jersey.

Dr. Leonard Tornheim, University of California, has accepted a position as a research mathematician with the California Research Corporation, Richmond, California.

Miss Eugenia I. Trapp, Tulane University, has a position as applied science representative for the International Business Machines Corporation, Houston, Texas.

Mr. C. H. Tross, Lockheed Aircraft Corporation, has accepted a position as head mathematician of the research laboratory, American Bosch Arma Corporation, Garden City, New York.

Assistant Professor R. Z. Vause, Jr., Vanderbilt University, has been appointed Assistant Professor at the University of Kansas City.

Professor J. A. Ward, University of Kentucky, has a position as a mathematician at Holloman Air Force Base, New Mexico.

Dr. G. P. Weeg, previously employed by the Sperry Rand Corporation, Remington Rand UNIVAC, St. Paul, Minnesota, has been appointed Assistant Professor at Michigan State University.

Dr. Chien Wenjen, Knoxville College, has been appointed Assistant Professor at Texas Technological College.

Mr. R. E. Wild, Teaching Assistant, University of California, Los Angeles, has a position as a computer analyst for Douglas Aircraft Corporation, Santa Monica, California.

Associate Professor A. G. Wootton, State University of New York, has been appointed Lecturer at the University of Maine.

Dr. Fumio Yagi, Ballistic Research Laboratories, Aberdeen Proving Ground, New Mexico, has accepted a position as a senior research engineer with the Jet Propulsion Laboratory, California Institute of Technology.

Assistant Professor F. H. Young, Portland State College, is now with the Autonetics Division of North American Aviation Company, Downey, California.

Professor Witold Hurewicz, Massachusetts Institute of Technology, died on September 6, 1956.

Professor Walter Reynolds, Georgia Institute of Technology, died on September 10, 1956.

Professor Emeritus L. W. Smith, Washington and Lee University, died on August 9, 1956. He was a charter member of the Association.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The thirty-fifth annual meeting of the National Council of Teachers of Mathematics will be held at the Bellevue-Stratford Hotel, Philadelphia, March 27–30, 1957. Inquiries concerning the meeting may be sent to the National Council, 1201 Sixteenth Street, N. W., Washington 6, D.C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1956–57

A. W. Tucker

Clarkson College of Technology	Potsdam, N. Y.	Oct. 1–3
St. Lawrence Univ.	Canton, N. Y.	Oct. 4–5
Georgetown Univ. and Catholic Univ. of America	Washington, D. C.	Oct. 15–20
North Carolina State College	Durham, N. C.	Oct. 22–24
Randolph-Macon Woman's Coll. and Lynchburg Coll.	Lynchburg, Va.	Nov. 12–14
Univ. of Richmond	Richmond Va.	Nov. 15–17
Univ. of Tennessee	Knoxville, Tenn.	Nov. 26–27
Georgia Institute of Technology	Atlanta, Ga.	Nov. 28–Dec. 1
Univ. of Georgia	Athens, Ga.	Nov. 28–Dec. 1
Alabama Polytechnic Institute	Auburn, Ala.	Dec. 3, 4, 5
Univ. of Alabama	Tuscaloosa, Ala.	Dec. 6–8
Univ. of Mississippi	Oxford, Miss.	Dec. 10–11
Univ. of Houston	Houston, Tex.	Dec. 13–14
North Texas State Coll.	Denton, Tex.	Dec. 17–19
Tulane Univ.	New Orleans, La.	Dec. 20–21
Dartmouth Coll.	Hanover, N. H.	Jan. 21–23
Franklin and Marshall Coll.	Lancaster, Pa.	Feb. 11–12
State Teachers Coll. and Wilson Coll.	Shippensburg, Pa. and Chambersburg, Pa.	To be arranged
Vassar Coll.	Poughkeepsie, N. Y.	Feb. 25–27
Drew Univ.	Madison, N. J.	March 4–6
Allegheny Coll.	Meadville, Pa.	April 4–5
Univ. of Buffalo	Buffalo, N. Y.	April 8–11
Kent State Univ.	Kent, Ohio	April 25–27
Miami Univ.	Oxford, Ohio	April 29–30
Univ. of Cincinnati	Cincinnati, Ohio	May 1–2
Xavier Univ.	Cincinnati, Ohio	May 3–4
Purdue Univ. (Fort Wayne Center)	Fort Wayne, Ind.	May 6–7
Valparaiso Univ.	Valparaiso, Ind.	May 8–9
DePauw Univ.	Greencastle, Ind.	May 10–11
Knox Coll.	Galesburg, Ill.	May 13–14
Southern Illinois Univ.	Carbondale, Ill.	May 16–17

Edwin Hewitt

Sacramento State Coll.	Sacramento, Calif.	March 27–28
Fresno State Coll.	Fresno, Calif.	March 28–29
California Institute of Technology	Pasadena, Calif.	April 1–2
Univ. of Redlands	Redlands, Calif.	April 3–5
Arizona State Coll.	Tempe, Ariz.	April 8–10
Univ. of Arizona	Tucson, Ariz.	April 11–12
New Mexico Coll. of Agriculture and Mechanic Arts	State College, N. M.	April 15–17
Univ. of Colorado and environs	Boulder, Colo.	April 22–24
Univ. of Nebraska	Lincoln, Neb.	April 25–26
Univ. of Utah and Utah State Agricultural Coll.	Salt Lake City and Logan, Utah	April 29–May 3
Montana State Univ.	Missoula, Mont.	May 7–9
State Coll. of Washington and Univ. of Idaho	Pullman, Wash. and Moscow, Idaho	May 13–17
Coll. of Idaho	Caldwell, Idaho	May 20–21

Kurt Mahler

N. J. State Teachers Coll.	Montclair, N. J.	March 28
Vanderbilt Univ. and George Peabody Coll. for Teachers	Nashville, Tenn.	April 8–11
Bowling Green State Univ.	Bowling Green, Ohio	April 12–15
Kenyon Coll. and Denison Univ.	Gambier and Granville, Ohio	April 16–19
Emmanuel Missionary Coll.	Berrien Springs, Mich.	April 22–23
Univ. of North Dakota	Grand Forks, N. D.	April 25–26
Concordia Coll.	Moorhead, Minn.	April 29–30
Carleton Coll. and St. Olaf Coll.	Northfield, Minn.	May 1–3
Iowa State Coll., State Univ. of Iowa and Iowa State Teachers Coll.	Ames, Iowa City, and Cedar Falls, Iowa	May 6–10
Central State College	Edmond, Okla.	May 13–14
Univ. of Kansas, Washburn Univ., Municipal Univ. of Wichita, and Bethel Coll.	Lawrence, Topeka, Wichita, and North Newton, Kan.	May 15–24

J. L. Walsh

Canisius Coll.	Buffalo, N. Y.	April 1–2
Univ. of Rochester	Rochester, N. Y.	April 3–5
Univ. of Vermont	Burlington, Vt.	April 8–9
Middlebury Coll.	Middlebury, Vt.	April 10
Skidmore Coll.	Saratoga Springs, N. Y.	April 11
Cardinal Stritch Coll.	Milwaukee, Wis.	April 12
Univ. of Wisconsin (Milwaukee Ext.)	Milwaukee, Wis.	April 15–16
Amherst Coll. and Univ. of Massachusetts	Amherst, Mass.	April 17–19
Wesleyan Univ. and Trinity Coll.	Middletown and Hartford, Conn.	April 22–24
Univ. of Delaware	Newark, Del.	April 26–29
St. John's College and St. John's Univ.	Jamaica, N. Y.	April 30

Continuation of the Program of Visiting Lecturers

The National Science Foundation has granted the Association the sum of $55,200 for the support of a continuation of the Program of Visiting Lecturers for a two year period beginning in September 1957. This is the third grant for this purpose from the National Science Foundation. The present grant will allow the continuation of the program after the conclusion of the period covered by the previous grants.

The Association's Committee on Visiting Lecturers consists of Professors G. B. Huff, R. A. Rosenbaum, D. E. Richmond, and B. W. Jones, Chairman. The Committee is authorized to select lecturers and to arrange their itinerary. Any correspondence on either of these topics should be addressed to Professor Jones as Chairman of the Committee.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 67 persons have been elected to membership by the Board of Governors on applications duly certified.

C. D. ABBATE, B.A. (Western S.C.) Math., Holloman Air Force Base, N. Mex.

G. C. ANDERSON, B.A. (Concordia) Grad. Asst., University of Nebraska

H. B. ANDERSON, M.A. (Michigan) Asst. Professor, Michigan College of Mining and Technology.

R. S. BACKEN, B.S. (North Dakota Agric. C.) Grad. Asst., North Dakota Agricultural College.

E. A. BEHRENS, Student, University of Washington.

D. J. BOYCE, B.S. (Central S.C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

D. A. BREAULT, Student, Carnegie Institute of Technology.

J. W. CASON, M.A. (Southern Methodist) Instr., Cuttington College, Suakoko, Liberia.

P. L. DUREN, A.B. (Harvard) Res. Asst., Massachusetts Institute of Technology.

R. S. ELLZEY, Student, University of Mississippi.

J. R. ENO, JR., B.A. (Whittier) Acting Instr., University of Idaho.

L. F. EPSTEIN, Ph.D. (M.I.T.) Res. Asso., Knolls Atomic Power Lab., General Electric Co., Schenectady, N. Y.

VILLA E. FENDER, A.M. (Kansas) Head, Department of Mathematics, College High School, Bartlesville, Okla.

N. H. FISHER, JR., M.A. (California) Math. & Statistical Aid, Ballistics Research Lab., Aberdeen Proving Ground, Md.

L. D. FOUNTAIN, S.M. (Chicago) Grad. Asst., University of Nebraska.

NANCY E. FRENCH, B.A. (Russell Sage) Lab. Asst., Knolls Atomic Power Lab., General Electric Co., Schenectady, N. Y.

K. D. FRYER, Ph.D. (Toronto) Asso. Professor, Royal Military College of Canada.

D. J. GARDA, Student, Lebanon Valley College.

H. R. GILLETTE, B.A. (Reed) Res. Engr., North American Aviation, Los Angeles, Calif.

F. M. C. GOODSPEED, Ph.D. (Cambridge, England) Asso. Professor, University of British Columbia.

L. A. HART, B.S. (Loras) Instr., Loras College.

PATRICIA HAUSS, A.B. (Indiana) Math. Teacher, Clinton High School & Junior College, Iowa.

ROY HEATH, B.A. (Omaha) Grad. Asst., University of Nebraska.

G. F. HECK, II, Student, Lebanon Valley College.

D. M. HESS, Student, Fordham University.

G. A. HEUER, M.A. (Nebraska) Instr., Concordia College.

RAYMOND HIRSCHKOP, B.S. (Rutgers) Instr., Pratt Institute.

FLORENCE E. INGHAM, M.A. (Teachers C., Columbia U.) Math. Teacher, College High School, Bartlesville, Okla.

I. M. ISAACS, Student, Polytechnic Institute of Brooklyn.

E. D. JACOBSON, B.S. (Columbia) Instr., State University of New York, Agricultural and Technical Institute, Alfred.

M. S. JOHNSON, B.A. (N.Y.U.) Math., Melpar, Falls Church, Va.

C. D. KEIM, Student, Gannon College.

R. F. KELLER, B.S. in Ed. (Southeast Missouri S.C.) Instr., School of Mines and Metallurgy, University of Missouri.

G. H. KIEL, Student, Polytechnic Institute of Brooklyn.

REV. R. F. KING, O.F.M., B.S. (Siena) Asst. Professor, Siena College.

MARTIN LEVEY, Ph.D. (Dropsie C.) Instr., Temple University.

J. S. LINNEKIN, Student, Lebanon Valley College.

R. S. LOCKHART, M.A. (Teachers C., Columbia U.) Head, Department of Mathematics, Madison High School, N. J.

G. G. LORENTZ, Ph.D. (Tübingen, Germany) Professor, Wayne State University.

F. J. LORENZEN, JR., S.B. (M.I.T.) Grad. Asst., University of New Hampshire.

J. B. LOVE, B.A. (Pennsylvania) Instr., Lebanon Valley College.

K. B. MACMURRAUGH, Student, Stanford University.

HELEN M. MCPHERSON, Student, University of Mississippi.

E. J. MILLER, B.S. (Calif. I.T.) Comptroller's Staff, C. F. Braun & Co., Alhambra, Calif.

H. W. MOORE, M.A. (Missouri) Numerical Analyst, General Electric Co., Cincinnati, Ohio.

H. E. MORRISON, B.S. (Johns Hopkins) Electronics Components Design Engr., Glenn L. Martin Co., Baltimore, Md.

D. M. OLSON, B.A. (Dana) Grad. Asst., Oklahoma Agricultural and Mechanical College.

R. K. OTNES, B.A. (Nebraska) Grad. Asst., University of Nebraska.

DON PASSMAN, Student, Polytechnic Institute of Brooklyn.

W. H. PEIRCE, Ph.D. (Wisconsin) Asst. Professor, Michigan State University.

R. S. PINKHAM, Ph.D. (Harvard) Res. Asst., Princeton University.

H. O. POLLAK, Ph.D. (Harvard) Res. Math., Bell Telephone Labs., Murray Hill, N. J.

V. R. RAIL, M.S. (S.U. of Iowa) Asst. Professor, Parsons College.

FRANK RAUNIKAR, B.S. (Oklahoma) Grad. Asst., Oklahoma Agricultural and Mechanical College.

WALTER ROTH, M.S. (S.U. of Iowa) Asst. Professor, Southeast Missouri State College.

DIRAN SARAFYAN, Mech. Eng. (Toulouse, France) Asso. Professor, Lamar State College of Technology.

E. G. SCHULD, M.A. (Wisconsin) Instr., University of Wisconsin, Milwaukee.

C. S. SMITH, M.A. (S.U. of Iowa) Asst. Professor, Drury College.

R. F. STEINEN, M.A. (Montclair S.T.C.) Asst. Professor, Montclair State Teachers College.

D. C. TERRELL, M.A. (Arkansas) Asst. Professor, Southern State College.

G. A. THOMAS, Student, Lebanon Valley College.

V. D. TURNER, M.A. (Illinois) Instr., Mankato State Teachers College.

MERLYN VANDERBEEK, B.S. (Nebraska) Grad. Asst., University of Nebraska.

C. K. VILIM, JR., B.S. (Northwestern) Math., Analog Computer Div., Aerial Measurements Lab., Evanston, Ill.

W. C. WALTER, A.B. (New York S.T.C., Albany) Grad. Asst., University of Wisconsin.

C. W. WILLIAMS, Ph.D. (Virginia) Asso. Professor, Washington and Lee University.

M. E. WINGER, M.S. (North Dakota) Instr., University of North Dakota.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at Concordia College, Moorhead, Minnesota, on October 6, 1956. Professor

Sigurd Mundhjeld presided at the morning session and Professor Walter Fleming, Chairman of the Section, presided at the afternoon session. The meeting was attended by 44 persons, including 30 members of the Association.

By invitation of the Executive Committee, Mr. G. A. Heuer, Concordia College, delivered an address at the morning session entitled "Invariant Measure on Topological Groups." An abstract of this address follows:

This paper outlines a development of Haar measure on locally compact groups. Neither the results nor the methods are new. The Daniell extension of an elementary integral to a Lebesgue type integral is discussed (*Ann. of Math.*, vol. 19; pp. 279–294). The elementary integral is an additive, homogeneous, non-negative functional which is continuous under the taking of monotone limits, defined on a vector space L of bounded real-valued functions on a set, where L is also closed under the lattice operations. An elementary integral is then obtained on the vector space L of the real-valued continuous functions with compact support on a locally compact group. Daniell's extension is the Haar integral. The paper concludes with the uniqueness theorem for the Haar integral and with the observation that as examples of the Haar integral on the real numbers, the integers and finite discrete groups, the Lebesgue integral, infinite summation and finite summation, respectively, are special cases of a unifying theory.

The following short papers were presented:

1. *A slide rule hint*, by Mr. Louis Van Slyck, North Dakota Agricultural College, introduced by Professor A. G. Hill.

A slide rule operation which is very important in electrical engineering is that of determining the magnitude and direction of a resultant from its components. This can be done in one motion of the slide on any rule which has the tangent and sine scales on the slide and a D scale on the body of the rule, provided the ratio of the smaller to the larger quantity is not less than 0.1. A description of this operation is given in this paper and the exceptional case is discussed.

2. *Imaginary roots from graphs*, by Professor P. A. Rognlie, University of North Dakota.

It was shown that certain properties of the graphs of quadratic and cubic functions can be used to determine the imaginary roots of the corresponding equations.

3. *A model of a cycloidal pendulum*, by Professor E. J. Camp, Macalester College.

A physical model was presented with a cycloidal pendulum and a simple pendulum suspended from the same frame. The cycloidal pendulum was constrained to move along a cycloidal path by suspending it from the point of tangency of the two arcs of the evolute to the path. The periods of the two pendulums were compared for small and large oscillations. A stop watch was used to show that the period of the cycloidal pendulum remained constant while the period of the simple pendulum increased with the amplitude. The differential equation of the path and the formula for the period of each pendulum was developed.

4. *Plane linkage models*, by Mr. M. E. Winger, University of North Dakota.

Plane linkages were described and illustrated by examples of models constructed by the author. The history and uses of linkages were discussed briefly beginning with the problem of producing rectilinear motion. Mentioned and illustrated were Watt's Parallel Motion, the Peaucellier Cell, the Hart Cell and several linkages producing special curves.

5. *Some undergraduates discover "mathematics,"* by Professor J. M. Calloway, Carleton College.

The paper reports the work of three students with little mathematical background on the representation of an integer as the sum of consecutive integers. The interest of this paper is in the

circumstances of the work rather than the result which is: An integer can be represented as the sum of consecutive integers if and only if it is not a power of two, and the number of representations is equal to the number of odd divisors (one excluded).

6. *Use of a high-speed computer in evaluating a probability*, by Dr. David Gosslee, North Dakota Agricultural College, introduced by Professor A. G. Hill.

The author presented a statistical problem which led to experimental sampling using an IBM 650 computer to obtain probabilities related to the distribution of the statistic involved. In one case experimental sampling was more efficient than direct calculating performed with a high-speed computer. In the second case, the use of experimental sampling was necessary since the distribution of the statistic was unknown. The manner in which the sampling experiment was conducted on the computer and some capabilities of the computer were discussed.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS, University of Kansas, Lawrence, April 13, 1957.

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.

METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.

MICHIGAN, Wayne State University, Detroit, March 23, 1957.

MINNESOTA, Carleton College, Northfield, May 11, 1957.

MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, University of Cincinnati, April 20, 1957.

OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

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CONTENTS

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RALPH D. JAMES, *Editor*

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FLEXAGONS

C. O. OAKLEY AND R. J. WISNER, Haverford College

1. Introduction. In 1939, four graduate students at Princeton University (R. Feynman, A. H. Stone, B. Tuckerman, and J. W. Tukey) discovered how to fold a piece of paper into what are now known as *flexagons*—hexagonal gadgets which "flex" under an operation we call pinching to exhibit several faces. Three short notes [3], [4], [6] merely show how to construct two of these paper models. So far as we know there is no other printed literature.

In order to motivate the definitions of Section 4, where abstract flexagons are discussed, we first describe informally how to construct physical models of the simplest abstract flexagons to be called regular flexagons of orders 3, 6, and 9.

2. Regular flexagons of orders 3 and 6. To construct the regular flexagon of order 3, RF_3, take a rectangular strip of paper* about an inch and a half wide and about a foot long, and from one end cut off a 30°, 60° triangle. Next, score

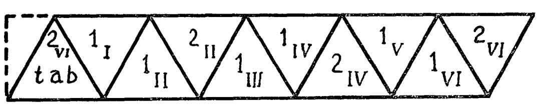

FIG. 1

the strip *very carefully* into ten equilateral triangles; discard any surplus material and label on both sides of the paper as in Figure 1. The first triangle on the left is a tab and is to be glued later to the last triangle on the right. With the strip oriented as in the figure, hold the tab in the left hand and with the right hand fold 2_{II} over on top of 1_{II}, fold 2_{IV} over on top of 1_{IV}, and fold 2_{VI} over on top of 1_{VI}. Now glue tab 2_{VI} onto 2_{VI} and the model is completed.

Hold the flexagon in the position of Figure 2 so that the Roman numerals I to VI, indicating what we shall call *pats*, run clockwise. Each of pats I, III, V contains a single triangle (piece of paper), each of pats II, IV, VI contains two triangles, and the pats are arranged in the form of a hexagon. Mark the vertices of the upper face of 2_{II} with the letters a, b, c clockwise as in Figure 2. Now with pat II to the north, pinch along the east radius (the east half of the east-west diagonal) forcing pats III and IV down. While holding these together push the west end of the west radius down. Actually this causes a folding east and along alternate radii and the flexagon begins to open at the center (now at the top). Releasing the pinching finger and thumb will permit the flexagon to "open" and lie flat again but this time displaying a new face (set of six triangles which will always appear together). The total operation is called a (physical) pinch. While

* Adding machine tape is satisfactory.

2_{II} is at the north, another pinch (east) is impossible since 1_{IV} and 2_{IV} are joined along the east radius. Rotate the whole model $-60°$ and label the vertices of 1_{II}, now north, with a, b, c as before. Pinch east. Rotate $-60°$ and label 1_I (north) with a, b, c. Another pinch brings the original face up again so that three distinct faces have appeared. If the flexagon is turned over, three pinches will exhibit three new (mathematical) faces since now the pat numbers run

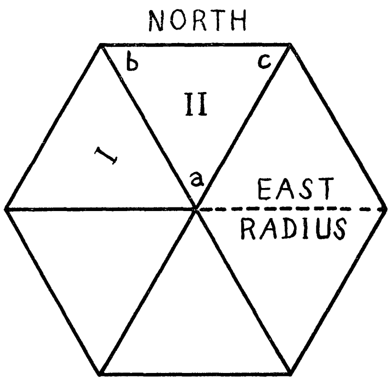

FIG. 2

counterclockwise. Moreover, the letter c is now at the center of the model.

The degree of a pat is the number of triangular pieces of paper in that pat. The order of a flexagon is the sum of the degrees of any two adjacent pats. Here the order is three.

To make the regular flexagon of order 6, RF_6, prepare a strip of nineteen triangles marked as in Figure 3. The construction is accomplished by "winding" the strip up pat by pat. In the folding of the triangles into pats, *the direction of the winding motion of the right hand is that of a wheel rolling on the ground toward you.* In what follows we suppress the Roman subscripts, which are the pat indicators, although they occur in the figure of the strip.

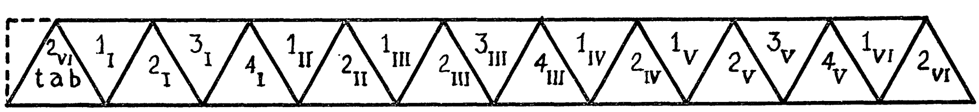

FIG. 3

To wind pat I:

(a) Hold strip as in Figure 3, tab in left hand;
(b) Fold triangle 2 back under triangle 1 by winding motion of right hand;
(c) Fold triangle 4 over on top of triangle 3 by winding motion;
(d) Fold the pair 4, 3 over on top of pair 1, 2 so that, from top down through

the pat, the strip numbers now read 3, 4, 1, 2. This completes the pat; it is of degree 4.

The order 3, 4, 1, 2 of this completed pat is unique; that is, there is no other way to assemble the four triangles in a 4-pat. For the moment we write this order 34,12 with a single comma placed where it tells us exactly what the final folding was in the winding process.

Pat II is a 2-pat and is wound as in pat II of RF_3 above. From top down through the pat the strip numbers are 2, 1 and there is no other way to assemble the two triangles into a 2-pat.

Now wind in order pat III the same as pat I, pat IV the same as pat II, pat V the same as pat I, and pat VI the same as pat II. The windings are therefore a triplication of pats I and II. Glue tab to 2_{VI} and the flexagon is complete. The pinching operation is the same for any flexagon. RF_3 and RF_6 are so special that we omit further discussion, but experimentation with them before proceeding would be helpful.

3. Regular flexagon of order 9. To make a model of the regular flexagon of order 9, RF_9, prepare a strip with a tab and 27 other triangles. Reserving the leftmost triangle for the tab, mark the others, from left to right, 1, 2, 3, 4; 1, 2, 3, 4, 5; $\cdots$, in triplicate. Wind pat I as in RF_6.

To wind pat II:

(a) Hold tab and pat I in left hand;
(b) Put 2 on top of 1;
(c) Put 4 under 3;
(d) Fold pair 3, 4 under pair 2, 1;
(e) Bring 5 over on top of 2 so that, from top down through the pat, the strip numbers read 5,2143. This completes the pat; it is of degree 5.

The position of the comma after 5 tells us that everything to the left of it was folded over 2143 in the last operation. The order 5,2143 is not unique for the 5-pat and this is explained in Section 5.

Triplicate these windings and glue tab. The strip numbers showing on the upper face should be 3's and 5's.

In order to keep track of all of the following pinches, ignore the original strip numbers and place a 1 in the middle of the top triangle of pat II; further, mark this triangle with a, b, c as in RF_3. There is no need of marking the other triangles since the same six pieces of paper will always appear together. This face is now designated as face $1abc$. Record this face on another piece of paper as $1a$ using only the letter at the center. With pat II north, pinch east; label north $2abc$ recording this new face as $2a$, and note that it is impossible to pinch east again. Rotate the flexagon $-60°$, pinch east, label new face (north) $3abc$ and record as $3a$. Pinch, label north $4abc$, record $4a$. Pinch, label north $5abc$, record $5a$. Rotate, pinch; face $3bca$ appears but we record this as $3b$, using only the letter at the center of the model.

Repeat the process: pinch east as many times as possible, then rotate and

pinch east, recording new faces as they arise. The succession of faces, from the beginning, is the following, the r indicating that a rotation is to be made before another pinch: $1a$, $2ar$, $3a$, $4a$, $5ar$, $3br$, $4a$, $2c$, $6ar$, $4cr$, $2c$, $3a$, $1a$, $7a$, $8ar$, $1cr$, $7a$, $3c$, $9ar$, $7cr$, $3c$; $1a$.

Remark 1. At any time a pinch following a rotation of $\pm 120°$ will yield the same face as the pinch without such a rotation. That is, pinching along any one of three alternate radii will open the flexagon to the same face. If the flexagon can also be pinched along one of the other set of alternate radii, a second face is exhibited.

Remark 2. In the middle of the sequence of pinches above, the original face $1a$ reappeared. But the next pinch led to the new face $7a$, and not to $2a$, because the flexagon was so oriented that one of the other set of alternate radii lay east. This situation is standard: in every complete sequence of pinches, a given face from which two pinches are possible will appear twice and in the proper orientations to yield the two possible faces. In any flexagon, the faces with no more than a single opening are those and only those where there is but a single paper triangle in alternate pats.

The following definitions apply to all flexagons. A *physical face* is that collection of the six uppermost triangles, one to a pat, regardless of their orientation. Each different orientation of these six triangles, with respect to each other, determines what is called a *mathematical face*.

The above RF_9 has the following properties:

(a) It has 9 physical faces;
(b) It is a Möbius band of 21 half-twists;
(c) It requires 9 rotations and 21 pinches to run through a complete cycle on one side;
(d) It has a total of 30 mathematical faces, 15 on either side. (The pat labels run counterclockwise for each back side mathematical face.)

4. Definition of an abstract flexagon. Take a new flexagon strip, mark with tab, 1, 2, 3, 4, 5 and wind a 5-pat as in RF_9. Now turn the whole strip over noting that the order 52143 has been reversed to read 34125. Mark the next two triangles in the strip 6 and 7, and with these wind a 2-pat which will of course read 76. If this 2-pat is now folded over onto the reversed 5-pat, the numbers 76 are reversed. The total process is that of winding a 7-pat reading, top to bottom, 67,34125. This helps to motivate the following definitions, which in this section are concerned with abstract flexagons. But we do not continue to carry the adjectives "abstract" and "physical" except where confusion might arise.

Let m be a positive integer. For $m=1$, the single permutation of the integer 1 is called a *pat of degree* 1. For $m=r+s>1$, the permutation $A_rA_{r-1}\cdots A_2A_1b_sb_{s-1}\cdots b_2b_1$, where $A_i=a_i+s$, of the integers 1 to m is called a *pat of degree m* if the permutation $a_1a_2\cdots a_r$ of the integers 1 to r and the permutation $b_1b_2\cdots b_s$ of the integers 1 to s are pats of degree r and s, respectively. We define an *abstract flexagon* F to be an ordered pair of pats, $F=(P, Q)$. If the pats

are of degree p (a p-pat) and q (a q-pat), then $N=p+q$ is called the *order* of the flexagon F_N.

Two important operations on F_N which preserve the order N are a pinch and a rotation. If the degree of Q is at least two, a *pinch* is that transformation which carries F into F', where $F=(A_rA_{r-1}\cdots A_1b_sb_{s-1}\cdots b_1,\ C_tC_{t-1}\cdots C_1d_ud_{u-1}\cdots d_1)$; $F'=(D_uD_{u-1}\cdots D_1b_1b_2\cdots b_sA_1A_2\cdots A_r,\ c_1c_2\cdots c_t)$; $A_i=a_i+s$, $C_i=c_i+u$, $D_i=d_i+r+s$, and it is easily seen that F' is a flexagon. A *rotation* of a flexagon is the transposition of its pats.

Two flexagons are said to be *equivalent* if one is obtainable from the other by a sequence of pinches or rotations, and this is an equivalence relation.

A flexagon can be represented physically as a triplication of a pair of pats, the arrangement in the paper model being such that the three pairs form a flexible hexagon which, under a pinch, exhibits another face. It can be constructed from a straight piece of paper by folding and gluing. For $N=1+1$, F_2 is the ordinary hexagon.

There is a subclass of flexagons, closed under the operations of pinching and rotating, which constitutes a universal class from which all flexagons arise. A member of this class is called a *regular flexagon* RF_N and is defined as above but in terms of *regular pats* which require $m\not\equiv 0$, mod 3. Further, the degrees p and q of the ordered pair of regular pats must belong to different residue classes, mod 3. Hence p, q are of the form $3k+1$, $3k+2$ and $N=p+q\equiv 0$, mod 3. A model of a regular flexagon can be constructed from a straight piece of paper by folding only.

The next two sections are devoted to regular flexagons.

5. Regular pats. We now consider the number of distinct ways to wind some regular pats of degree m ($\not\equiv 0$, mod 3). Pats of degree $m=1$ and $m=2$ can be constructed in only one way.* A pat of degree $m=4$ is also unique, namely 34,12.

If $m=5$, there are two and only two distinct ways. To make a 5-pat, we add one more triangle to the 4-pat. But clearly this can be done in two ways: first, to the whole 4-pat, *after it has been turned over*, can be added triangle 5, which is now on top. In turning the 4-pat over, we have reversed its sequential order so that the whole 5-pat now reads, top to bottom, 5,2143. Second, a 4-pat could be wound on triangles 2, 3, 4, 5 and its order (3412) would then be 4523. Now *this* can be turned over, becoming 3254, and folded over triangle 1 making the 5-pat read 3254,1. Note that the ordered binary partitions of $5=r+s$, where neither r nor s is congruent to 0, mod 3, are $1+4$ and $4+1$. These must be considered in winding the 5-pat which is necessarily made up of $1+4$ triangles or $4+1$ triangles. There are no other ways to wind a regular 5-pat.

If $m=7$, there are four ways. The permissible ordered binary partitions are $7=2+5$ and $7=5+2$. We take *one* of the 5-pats, say 52143 made from triangles

* We are not concerned here with $m\equiv 0$, mod 3. If you try to wind a 3-pat, for example, you will see that the paper folds back over the tab and the winding cannot proceed. (But see Section 7.)

1, 2, 3, 4, 5, and we take the only 2-pat now to be called 76 because it is made from triangles 6 and 7 in the strip, and combine them. Physically each of these must be turned over to be combined and therefore, sequentially, the 7-pat reads 67,34125. Using, in the same way, the 32541, we get another 7-pat, 67,14523. Or, for the binary partition $7=5+2$, we write the $5=1+4$ sequence, namely 52143, on the triangles 3, 4, 5, 6, 7, thus adding 2 to each member of the sequence which now becomes 74365. Combining this with 21 (reversing each since, physically, they have to be turned over) we arrive at the first sequence corresponding to the partition $7=5+2$, namely, 36745,12. Similarly for the other $7=5+2$, namely, 56347,12. There are no more ordered partitions of 7 and, consequently, no more 7-pats.

We have been doing nothing more than forming regular pats from the definition. Following is a table of regular pats extending through $m=8$. The notation of a pat describes how to wind it. Consider for example the last entry of Table I, and remember the *direction* of winding. Put 5 and 4 together (5 under 4); 7 over 6; 7 (along with 6) over 4; 3 under 2; 8 over 5; 8 (and everything under it) on 2; 6 (and everything under it) on 1.

TABLE I. REGULAR PATS OF DEGREE m

m	Pat	m	Pat
1	1	$8=1+7$	8,5214376
$2=1+1$	2,1		8,2154763
$4=2+2$	34,12		8,2174365
$5=1+4$	5,2143		8,3254176
$=4+1$	3254,1	$=4+4$	6587,2143
$7=2+5$	67,34125	$=7+1$	3265874,1
	67,14523		6325487,1
$=5+2$	36745,12		4365287,1
	56347,12		3285476,1

The *thumbhole* in any pat is that unique place such that each number to the right of it is less than each number to the left. Actually, the comma as it has been used indicates the thumbhole. Since the thumbhole is unique, the comma will now be omitted. The thumbhole separates a pat according to the partitioning used in the winding and is the one place in the physical pat where the thumb can be inserted without encountering a pocket.

Under a pinch east of any flexagon, with pat II north:

(a) Those triangles in I are retained in I but are reversed;

(b) Those triangles above (to the right of) the thumbhole in II are retained in II but are reversed;

(c) Those triangles below (to the left of) the thumbhole in II are slid out of II, without reversal, onto the top of I (reversed).

In the notation of the definition of a flexagon, our example RF_9 becomes (3412,52143). By (a), (b), and (c), the first pinch produces the ordered pair of

regular pats (65872143,1) where the structure of each pat is clearly exhibited. (Of course, we mentally relabel the triangles in writing new pat structures.) This "new pat I" is the one 8-pat derived from the partition $8=4+4$ (See Table I). This corresponds to face $2a$. We rotate, pinch, *etc.* The total sequence is as follows:

$1a$	3412,52143	$2c$	3674512,21		1,85214376
$2ar$	65872143,1	$6ar$	82154763,1	$1cr$	63254871,1
	1,65872143		1,82154763		1,63254871
$3a$	32541,3412	$4cr$	32658741,1	$7a$	21,6734125
$4a$	6714523,21		1,32658741	$3c$	5634712,21
$5ar$	83254176,1	$2c$	21,3674512	$9ar$	82174365,1
	1,83254176	$3a$	3412,32541		1,82174365
$3br$	43652871,1	$1a$	52143,3412	$7cr$	32854761,1
	1,43652871	$7a$	6734125,21		1,32854761
$4a$	21,6714523	$8ar$	85214376,1	$3c$	21,5634712

Each pinch and rotation has produced another flexagon. As a matter of fact, each pat in Table I has been used. This equivalence class of flexagons could have been made originally according to any entry in the above sequence. For example, look at $3a$ 3412,32541; this is an RF_9 wound with the *other* regular 5-pat.

6. Number of regular flexagons. To find the number of regular flexagons of a given order we must first determine the number u_{3k+1} and the number u_{3k+2} of distinct regular pats of degree $3k+1$ and $3k+2$. We have,

$$u_1 = 1, \tag{1}$$

$$u_{3k+1} = u_2u_{3k-1} + u_5u_{3k-4} + \cdots + u_{3k-1}u_2, \tag{2}$$

$$u_{3k+2} = u_1u_{3k+1} + u_4u_{3k-2} + \cdots + u_{3k+1}u_1, \tag{3}$$

where the subscripts describe the ordered partitions. We define the two generating functions $f(x) = \sum_{k=0}^{\infty} u_{3k+1}x^k$ and $g(x) = \sum_{k=0}^{\infty} u_{3k+2}x^k$ and it follows from (1), (2), and (3) that $g(x) = f^2(x)$, $f(x) = 1 + xg^2(x)$, so that

$$f(x) = 1 + xf^4(x), \tag{4}$$

$$g(x) = [1 + xg^2(x)]^2. \tag{5}$$

To compute the coefficients u_{3k+1} and u_{3k+2} we apply Lagrange's inversion formula [8] directly to (4) and after the transformation $h(t) = tg(t^2)$ to (5). The results* are

$$u_{3k+1} = \frac{1}{4k+1}\binom{4k+1}{k}, \qquad u_{3k+2} = \frac{1}{2k+1}\binom{4k+2}{k}.$$

* We are indebted to T. S. Motzkin and Hans Rademacher who, independently, transmitted them to us in correspondence.

The total number U_N of regular flexagons of order N, each being an ordered pair of regular pats, is therefore given by

$$U_N = U_{3\lambda} = 2(u_1u_{3\lambda-1} + u_4u_{3\lambda-4} + \cdots + u_{3\lambda-2}u_2)$$
$$= 2\sum_{\gamma=0}^{\lambda-1} \frac{1}{4\gamma+1}\binom{4\gamma+1}{\gamma}\cdot\frac{2}{4(\lambda-\gamma-1)+2}\binom{4(\lambda-\gamma-1)+2}{\lambda-\gamma-1}.$$

This can be summed by making use of a result of Gould [2], namely: $\sum_{k=0}^{n} A_k(a, b)A_{n-k}(c, b) = A_n(a+c, b)$, where

$$A_m(\alpha, \beta) = \frac{\alpha}{\alpha+\beta m}\binom{\alpha+\beta m}{m}.$$

Therefore,

$$U_{3\lambda} = 2\sum_{\gamma=0}^{\lambda-1} A_\gamma(1, 4)A_{\lambda-\gamma-1}(2, 4) \qquad = \frac{6}{4\lambda-1}\binom{4\lambda-1}{\lambda-1}.$$

But these are divided into equivalence classes, and so in that sense the number of unique regular flexagons is considerably smaller than the number of ways in which they may be constructed. We shall now compute the number of equivalence classes.

Each (physical) triangle must—at some stage of pinching—constitute a pat by itself. For if in any pat P of degree exceeding one, we fix on any particular triangle T, then the flexagon can be held (turned over if necessary) so that T is above the thumbhole in P (north) and a pinch (east) reduces the degree of P which, of course, still contains T. Repetition of this process will reduce the pat P containing T to degree one. (Here, P is used as a generic notation for the new pats containing T; and the turning over of the flexagon causes no loss of generality.)

In each adjacent pair of pats there is a sum of N triangles and each time one of them constitutes a pat by itself, a rotation is necessary to continue the normal course of pinching. Hence, there are at most N rotations in the course of pinching through all the physical faces. Furthermore, for each pair of adjacent pats, each pinch annexes triangles to one of the pats by a half-twist and at the same time removes triangles from the other by means of removing a half-twist from that pat. Since a flexagon of order N is a Möbius band of $3N-6$ half-twists, there will be at most $3N-6$ pinches in running through a class of equivalent flexagons. Therefore, a flexagon runs through at most $4N-6$ stages by means of rotations and pinches. For RF_N it is easy to show that the number of stages is either $4N-6$, (full period), or $(4N-6)/3$, (1/3 period). Period 1/3 occurs when and only when the flexagon is equivalent to $(a_1a_2 \cdots a_m, A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$. For the argument see Section 7.

The number of equivalence classes is therefore $U_N/(4N-6)$ in case the period

is full. But in cases where the period is 1/3 this number must be increased by $2u_m/3$. Where $N=3\lambda$, we now take cases.

(a) $\lambda=3k$. Since no regular pat has degree a multiple of three, no flexagon of period 1/3 is possible in this case and the number of equivalence classes is

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} = \frac{1}{(6k-1)(12k-1)}\binom{12k-1}{3k-1}.$$

(b) $\lambda=3k+1$. Here we must add $2u_{3k+1}/3$.

$$U_N^* = \frac{6}{(12\lambda-6)(4\lambda-1)}\binom{4\lambda-1}{\lambda-1} + \frac{2}{3(3k+1)}\binom{4k}{k}$$
$$= \frac{1}{3(4k+1)(6k+1)}\binom{12k+3}{3k} + \frac{2}{3(3k+1)}\binom{4k}{k}.$$

(c) $\lambda=3k+2$. Similar to (b), we obtain

$$U_N^* = \frac{1}{3(2k+1)(12k+7)}\binom{12k+7}{3k+1} + \frac{4}{3(3k+2)}\binom{4k+1}{k}.$$

7. General flexagons. We now wish to discuss non-regular pats and hence remove the restriction that the order of a flexagon be a multiple of three. Consider the regular pat 3412; if we identify the triangles numbered 3 and 4, we then have a pat of degree three which we may write as $\overline{34}12$, or simply 312. This topological identification of triangles may be executed physically by gluing triangles 3 and 4 together. It is easily seen that such identification may be made on any pair of consecutive integers which are adjacent numbers of the pat. Hence, by identifying 1 and 2 in 3412, we obtain $34\overline{12}$, or simply 231. These are the only possibilities for a general pat of degree three. To illustrate further, we obtain the general pats derivable from the regular 5-pat 52143. These are $5\overline{21}43=4132$, $521\overline{43}=4213$, and $5\overline{21}\overline{43}=312$. All the non-regular pats may be obtained from the regular ones in this manner, and they may be combined without restriction (physically: by gluing when necessary) to form new pats of any degree and flexagons of any order. The definitions of a pat, a flexagon, a pinch, and a rotation are given in Section 4.

Pats of degree m are dependent on the ordered binary partitions of $m=r+s$. For example, two 7-pats are obtained from 4213 and 312 by first taking $a_1a_2a_3a_4=4213$, $b_1b_2b_3=312$ (yielding 6457213) and by next taking $a_1a_2a_3=312$, $b_1b_2b_3b_4=4213$ (yielding 6573124). The pats of degrees 1 through 5 are given by: **1**; **21**; 312, 231; 3241, 2431, **3412**, 4213, 4132; 25341, 24531, **32541**, 42351, 34251, 43512, 35412, 45213, 45132, 51423, 51342, **52143**, 53124, 52314. The regular pats are given in bold face type.

We now obtain the number of non-equivalent flexagons of order N. If v_m is

the number of pats of degree m, then $v_m = \sum_{k=1}^{m-1} v_{m-k}v_k$. This recursive convolution may be solved by consideration of the generating function $\phi = \sum_{k=1}^{\infty} v_k x^{k-1}$, which satisfies $x\phi^2 - \phi + 1 = 0$ and for which we impose the condition $v_1 = 1$ as demanded by the definition of a 1-pat. Again, by use of Lagrange's inversion formula, we find

$$v_m = \frac{1}{2m-1}\binom{2m-1}{m}.$$

The number V_N of flexagons of order N is the sum of the number of ways in which ordered pairs of pats may be taken, and hence

$$V_N = v_N = \frac{1}{2N-1}\binom{2N-1}{N}.$$

Notice that $V_N = \{(4N-6)/N\} V_{N-1}$.

As before, each flexagon may be pinched or rotated into $4N-6$ equivalent flexagons except that the flexagon $(a_1a_2 \cdots a_m,\ A_mA_{m-1} \cdots A_1a_ma_{m-1} \cdots a_1)$ has 1/3 period and the flexagon $(a_1a_2 \cdots a_m,\ a_1a_2 \cdots a_m)$ has 1/2 period. We must show that these are the only special cases to worry about in computing V_N^*, the number of non-equivalent flexagons of order N.

First, the two cases cited never coincide. That is, if $N = 6M$, then no flexagon of the form $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ is equivalent to a flexagon of the form $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$. For this to occur, the $2M$-pat $a_1a_2 \cdots a_{2M}$ would have to be a duplication of an M-pat $P = c_1c_2 \cdots c_M$ (*i.e.*, $a_1a_2 \cdots a_{2M} = C_MC_{M-1} \cdots C_1c_Mc_{M-1} \cdots c_1$), and $b_1b_2 \cdots b_{3M}$ would have to be a triplication of the same pat P. Now if we identify the triangles of P as being the same triangle, then the flexagon $(a_1a_2 \cdots a_{2M},\ A_{2M}A_{2M-1} \cdots A_1a_{2M}a_{2M-1} \cdots a_1)$ becomes (21,3412) and $(b_1b_2 \cdots b_{3M},\ b_1b_2 \cdots b_{3M})$ becomes either (312,312) or (231,231). But (21,3412) is a regular flexagon, and hence is not equivalent to a non-regular one.

Second, let F_N be a flexagon of order N. We must establish that F_N is full period, 1/2 period, or 1/3 period. We consider the case in which F_N is a 1-pat and an $(N-1)$-pat, and no generality is lost since every flexagon is equivalent to one of this type. It is clear that if F_N is equivalent to fewer than $4N-6$ flexagons, it is necessarily equivalent to one which is composed of an m-pat P_m, $m > 1$, and a k-plication of that same P_m. By identifying the triangles of P_m, we arrive at a derived flexagon which is a 1-pat and a k-pat. If $k = 1$, then the derived flexagon is F_2, the hexagon, which has 1/2 period; F_N also has 1/2 period. If $k = 2$, the derived flexagon is RF_3, of 1/3 period and F_N has 1/3 period. If $k > 2$ and if the derived flexagon were full period, then F_N would also be full period. Since this is contrary to the hypothesis, the derived flexagon is not full period, and we repeat the process. Thus, we arrive at a flexagon which is either 1/2 period or 1/3 period, and F_N has the corresponding periodicity. Since RF_N is never of the form (P, P), a *regular* flexagon cannot be of 1/2 period.

Therefore, in computing V_N^*, we must add to V_N the quantity $\frac{1}{2}v_{N/2}V_2^*$ whenever $N\equiv 0$, mod 2, and $\frac{2}{3}v_{N/3}V_3^*$ whenever $N\equiv 0$, mod 3. But since V_2^* and V_3^* are both 1, we have

$$V_N^* = \frac{1}{4N-6}V_N + \left\{\frac{1}{2}v_{N/2}\right\} + \left\{\frac{2}{3}v_{N/3}\right\}$$
$$= \frac{1}{N}V_{N-1} + \left\{\frac{1}{2}v_{N/2}\right\} + \left\{\frac{2}{3}v_{N/3}\right\},$$

where the braces indicate inclusion of that term when and only when applicable.

We now give a table showing some values of the various numbers considered, and it is interesting to note how small the class of regular flexagons is in comparison with the class of all flexagons.

TABLE II

N	u_N	U_N	U_N^*	$V_N(=v_N)$	V_N^*
2	1	0	0	1	1
3	0	2	1	2	1
4	1	0	0	5	1
5	2	0	0	14	1
6	0	6	1	42	4
7	4	0	0	132	6
8	9	0	0	429	19
9	0	30	1	1,430	49
10	22	0	0	4,862	150
11	52	0	0	16,796	442
12	0	182	5	58,786	1,424
13	140	0	0	208,012	4,522
14	340	0	0	742,900	14,924
15	0	1,224	24	2,674,440	49,536
16	969	0	0	9,694,845	167,367
17	2394	0	0	35,357,670	570,285
18	0	8,778	133	129,644,790	1,965,058

8. Remarks. A. There is an essential difference between the class of all flexagons and the subclass of regular ones. The statement in Section 7 that $V_N=v_N$ gives the hint: the general flexagon $(a_1a_2 \cdots a_r,\ b_1b_2 \cdots b_s)$ could be studied and considered as just the pat $A_rA_{r-1} \cdots A_1b_sb_{s-1} \cdots b_1$. The regular flexagon, RF_N, $N\equiv 0$, mod 3, could not be studied by means of studying regular pats since no regular pats have degree $m\equiv 0$, mod 3.

B. Notice that if $P=a_1a_2 \cdots a_m$ is a pat, so also is $P'=a_1'a_2' \cdots a_m'$, where $a_j'+a_{m-j+1}=m+1$. Thus, pats occur in *conjugate pairs* or are self-conjugate (*e.g.*, 3412). To construct P', one may label the triangles in the flexagon strip from right to left instead of from left to right, and then carry out the instructions for winding P. Suppose $F=(P, Q)$ is a model of a flexagon. We have discussed

pinching and rotating F, but another obvious operation is that of revolving F 180° about an axis through opposite vertices. This carries F into the *conjugate flexagon* (Q', P').

C. We have constructed a "one-sided" theory of flexagons, and this has led to a simple analysis by convolutions. In the theory of pats, the conjugate pats arose by considering *ordered* partitions. Hence, in computing V_N^*, we have counted twice those *models* arising from ordered binary partitions with unequal components. However, when the components of the partitions are equal, the models have been counted only once. In order to compute the number W_N^* of models of flexagons of order $N=2M$, we write $W_{2M}^*=(V_{2M}^*+v_M)/2$. There is no other corresponding change in Table II.

D. In much the same manner as one labels symmetry operations on the faces of a regular polyhedron to obtain the group of symmetries of that configuration, one can find groups associated with flexagons. For example, one group associated with F_3 is S_3. These usually turn out to be dihedral groups, and finding them is useful and interesting in the teaching of elementary group theory.

E. The number v_m of flexagons of order m adds to the very long list of combinatorial interpretations (ranging from election possibilities and postage stamps to continued fractions) of the recursion formula $v_1=1$; $v_m=\sum_{k=1}^{m-1} v_{m-k}v_k$ which has appeared—with variations—many times in the literature. Some interpretations are mentioned by Becker [1], and a paper by Motzkin [5] gives interpretations and generalizations. The numbers u_N of this paper constitute a variation, and the relevance of continued fractions is given by Touchard [7]. Other references are given in these papers.

Added in proof: An article by Martin Gardner on flexagons appeared in *Scientific American*, December, 1956. While non-mathematical in nature, the article indicates rather complete unpublished work by the inventors, and is an account of the interesting history of the gadgets.

References

1. H. W. Becker, Discussion of problem 4277, this MONTHLY, vol. 56, 1949, pp. 697–699.

2. H. W. Gould, Some generalizations of Vandermonde's convolution, this MONTHLY, vol. 63, 1956, pp. 84–91.

3. Margaret Joseph, Hexahexaflexagrams, Math. Teacher, vol. 44, 1951, pp. 247–248.

4. F. G. Maunsell, The flexagon and the hexahexaflexagram, Math. Gaz., vol. 38, 1954, pp. 213–214.

5. Th. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bull. Amer. Math. Soc., vol. 54, 1948, pp. 352–360.

6. W. R. Ransom, A six-sided hexagon, Sch. Sci. and Math., vol. 52, 1952, p. 94.

7. Jacques Touchard, Sur un problème de configurations et sur les fractions continues, Canad. J. Math., vol. 4, 1952, pp. 2–25.

8. Whittaker and Watson, A Course of Modern Analysis, 4th Edition, 1927, p. 133.

A DERIVATION OF THE EQUATION FOR A VIBRATING STRING

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Consider an *inextensible* string tautly stretched between two *fixed* points. If the italicized words are understood literally, then no displacement of the string is possible, for any displacement leaving the endpoints fixed would increase the length of some part of the "inextensible" string.

In many derivations of the differential equation for the vibrating string (the one-dimensional wave equation) it is not clear in what way, precisely, the literal meanings of the words "fixed" and "inextensible" are relaxed. We present here a discussion, similar in some respects to one in Osgood's *Advanced Calculus*, in which "fixed" is still understood literally, while "inextensible" is replaced by "slightly extensible" in a sense that will be clarified presently.

The string treated here is a continuous curvilinear distribution of matter of uniform (unstretched) density and uniform and constant elasticity. The string offers no resistance to bending; hence the only body force is the tension acting in the direction of the tangent. The string moves in such a way that no portion ever becomes slack and the elastic limit is never exceeded; the tension is therefore related to the extension by Hooke's law. All other forces are neglected.

The ends of the string are attached to two fixed points whose distance apart exceeds the unstretched length of the string, which may then rest in equilibrium along the line joining these points. This equilibrium position is taken to be horizontal, and the motion is assumed to take place in a vertical plane.

We use rectangular coordinates in the plane of the motion, with x-axis along the equilibrium line and origin at the left endpoint of the string; unit vectors along the axes are denoted by $\mathbf{i}$ and $\mathbf{j}$. We use the same symbol $\mathbf{R}$ both for a point and for its position vector with respect to the origin. Partial differentiation is indicated by subscript letters.

Let the point with equilibrium position $x\mathbf{i}$ be located, at time t, at $\mathbf{R}(x, t) = [x+u(x, t)]\mathbf{i}+v(x, t)\mathbf{j}$. We assume that $\mathbf{R}_{xx}$ and $\mathbf{R}_{tt}$ exist. At each instant the distance along the string from the origin to $\mathbf{R}(x, t)$ is given by

$$s(x) = \int_0^x [(1 + u_\xi)^2 + v_\xi^2]^{1/2} d\xi.$$

An explicit expression for the magnitude τ of the tension will now be obtained. Let a segment have unstretched length Δq, equilibrium length Δx, and displaced length Δs, and let ϵ denote the elasticity (Young's modulus, of dimensions force/elongation/length). In equilibrium, at tension τ_0, Hooke's law states that

$$\tau_0 = \epsilon(\Delta x - \Delta q)/\Delta q;$$

in fact, $x = q(1+\tau_0/\epsilon)$, where x and q are the equilibrium and unstretched lengths measured from the origin. In the displaced position $\tau = \epsilon(\Delta s - \Delta q)/\Delta q$. Eliminate

Δq and let $\Delta x \to 0$, getting $\tau = (\tau_0 + \epsilon)s_x - \epsilon = \tau_0 + (\tau_0 + \epsilon)(s_x - 1)$.

Since the string is never slack, $\tau > 0$. Hence $s_x > \epsilon/(\tau_0 + \epsilon) > 0$, s and x are smooth functions of each other, and $x_s s_x = 1$. (The displaced and equilibrium densities, ρ and ρ_0, are related by the equation $\rho s_x = \rho_0$.) The unit tangent $\mathbf{T}$ is given by

$$\mathbf{T} = [(1 + u_x)\mathbf{i} + v_x\mathbf{j}]x_s = \mathbf{R}_x/s_x,$$

and the tension at $\mathbf{R}$ is $\tau\mathbf{T}$.

The only force acting on a segment of equilibrium length Δx is the difference $\Delta(\tau\mathbf{T})$ between the tensions at the ends of the segment. By the mean value theorem, $\Delta(\tau\mathbf{T}) = (\tau\mathbf{T})_x\Delta x$, where the partial derivative is evaluated at an appropriate point inside the segment. Let the centroid of the displaced segment be denoted by $\overline{\mathbf{R}}$. Then Newton's law states that

$$(\rho_0 \Delta x \overline{\mathbf{R}}_t)_t = (\tau\mathbf{T})_x \Delta x.$$

Cancel Δx and then let $\Delta x \to 0$, getting the differential equation $\rho_0 \mathbf{R}_{tt} = (\tau\mathbf{T})_x$. A short calculation using this equation shows that the equation for angular acceleration is satisfied identically.

In order to obtain a simple explicit expression for $(\tau\mathbf{T})_x$, we impose three further assumptions.

From the expression for τ it is clear that the following assumptions (SE: slightly extensible) are equivalent:

SE 1 $$|\tau - \tau_0| \ll \tau_0;$$

SE 2 $$(\tau_0 + \epsilon)|s_x - 1| \ll \tau_0.$$

Here "$\ll$" means "can be neglected compared with." It follows in particular that $|s_x - 1| \ll 1$. Although SE restricts both the physical characteristics of the string and the kinematic characteristics of the motion, we state it verbally by saying merely that the string is slightly extensible.

Our second assumption, SI, is that the string is "slightly inclined" to the horizontal. More explicitly, if θ is the angle of inclination, we assume that:

SI $$\sin^2 \theta \ll 1, \quad \text{or} \quad v_x^2 \ll s_x^2.$$

It follows that $1 - \cos^2 \theta = 1 - (1 + u_x)^2/s_x^2 \ll 1$, whence, *a fortiori*, $1 - |1 + u_x|/s_x \ll 1$.

Finally, let us record explicitly the restrictions on the second derivatives actually used in order to arrive at the wave equation:

A (1) $$|v_x v_{xx}| \ll |u_{xx}| \quad \text{and} \quad (2) \quad \epsilon|v_x u_{xx}| \ll \tau_0 |v_{xx}|.$$

No clear physical or geometric motivation for this assumption has appeared. It is a useful consequence of A(1), in the presence of SE and SI, that $s_{xx} = u_{xx}$ approximately, as is clear from the equation $s_x s_{xx} = (1 + u_x)u_{xx} + v_x v_{xx}$.

Returning to the differential equation, we write

$$\begin{aligned}\rho_0\mathbf{R}_{tt} &= (\tau\mathbf{T})_x = [(\tau_0 + \epsilon - \epsilon/s_x)\mathbf{R}_x]_x \\ &= (\tau_0 + \epsilon - \epsilon/s_x)\mathbf{R}_{xx} + (\epsilon s_{xx}/s_x^2)\mathbf{R}_x,\end{aligned}$$

and approximate s_{xx} by u_{xx}. The result for the longitudinal component is

$$\rho_0 u_{tt} = [\tau_0 + \epsilon - (\epsilon/s_x)(1 - (1 + u_x)/s_x)]u_{xx}.$$

By SE and SI the term involving the parentheses may be neglected in comparison with ϵ, leaving $\rho_0 u_{tt} = (\tau_0 + \epsilon)u_{xx}$.

The result for the transverse component is

$$\rho_0 v_{tt} = (\tau_0 + \epsilon - \epsilon/s_x)v_{xx} + (\epsilon v_x/s_x^2)u_{xx}.$$

By SE, $(s_x - 1)/s_x$ may be replaced by $s_x - 1$ and then, by SE(2), $\epsilon(s_x - 1)$ may be neglected in comparison with τ_0. Finally, by SE and A, the last term may be neglected in comparison with the first, leaving $\rho_0 v_{tt} = \tau_0 v_{xx}$.

Thus the desired equations have been obtained. One notes that the speeds of propagation are different.

It may be of some interest to verify, as would indeed be expected, that these equations are the Euler equations for the problem of minimizing the integral

$$\int_{t_0}^{t_1} (T - V)dt,$$

where T and V are the kinetic and potential energies of the string.

If a segment of the string has unstretched length a then the work done in stretching it from length $z > a$ to length $z + \Delta z$ is $\tau\Delta z = \{\epsilon(z - a)/a\}\Delta z$. Hence the work done in stretching it from length a to length b is

$$\int_a^b \{\epsilon(z - a)/a\}dz = (\epsilon/2a)(b - a)^2.$$

In our earlier notation, then, the energy in a segment of equilibrium length Δx is $(\epsilon/2\Delta q)(\Delta s - \Delta q)^2 = (\epsilon/2q_x)(s_x - q_x)^2\Delta x = \{\tau^2/2(\tau_0 + \epsilon)\}\Delta x$.

Hence, if l denotes the equilibrium length of the string,

$$V = 1/\{2(\tau_0 + \epsilon)\} \int_0^l \tau^2 dx$$

and the integral to be minimized is

$$\frac{1}{2}\int_{t_0}^{t_1}\int_0^l [\rho_0(u_t^2 + v_t^2) - \tau^2/(\tau_0 + \epsilon)]dxdt.$$

The Euler equations are now easily seen to be precisely the components of the equation $(\rho_0\mathbf{R}_t)_t = (\tau\mathbf{R}_x/s_x)_x$ obtained above.

THE EXPONENTIAL FUNCTION

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The purpose of this paper is to discuss a slightly unorthodox definition of the exponential function.† This definition allows us to derive the basic properties of e^x and $\log x$ by very elementary means. During the discussion I will make use of the following four results:

AI. If $a>b>0$, then $a^n>b^n$.

AII. If $1+x>0$, then $(1+x)^n \geqq 1+nx$.

AIII. $\lim\limits_{n\to\infty} (1 + x/n^2)^n = 1$.

AIV. If $f(x)$ has a positive derivative for all x, then f has an inverse function g defined for all values of $f(x)$, and if $y=f(x)$ then $g'(y)=1/f'(x)$.

The first two results are simple exercises in mathematical induction. The third result can be established by bounding the expression by $1+x/n$ from below and by 1 or $1/(1-x/n)$ from above. Finally, AIV is a standard theorem in the calculus.

As motivation for the discussion I suggest a consideration of compound interest and radioactive decay. At simple interest, of rate x, a dollar yields $(1+x)$; if compounded twice, it yields $(1+x/2)^2$; and if we keep halving the time interval, we arrive at the formula $(1+x/2^n)^{2^n}$. If x is negative, then the same formula applies to decay. The function $E(x)$ defined below may be thought of as interest compounded continuously or as continuous decay.

Definitions.

$$E_n(x) = (1 + x/2^n)^{2^n}. \tag{1}$$

$$F_n(x) = 1/E_n(-x). \tag{2}$$

$$E(x) = \lim_{n\to\infty} E_n(x). \tag{3}$$

$$F(x) = \lim_{n\to\infty} F_n(x). \tag{4}$$

The first definition holds for all x and n. The second holds for any x for sufficiently high n, high enough to guarantee that $E_n(x)$ is positive. That the two limits exist will be shown below.

THEOREM I. *$E_n(x)$ is identically 1 if $x=0$; monotone increasing in n for other values of x, for sufficiently high values of n.*

This result can be read off from the identity $E_{n+1}(x)=(1+x/2^n+x^2/2^{2n+2})^{2^n}$.

* I wish to express my thanks to James K. Schiller, on whose notes covering a lecture of mine I have based this article.

† The definition was suggested by E. Artin.

For sufficiently high n, $1+x/2^n>0$, and $E_{n+1}(x)>E_n(x)$ by AI, the equality holding only for $x=0$. Hence all we need to show, to prove convergence, is that $E_n(x)$ is bounded from above.

THEOREM II. *$F_n(x)$ is identically 1 if $x=0$; monotone decreasing in n for other values of x, for sufficiently high values of n.*

For sufficiently high n we have that $E_{n+1}(x)>E_n(x)>0$. Hence $1/E_{n+1}(x) <1/E_n(x)$. This holds for all x, including $-x$, which yields the theorem.

THEOREM III. *$E_n(x)<F_n(x)$ for all sufficiently high n.*

This follows from the identity $E_n(x)/F_n(x)=(1-x^2/2^{2n})^{2^n}<1$.

THEOREM IV. *The limits $E(x)$ and $F(x)$ exist and are positive for all x.*

They are limits of monotone increasing (decreasing) sequences which are bounded from above (below), and which stay above 0 for sufficiently large n.

THEOREM V. $F(x)=E(x)$.

We computed $E_n(x)/F_n(x)$ above. By AIII this tends to 1. Hence the numerator and the denominator have the same limit.

THEOREM VI. $E(-x)=1/E(x)$, $E(0)=1$.

The first result follows from the fact that $E_n(-x)=1/F_n(x)$, and Theorem V. The value of $E(0)$ can be found from this, or from the fact that $E_n(0)\equiv 1$.

THEOREM VII. $1+x\leqq E(x)\leqq 1/(1-x)$, *the right side holding for $x<1$.*

The left side inequality is easily derived from the definition of $E_n(x)$ and AII. Then we can substitute $-x$ for x in the inequality, and take reciprocals to get the right side.

THEOREM VIII. $E(x)\cdot E(y)=E(x+y)$.

If we let $x+y+z=0$ and compute $E_n(x)\cdot E_n(y)\cdot E_n(z)$, we find that it is of the form $(1+k/2^{2n})^{2^n}$, and hence by AIII it tends to 1. This means that $E(x)\cdot E(y)=1/E(z)=E(-z)$. But $-z=x+y$.

THEOREM IX. $D_xE(x)=E(x)$.

From the previous theorem, $[E(x+h)-E(x)]/h=E(x)[E(h)-1]/h$. Hence we know that the derivative is proportional to $E(x)$; we need only find the proportionality constant, which is $\lim_{h\to 0} [E(h)-1]/h$. For small h, Theorem VII gives the bounds $(1+h-1)/h$ and $(1/(1-h)-1)/h$ for the quotient. Both bounds tend to 1 as h tends to 0.

THEOREM X. *$E(x)$ is continuous, monotone increasing, concave upward.*

Since a derivative exists everywhere, the function is continuous. Since $E(x)$ is positive and is both the first and second derivative of itself, the rest of the

theorem follows.

THEOREM XI. $E(x)^y = E(xy)$.

If y is an integer, then $E(x)^y$ is defined by repeated multiplication. The theorem follows from repeated use of Theorem VIII. If $y = p/q$, then $E(x)^y$ is the positive number which, when raised to the qth power, yields $E(x)^p$. It is easy to verify that $E(xp/q)$ has this property. Irrational powers are defined as limits of rational powers. Since the above equality holds for all rational y, we can infer its truth for all y from the continuity of the function E.

We have now found the most important properties of the function E. It is worth noting that we also have a method for getting good numerical approximations for $E(x)$. By Theorem VI we may restrict ourselves to positive x. Since $E(x) = E(x/k)^k$, we may restrict ourselves to very small x. For these, Theorem VII furnishes a very good approximation, namely $(1 + x/k)^k$. This provides the connection with the customary definition.

From Theorem XI we see that $E(x) = E(1)^x$. Hence E is an exponential function. Let us call the base $E(1) = e$, then $E(x) = e^x$. (Among other possible definitions of e, we find that e is the yield of one dollar compounded continuously at 100% interest for a year.) We can now restate Theorems VI–XI in more traditional form.

From AIV we can infer that e^x has an inverse function defined for all positive values of x. (It is easy to see from Theorem VII that e^x takes on all positive values.) This function is $\log x$, the natural logarithm. The usual properties of the logarithm follow from the properties of e^x, together with the fact that $\log x$ is its inverse function. I will consider only two analytic properties.

THEOREM XII. $D_x \log x = 1/x$.

From AIV, $D_y \log y = 1/D_x e^x = 1/e^x = 1/y$. The theorem follows by putting x for y.

THEOREM XIII. *The function log is continuous, monotone increasing, and concave downward.*

Continuity follows from the existence of a derivative. Since the derivative $1/x$ is positive (for $x > 0$ where log is defined), the function is monotone increasing. Since the second derivative $-1/x^2$ is clearly negative, the function is concave downward.

The discussion should be completed by considering a^x and $\log_a x$ for any positive a. This is easily accomplished once we know that e^x takes on all positive values. Then $a^x = e^{(\log a)x}$, and all the properties of a^x and $\log_a x$ follow from the properties discussed above.

This approach to the exponential and logarithmic functions has appealed to me as one suitable for a bright freshman class, allowing me to prove all the important properties of these functions, without using up an undue amount of classroom time.

PARTIAL LINEAR DIFFERENCE EQUATIONS

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1. We shall consider a linear difference equation with two independent variables only, although much that is written can be immediately generalized. The independent variables are restricted to integral values and the coefficients of the dependent variable are defined at points of a rectangle called *defining points*. The rectangle will be referred to as the *defining rectangle*. The equation itself serves to connect the values of the dependent variable at certain neighboring points in the plane of the independent variables. It is frequently most easily described by a pattern. Thus an equation,

$$
\begin{aligned}
(1) \qquad & k_1(i,j)y(i-1,j) + k_2(i,j)y(i,j-1) + k_3(i,j)y(i+1,j) \\
& + k_4(i,j)y(i,j+1) + k_5(i,j)y(i,j) = R(i,j),
\end{aligned}
$$

is readily described by means of the five point pattern

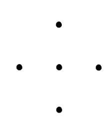

FIG. 1

Equation (1) will be referred to as a five point equation. Here, as usual, the points of the pattern will be said to cluster about (i, j). We may simply refer to the pattern of the equation or the pattern about (i, j). The terms linear, homogeneous, *etc.*, have the usual import.

DEFINITION 1. *The* order *of a linear difference equation is the maximum number of steps required to get from one point of the pattern to another where, moreover, the coefficients of the dependent variable in the outlying points of the outlying rows and columns of the pattern are never zero.*

Thus equation (1) is of the second order provided that neither k_1, k_2, k_3, nor k_4 is ever zero. The equation,

$$
\begin{aligned}
& k_1(i,j)y(i,j) + k_2(i,j)y(i+1,j) + k_3(i,j)y(i+2,j) \\
(2) \qquad & + k_4(i,j)y(i,j+1) + k_5(i,j)y(i+1,j+1) + k_6(i,j)y(i+2,j+1) \\
& + k_7(i,j)y(i,j+2) + k_8(i,j)y(i+1,j+2) + k_9(i,j)y(i+2,j+2) = 0,
\end{aligned}
$$

is described by a nine point pattern as illustrated:

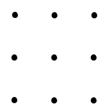

FIG. 2

Equation (2) is of the fourth order provided neither k_1 nor k_3 nor k_7 nor k_9 is ever zero.

The requirement of the nonvanishing of coefficients of outlying terms seems somewhat artificial at first and is, in fact, at times more than is necessary. However, it is to be recognized that in this work there is nothing corresponding to a singular point of a differential equation. The vanishing of a leading coefficient simply destroys the equation at the point in question.

It is not found advisable to describe the equation that we are considering more precisely than to say that it is linear of fixed patterns. We shall denote it by

$$L(y) = R(i, j). \tag{3}$$

DEFINITION 2. *We shall designate by the term* basic points *those points which fall within the pattern when* (i, j) *is any defining point.*

For equation (1) the basic points constitute a rectangle without the four corner points. It is pictured in Figure 3 for a five by seven defining rectangle. Note that the basic rectangle is seven by nine except for corner points.

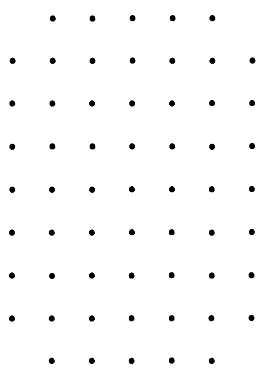

FIG. 3

On the other hand the basic points for an equation described by the nine point pattern of Figure 2 constitute all points of a rectangle.

We shall universally mean by the term *solution* a function that satisfies the equation at all points of a defining rectangle. However, it should be noticed that the solution is determined at more points than the coefficients are defined at,

namely the basic points. The reader will also notice that much that is said applies to other regions than rectangles.

DEFINITION 3. *A* fundamental domain *for a particular equation consists of points such that there exists one and only one solution taking on arbitrary prescribed values at each point of the domain.*

A fundamental domain for equation (1) with defining rectangle determined by $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$ consists of the points $(i, 0)$, $(i, 1)$, $i = 1, 2, \cdots, m$ and $(0, j)$, $(m+1, j)$, $j = 1, 2, \cdots, n$. These points can be described as the two bottom rows of the basic rectangle plus the first and last columns. There are other fundamental domains for this equation. To prove that the points listed constitute a fundamental domain simply mark them on the basic rectangle and notice that values for y at all points can be successively determined. We, of course, are assuming that k_1, k_2, k_3, k_4 are never zero. As a second illustration consider the nine point equation given by (2) with the same defining points. We readily show that the points of the two bottom rows and first two columns of the basic rectangle form a fundamental domain.

THEOREM 1. *If an equation involves a point pattern containing points from p consecutive columns and q consecutive rows then the points of $p-1$ columns and $q-1$ rows of the basic rectangle chosen as now to be described form a fundamental domain.*

Let the points in the top row of the pattern farthest to the right be $(i+s$ $j+q-1)$ assuming (i, j) in the lower left corner. Then the following points form a fundamental domain; $q-1$ rows at the bottom of the basic rectangle; s columns at the extreme left and $p-s-1$ columns at the extreme right. Proof is by constructing the basic rectangle, marking the points just described and then filling out the basic rectangle point by point by means of the equation. There are other fundamental domains.

2. The homogeneous equation. Terms in common use with reference to differential or other types of equations will be used. It will not always be thought necessary to take the space to define them. The same will hold for simple theorems common to other homogeneous equations.

Suppose a certain fundamental domain for a homogeneous equation to consist of t points.

DEFINITION 4. *A* fundamental system *of solutions consists of t solutions, $y_1, \cdots, y_t$ which are such that when taken over the t points of some fundamental domain they constitute t sets of t linearly independent constants.*

For this it is, of course necessary and sufficient that the determinant of the t sets of the constants be different from zero.

THEOREM 2. *Any solution is a linear combination with constant coefficients of a fundamental system of solution.*

It is immediate that any solution is expressible linearly in terms of a fundamental system over a fundamental domain pertaining to that fundamental system. However, the determination of all values from the values over the fundamental domain is, by hypothesis, possible. This determination is from a set of linear equations and can be done by Cramer's rule. This will preserve the linearity.

THEOREM 3. *All fundamental systems contain the same number of solutions.*

Suppose $y_1, y_2, \cdots, y_t$ to be a fundamental system of solutions and $y_1', y_2', \cdots, y_{t'}'$ to be a second set. Suppose $t'>t$. We know that

$$y_1' = c_{11}y_1 + c_{12}y_2 + \cdots + c_{1t}y_t,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_{t'}' = c_{t'1}y_1 + c_{t'2}y_2 + \cdots + c_{t't}y_t.$$

This is true throughout the basic rectangle and hence in particular on a fundamental domain of $y_1', \cdots, y_{t'}'$. But $t'>t$. Hence $y_1', \cdots, y_{t'}'$ are linearly dependent over a fundamental domain of theirs. This contradicts our definition of a fundamental system. Hence $t' \leqq t$. Similarly $t \leqq t'$.

THEOREM 4. *All fundamental domains for a given equation contain the same number of points.*

This follows from the previous theorem since the number of solutions in a fundamental system is the same as the number of points in its fundamental domain.

THEOREM 5. *Let $y_1, y_2, \cdots, y_t$ be a fundamental system of solutions and let $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$ be a fundamental domain for them. Let $y_1', y_2', \cdots, y_t'$ be a second fundamental system. Then $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$ are a fundamental domain for $y_1', y_2', \cdots, y_t'$.*

If we assume the contrary to this theorem, then $y_1', y_2', \cdots, y_t'$ are linearly dependent over $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$. But $y_1 = d_{11}y_1' + d_{12}y_2' + \cdots + d_{1t}y_t', \cdots, y_t = d_{t1}y_1' + d_{t2}y_2' + \cdots + d_{tt}y_t'$. Hence $y_1, \cdots, y_t$ are linearly dependent over $(i_1, j_1), (i_2, j_2), \cdots, (i_t, j_t)$. This contradicts our hypothesis.

DEFINITION 5. *Let $y_1, \cdots, y_t$ be solutions of an equation and let a fundamental domain for the equation contain t points. Let $(i_1, j_1), \cdots, (i_t, j_t)$ be any t points. Then the t by t determinant,* det $(y_m(i_n, j_n))$ *is called the* Wronskian *of $y_1, \cdots, y_t$. We denote it by $W(y_1, \cdots, y_t)$.*

THEOREM 6. *A necessary and sufficient condition that $y_1, \cdots, y_t$ constitute a fundamental system is that $W(y_1, \cdots, y_t)$ does not vanish on any fundamental domain.*

Proof is immediate.

DEFINITION 6. *A formula which satisfies a difference equation and which has every solution of the difference equation as a special case is called the general solution.*

THEOREM 7. *If $y_1, \cdots, y_t$ constitute a fundamental system of solutions then $c_1y_1+\cdots+c_ty_t$, where $c_1, \cdots, c_t$ are arbitrary constants is the general solution.*

3. The non-homogeneous equation.

THEOREM 8. *The general solution of a non-homogeneous equation consists of a particular solution plus the general solution of the reduced (homogeneous) equation.*

4. Auxiliary conditions.

Write equation (3) for each defining point (i, j). Suppose that there are mn of them. We next assume that the basic rectangle contains $mn+\tau$ points. What we have is mn linear equations in $mn+\tau$ variables. We expect to be able, in some instances at least, to impose τ additional linear conditions and have all $mn+\tau$ equations satisfied. The most usual such conditions are boundary conditions.

We let the auxiliary condition be represented by

$$\bar{V}_1(y) = g_1, \cdots, \bar{V}_\tau(y) = g_\tau. \tag{4}$$

Here $g_1, \cdots, g_\tau$ are constants not all zero. Equations (4) together with (3) will be referred to as the *non-homogeneous* system. We shall call by the name *homogeneous* the system consisting of

$$L(y) = 0, \bar{V}_1(y) = 0, \cdots, \bar{V}_\tau(y) = 0. \tag{5}$$

It is, of course, assumed that $L, \bar{V}_1, \cdots, \bar{V}_\tau$ are linear and homogeneous.

Let $L(y)=0$ be written down for each of the defining points. We denote these equations by $L_{ij}(y)=0$. Now

$$L_{ij}(y) = 0, \bar{V}_1(y) = 0, \cdots, \bar{V}_\tau(y) = 0 \tag{6}$$

constitute $mn+\tau$ linear homogeneous equations in $y(i, j)$.

THEOREM 9. *In order for the equations* (5) *to be satisfied by a solution not identically zero it is necessary and sufficient that the determinant of the coefficients of* (6) *vanish.*

DEFINITION 7. *The determinant of the coefficients of* (6) *is called the* characteristic determinant *for system* (5).

DEFINITION 8. *If the characteristic determinant of system* (5) *vanishes then system* (5) *is said to be* compatible. *It is said to be* incompatible *in the contrary case.*

DEFINITION 9. *System* (5) *is said to be* K-fold compatible *in case the matrix of the coefficients is of rank $mn+1-K$.*

THEOREM 10. *If* (5) *is K-fold compatible there exist K, and not more, linearly independent solutions of system* (5).

Proof is immediate. There may, of course, be more than one set of linearly independent solutions.

We shall now assume (as is certainly many times the case) that the number of auxiliary conditions is the same as the number of points in a fundamental domain ($\tau = t$).

THEOREM 11. *Let $y_1, \cdots, y_t$ be a fundamental system of solutions of $L(y)=0$. Then a necessary and sufficient condition that system* (5) *be compatible is that*

$$\det (V_m(y_n)) = 0. \tag{7}$$

Substitute $c_1y_1+c_2y_2+\cdots+c_ty_t$ in the auxiliary conditions. There result t equations for the determination of $c_1, c_2, \cdots, c_t$. The determinant of the coefficients of the c's is precisely the determinant of the theorem.

Consider next the non-homogeneous system

$$L(y) = R(i, j), \bar{V}_1(y) = g_1, \bar{V}_2(y) = g_2, \cdots, \bar{V}_t(y) = g_t. \tag{8}$$

THEOREM 12. *A necessary and sufficient condition that the system* (8) *have one and only one solution is that the* reduced system (5) *be incompatible.*

Proof follows immediately from Theorem 9.

5. Green's function. Consider the homogeneous system (5).

DEFINITION 10. *By the Green's function for* (5) *we shall mean a function $G(i, j; \xi, \eta)$, which as a function of (i, j) satisfies the auxiliary conditions and also satisfies the difference equation at every point of the defining rectangle except the point (ξ, η), and where, moreover, $L(G(\xi, \eta: \xi, \eta))=1$.*

THEOREM 13. *A necessary and sufficient condition that a Green's function $G(i, j; \xi, \eta)$ exist for every point of the defining rectangle is that* (5) *be incompatible.*

This theorem is again an immediate consequence of Theorem 9.

6. Construction of the Green's function for the five-point equation. We shall consider in some detail the construction of the Green's function for equation (1) subject to auxiliary conditions. The methods employed will be applicable to many other equations.

The defining rectangle is the rectangle determined by $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$. These points plus the boundary points $(0, j)$, $(m+1, j)$, $j=1, \cdots, n$ and $(i, 0)(i, n+1)$, $i=1, \cdots, m$ form what we have called the basic rectangle. Fundamental domains consists of certain $2m+2n$ points. Hence $t=2m+2n$.

The auxiliary conditions are $\bar{V}_1(y)=0, \cdots, \bar{V}_t(y)=0$.

Let $y_1, \cdots, y_t$ be a fundamental system of solutions. Let $u_1=c_1y_1+c_2y_2+\cdots+c_ty_t$ and $u_2=d_1y_1+d_2y_2+\cdots+d_ty_t$. Assume $G(i, j; \xi, \eta)=u_1$ when $j \leqq \eta$ and $G(i, j; \xi, \eta)=u_2$ when $j>\eta$. We attempt to determine the c's and the d's so that $u_1=u_2$ at all points where $j=\eta$ and where $j=\eta+1$ except at the point

$(\xi, \eta+1)$ where $u_1-u_2=1/k_4(\xi, \eta)$ also, so that $u_1=u_2$ at all points of the first and last columns of the basic rectangle, that is, at the points $(0, j)$ and $(m+1, j)$, $j=1, \cdots, n$. If this is done $G(i, j; \xi, \eta)$ will satisfy the difference equation at all points of the defining rectangle except (ξ, η). At this point when G is substituted in the equation the result 1 is obtained as desired. To show that such a determination of the c's and d's is possible, let $z_v=c_v-d_v$, $v=1, 2, \cdots, t$. Then substitute u_1 and u_2 in the conditions just enumerated and transpose. We have $z_1y_1(i, j)+\cdots+z_ty_t(i, j)=0$, where (i, j) runs through all points on the lines $j=\eta$, $j=\eta+1$ except $(\xi, \eta+1)$ and also at all points on the lines $i=0$ and $i=m+1$. Moreover, $z_1y_1(\xi, \eta+1)+z_2y_2(\xi, \eta+1)+\cdots+z_ty_t(\xi, \eta+1)=1/k_4(\xi, \eta)$. The points which we have just described constitute a fundamental domain. We consequently can solve for $z_1, z_2, \cdots, z_t$. Now as to auxiliary conditions! Assign any values to the d's. Substitute in the auxiliary conditions and transpose all c's to the left-hand member and constant terms to the right-hand member. Solution for the c's is now possible since the determinant of the coefficients of the c's is precisely the determinant in (7) which is not zero, since the system is incompatible.

7. An application of Green's function. Consider the system consisting of

$$L(y) = R(i, j),\ \bar{V}_1(y) = 0,\ \bar{V}_2(y) = 0, \cdots, \bar{V}_t(y) = 0. \tag{9}$$

We speak of this as the *semihomogeneous* system.

THEOREM. *If the system* (5) *is incompatible then the system* (9) *has one and only one solution which is given by*

$$y(i, j) = \sum_{\eta=1}^{n}\sum_{\xi=1}^{m} G(i, j; \xi, \eta)R(\xi, \eta).$$

Proof is a matter of direct verification:

$$L(y(i, j)) = L\left[\sum_{\eta=1}^{n}\sum_{\xi=1}^{m} G(i, j; \xi, \eta)R(\xi, \eta)\right]$$
$$= \sum_{\eta=1}^{n}\sum_{\xi=1}^{m} L[G(i, j; \xi, \eta)]R(\xi, \eta) = R(i, j).$$

This follows since $L[G(i, j; \xi, \eta)]=0$ at all points except where $\xi=i$, $\eta=j$. Here it is 1 by definition of the Green's function. The satisfaction of the auxiliary conditions is immediate. The fact that there is no other solution follows from Theorem 9.

SEMIMAGIC SQUARES AND NON-SEMISIMPLE ALGEBRAS

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1. Recently, L. M. Weiner [1] wrote an interesting note about the so-called semimagic squares, which are defined as follows:

DEFINITION. *A semimagic square* $A=(a_{ij})$ *of order* n *is an* n *by* n *matrix for which* $\sum_{i=1}^{n} a_{ij}=\sum_{i=1}^{n} a_{ji}=\sigma(A)$ *for every* j.

The set of all semimagic squares of order n forms a subalgebra $\mathfrak{A}_n$ of the total matric algebra $\mathfrak{M}_n$ of degree n which satisfies the following conditions:

$$\sigma(A+B)=\sigma(A)+\sigma(B), \tag{a}$$

$$\sigma(kA)=k\sigma(A), \tag{b}$$

$$\sigma(AB)=\sigma(A)\sigma(B). \tag{c}$$

He noticed a decomposition of $\mathfrak{A}_n$ into the direct sum of two ideals $\mathfrak{B}_n$ and $\mathfrak{C}_n$. The subalgebra $\mathfrak{B}_n$ consists of all matrices whose elements a_{ij} are all equal, and the subalgebra $\mathfrak{C}_n$, of all matrices whose $\sigma(A)$ is zero. Thus,

$$A=\begin{pmatrix} \sigma/n \cdots \sigma/n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \sigma/n \cdots \sigma/n \end{pmatrix}+\begin{pmatrix} a_{11}-\sigma/n & \cdots & a_{1n}-\sigma/n \\ \cdot & \cdots & \cdot \\ a_{n1}-\sigma/n & \cdots & a_{nn}-\sigma/n \end{pmatrix}. \tag{1}$$

Now, define A_{ij} to be the matrix with 1 in the first row and column, -1 in the first row and jth column, -1 in the ith row and first column, 1 in the ith row and jth column, and 0 elsewhere for $i, j=2, \cdots, n$. Then, as he remarked, the algebra $\mathfrak{C}_n$ has the dimension $(n-1)^2$, and the matrices A_{ij} form a basis for $\mathfrak{C}_n$.

These results are all he mentioned in his paper. But, in the following, it will be shown that $\mathfrak{C}_n$ is isomorphic to the total matric algebra of degree $n-1$ when the characteristic p of the field Ω of elements of matrices does not divide the order n, and that when n is divisible by p, the decomposition (1) does not hold, $\mathfrak{A}_n$ is not semisimple, and there is the radical $\mathfrak{N}$ such that $\mathfrak{N}^3=0$. Moreover, by taking half of the condition, either $\sum_{i=1}^{n} a_{ji}=\sigma(A)$ or $\sum_{i=1}^{n} a_{ij}=\sigma(A)$, for all j, we may obtain larger subalgebras of $\mathfrak{M}_n$, because the conditions (a), (b), (c) also hold. These algebras will be shown to be non-semisimple, having the radical $\mathfrak{N}$ such that $\mathfrak{N}^2=0$. The investigation of these algebras offers us a suitable illustration of the general theory of non-semisimple algebras [2].

2. We start from the half of the condition for semimagic squares.

DEFINITION. *A row-semimagic square* $A=(a_{ij})$ *of order* n *is a matrix of degree* n *for which* $\sum_{j=1}^{n} a_{ij}=\sigma(A)$ *for every* i.

The product C of two row-semimagic squares A and B is also row-semimagic, because

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} a_{ik}\sigma(B) = \sigma(A)\sigma(B).$$

Thus, the set of all row-semimagic squares of order n forms an algebra, which we denote by $\mathfrak{A}_n$, but, as we shall consider always a fixed order n, we may omit the subscript and simply write $\mathfrak{A}$.

Now consider the matrix $P=(p_{ij})$, where $p_{ii}=p_{in}=1$ $(i=1, \cdots, n)$ and $p_{ij}=0$ otherwise. When we transform a row-semimagic matrix A by P, then $A'=P^{-1}AP$ is given by

$$(2)\qquad A' = \begin{bmatrix} a_{11}-a_{n1} & \cdots & a_{1,n-1}-a_{n,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a_{n-1,1}-a_{n1} & \cdots & a_{n-1,n-1}-a_{n,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & \sigma \end{bmatrix}.$$

The set $\mathfrak{A}'$ of all $A'=(a'_{ij})$ forms an algebra isomorphic to $\mathfrak{A}$ and $\mathfrak{A}'$ consists of all matrices whose elements in the last column are all 0 except for a'_{nn}. Therefore, the dimension of $\mathfrak{A}$ is $n^2-(n-1)$. Now, we have

$$(3)\qquad A' = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sigma \end{bmatrix} + \begin{bmatrix} a'_{11} & \cdots & a'_{1,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix}.$$

The sum of the set of all matrices of the first and second types forms a subalgebra $\mathfrak{B}'$ of $\mathfrak{A}'$. The set $\mathfrak{N}'$ of all matrices of the third type is an ideal of $\mathfrak{A}'$, whereas $\mathfrak{B}'$ is not an ideal. Moreover, $\mathfrak{N}'$ is nilpotent and $\mathfrak{N}'^2=0$. The residue-class ring $\mathfrak{A}'-\mathfrak{N}'$ is isomorphic to $\mathfrak{B}'$, and is semisimple because $\mathfrak{B}'$ is the direct sum of a one-dimensional subalgebra and a subalgebra isomorphic to the total matric algebra of degree $n-1$. Hence, $\mathfrak{N}'$ is the radical of $\mathfrak{A}'$, and $\mathfrak{A}'=\mathfrak{B}'+\mathfrak{N}'$ is the supplementary sum of $\mathfrak{B}'$ and $\mathfrak{N}'$.

We return now to the algebra $\mathfrak{A}$. Here $\mathfrak{N}=P\mathfrak{N}'P^{-1}$ is the set of all matrices for which $\sigma(A)=0$ and $a_{1j}= \cdots =a_{nj}$ for $j=1, \cdots, n-1$, and consequently for $j=n$. Thus $\mathfrak{N}$ consists of all matrices of the type

$$(4)\qquad (c_{ij}),\ \text{where } c_{ij} = c_j\ (i, j = 1, \cdots, n) \text{ and } \sum_{j=1}^{n} c_j = 0.$$

Corresponding to (3) we have the following decomposition of $\mathfrak{A}$:

$$A = \begin{pmatrix} 0 & \cdots & 0 & \sigma \\ \cdot & \cdot\cdot\cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \sigma \\ 0 & \cdots & 0 & \sigma \end{pmatrix} + \begin{pmatrix} a_{11}-a_{n1} & \cdots & a_{1n}-a_{nn} \\ \cdot & \cdots & \cdot \\ a_{n-1,1}-a_{n1} & \cdots & a_{n-1,n}-a_{nn} \\ 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma \\ \cdot & \cdots & \cdot & \cdot \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn}-\sigma \end{pmatrix}.$$

THEOREM 1. *The set of all row-semimagic squares of order n forms a non-semisimple algebra and has the radical $\mathfrak{N}$ such that $\mathfrak{N}^2=0$. The radical $\mathfrak{N}$ consists of all matrices of the type* (4), *and $\mathfrak{A}-\mathfrak{N}$ is the direct sum of a one-dimensional subalgebra and a simple subalgebra isomorphic to the total matric algebra of degree $n-1$. We can take a subalgebra $\mathfrak{B}$ such that a supplementary decomposition, $\mathfrak{A}=\mathfrak{B}+\mathfrak{N}$, holds* [3].

Obviously, we may form another decomposition like the above, so that such an algebra $\mathfrak{B}$ is not uniquely determined.

3. We now consider the semimagic squares defined in Section 1. In this case, the transformed matrix A' of (2) must satisfy

$$\sum_{i=1}^{n-1} a'_{ij} = \sigma(A) - na_{nj} \qquad (j = 1, \cdots, n-1). \tag{5}$$

In this section we first consider the case in which the characteristic p of the field Ω of elements of matrices does not divide the order n. Then, from (5), we have

$$a_{nj} = \sigma(A)/n - \sum_{i=1}^{n-1} a'_{ij}/n \qquad (j = 1, \cdots, n-1),$$

so that the matrix A' is completely determined by its elements a'_{ij} for $i, j = 1, \cdots, n-1$ and $a'_{nn}=\sigma(A)$. Therefore, the dimension of $\mathfrak{A}$ is $(n-1)^2+1$.

Now consider the matrix $Q=(q_{ij})$, $q_{ii}=1$ $(i=1, \cdots, n)$, $q_{nj}=-1/n$ $(j=1, \cdots, n-1)$, and $q_{ij}=0$ otherwise. When we transform A' by Q, then $A''=Q^{-1}A'Q$ is given by

$$A'' = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdot & \cdot\cdot\cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sigma \end{pmatrix} + \begin{pmatrix} a'_{11} & \cdots & a'_{1,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Denote the set of all matrices of the first and second types by $\mathfrak{L}''$ and $\mathfrak{M}''$, respectively. Then, $\mathfrak{L}''$ and $\mathfrak{M}''$ are both two-sided ideals of $\mathfrak{A}''$, and $\mathfrak{A}''$ is the direct sum of $\mathfrak{L}''$ and $\mathfrak{M}''$. With reference to the algebra $\mathfrak{A}$, the ideal $\mathfrak{L}$ consists of all matrices for which $a_{ij}=a_{nj}=\sigma(A)/n$, namely, all elements are equal. The ideal $\mathfrak{M}$ consists of all matrices whose $\sigma(A)$ is 0. Thus we obtain the decomposition (1).

THEOREM 2. *In case $p\nmid n$, the algebra $\mathfrak{A}$ of all semimagic squares of order n is semisimple, and $\mathfrak{A}$ may be split into the direct sum of two ideals $\mathfrak{L}$ and $\mathfrak{M}$, where $\mathfrak{L}$ is a one-dimensional subalgebra and $\mathfrak{M}$ is a subalgebra isomorphic to the total matric algebra of degree $n-1$. This decomposition is that of* (1).

4. In the case when n is divisible by p, condition (5) becomes

$$\sum_{i=1}^{n-1} a'_{ij} = \sigma(A).$$

Therefore, in this case the algebra $\mathfrak{A}$ of semimagic squares of order n is isomorphic to the algebra of all matrices of the following type:

$$A' = \begin{pmatrix} a'_{11} & \cdots & a'_{1,n-1} & 0 \\ \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & \cdots & a'_{n-1,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & \sigma \end{pmatrix},$$

where, after removing the last row and column, A' becomes a column-semimagic square of order $n-1$.

Now consider the matrix $R=(r_{ij})$, where $r_{ii}=1$ $(i=1,\cdots,n)$, $r_{1j}=-1$ $(j=2,\cdots,n-1)$, and $r_{ij}=0$ otherwise. Then we have

$$A'' = R^{-1}A'R = \begin{pmatrix} \sigma & 0 & \cdots & 0 & 0 \\ a'_{21} & a'_{22}-a'_{21} & \cdots & a'_{2,n-1}-a'_{21} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a'_{n-1,1} & a'_{n-1,2}-a'_{n-1,1} & \cdots & a'_{n-1,n-1}-a'_{n-1,1} & 0 \\ a_{n1} & a_{n2}-a_{n1} & \cdots & a_{n,n-1}-a_{n1} & \sigma \end{pmatrix}.$$

The set $\mathfrak{A}''$ of all these matrices $A''=(a''_{ij})$ forms an algebra isomorphic to $\mathfrak{A}$, and $\mathfrak{A}''$ consists of all matrices whose elements in the first row and in the last column are all 0 except for a''_{11} and a''_{nn}. Now, the set of all matrices of the first type in (6) is a two-sided ideal $\mathfrak{N}''$ of $\mathfrak{A}''$, and is nilpotent with $\mathfrak{N}''^3=0$, as may easily be verified.

$$(6)\qquad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a''_{21} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a''_{n-1,1} & 0 & \cdots & 0 & 0 \\ a''_{n1} & a''_{n2} & \cdots & a''_{n,n-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 & \cdots & 0 & 0 \\ 0 & a''_{22} & \cdots & a''_{2,n-1} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & a''_{n-1,2} & \cdots & a''_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & \sigma \end{pmatrix}.$$

Also, $\mathfrak{A}''-\mathfrak{N}''$ is isomorphic to the algebra consisting of all matrices of the second

type in (6). Thus $\mathfrak{A}''-\mathfrak{N}''$ is semisimple and $\mathfrak{N}''$ is the radical of $\mathfrak{A}''$. The set of all matrices of the second type in (6) forms a subalgebra $\mathfrak{B}''$ of $\mathfrak{A}''$, although not an ideal, and $\mathfrak{A}''=\mathfrak{B}''+\mathfrak{N}''$ is the supplementary sum of $\mathfrak{B}''$ and $\mathfrak{N}''$.

If we return to the algebra $\mathfrak{A}$, the matrices of $\mathfrak{N}$ satisfy the conditions:

$$\sigma(A) = 0 \text{ and } a'_{ij} = a_{ij} - a_{nj} = \delta_i \quad (i = 2, \cdots, n-1, j = 1, \cdots, n-1).$$

This second relation holds also for $i=1$ and for $j=n$, since, considering that $p|n$, first for $j=n$,

$$a_{in} = -\sum_{j=1}^{n-1} a_{ij} = -\sum_{j=1}^{n-1} a_{nj} - (n-1)\delta_i = a_{nn} + \delta_i,$$

and next for $i=1$,

$$a_{1j} = -\sum_{i=2}^{n} a_{ij} = -(n-1)a_{nj} - \sum_{i=2}^{n-1} \delta_i = a_{nj} - \sum_{i=2}^{n-1} \delta_i = a_{nj} + \delta_1,$$

where $\delta_1 = -\sum_{i=2}^{n-1} \delta_i$. Thus the ith row differs from the nth row by δ_i, for every $i=1, \cdots, n-1$, and $\sum_{i=1}^{n-1} \delta_i=0$. If we put $a_{nj}=c_j+d_n$ for $j=1, \cdots, n$ and $d_i=\delta_i+d_n$ for $i=1, \cdots, n-1$, the matrix A may be written in the form

$$A = C + D = (c_{ij}) + (d_{ij}) = (c_{ij} + d_{ij}),$$

where

$$C = (c_{ij}), \qquad c_{ij} = c_j \ (i, j = 1, \cdots, n), \qquad \sum_{j=1}^{n} c_j = 0, \tag{7}$$

$$D = (d_{ij}), \qquad d_{ij} = d_i \ (i, j = 1, \cdots, n), \qquad \sum_{i=1}^{n} d_i = 0. \tag{8}$$

The sum of the set $\mathfrak{C}$ of all matrices C and the set $\mathfrak{D}$ of all matrices D proves to be the radical $\mathfrak{N}$ of $\mathfrak{A}$.

THEOREM 3. *In the case $p|n$, the algebra $\mathfrak{A}$ of semimagic squares of order n is non-semisimple and has the radical $\mathfrak{N}$ such that $\mathfrak{N}^3=0$. The radical $\mathfrak{N}$ consists of all linear combinations of all matrices* (7) *and* (8), *and $\mathfrak{A}-\mathfrak{N}$ may be split into the direct sum of a one-dimensional subalgebra and a simple subalgebra isomorphic to the total matric algebra of degree $n-2$.*

5. We see that these algebras give us suitable illustrations of the theory of non-semisimple algebras [2]. For example, consider a decomposition into the supplementary sum of indecomposable left ideals about the algebra $\mathfrak{A}$ of row-semimagic squares (Section 2).

Decompose the matrix A of $\mathfrak{A}$ into the sum $A=A_1+ \cdots +A_{n-1}+A_n$, where, for $j=1, \cdots, n-1$, the matrix A_j has $(a_{1j} \cdots a_{nj})^t$ as its jth column, $(-a_{1j} \cdots -a_{nj})^t$ as its nth column, and 0 elsewhere. The matrix A_n has all its

elements 0 except for the elements in the nth column, which are equal to σ. Then, this leads to a decomposition of $\mathfrak{A}$ into the supplementary sum of left ideals $\mathfrak{A}=\mathfrak{l}_1+\cdots+\mathfrak{l}_{n-1}+\mathfrak{l}_n$.

For $j=1, \cdots, n-1$, let E_j denote the matrix which has 1 in its jth row and jth column, -1 in its jth row and nth column, and 0 elsewhere. Then, E_j is an idempotent of $\mathfrak{A}$ and the left ideal it generates is $\mathfrak{l}_j$. The idempotent E_n, with every element of the nth column equal to 1 and every other element equal to 0, generates the ideal $\mathfrak{l}_n$. Thus we have $\mathfrak{l}_j=\mathfrak{A}E_j$ for every j.

The algebra $\mathfrak{A}$ contains the identity matrix as the unit element, which we denote by E, and we have $E=E_1+\cdots+E_{n-1}+E_n$ and $E_iE_j=0$ $(i\neq j)$.

First we consider the left ideals $\mathfrak{l}_1, \cdots, \mathfrak{l}_{n-1}$. If A_j temporarily denotes the matrix with $(a_1 \cdots a_n)^t$ as its jth column, $(-a_1 \cdots -a_n)^t$ as its nth column, and 0 elsewhere, the correspondence $A_j \leftrightarrow A_i$ shows that $\mathfrak{l}_j$ and $\mathfrak{l}_i$ are isomorphic to each other. For each j, $\mathfrak{l}_j$ contains the left ideal $\mathfrak{N}E_j$. Since $\mathfrak{N}E_j$ consists of all matrices with $(c \cdots c)^t$ as the jth column, $(-c \cdots -c)^t$ as the nth column, and 0 elsewhere, it is a one-dimensional vector space over the field Ω, while $\mathfrak{l}_j$ is an n-dimensional vector space.

As for the last ideal $\mathfrak{l}_n$, we have $AE_n=\sigma(A)E_n$, so that $\mathfrak{l}_n$ is also a one-dimensional vector space.

Now, $E_jAE_j=(a_{jj}-a_{nj})E_j$ for $j=1, \cdots, n-1$. Therefore, the ring $E_j\mathfrak{A}E_j$ is isomorphic to the field Ω of elements of matrices. Then, of course, it is completely primary. Hence it follows that the non-nilpotent left ideal $\mathfrak{l}_j=\mathfrak{A}E_j$ is indecomposable [2].

Next, considering the residue-class ring $\bar{\mathfrak{A}}=\mathfrak{A}-\mathfrak{N}$, we have $\bar{\mathfrak{A}}=\bar{\mathfrak{l}}_1+\cdots+\bar{\mathfrak{l}}_{n-1}+\bar{\mathfrak{l}}_n$, where, as an $\mathfrak{A}$-left space, $\bar{\mathfrak{l}}_j$ is isomorphic to $\mathfrak{A}E_j/\mathfrak{N}E_j$ for $j=1, \cdots, n-1$, and $\bar{\mathfrak{l}}_n$ to $\mathfrak{l}_n$. Accordingly, all ideals $\bar{\mathfrak{l}}_j$ are irreducible, $\bar{\mathfrak{l}}_1, \cdots, \bar{\mathfrak{l}}_{n-1}$ are mutually isomorphic, and $\bar{\mathfrak{l}}_1+\cdots+\bar{\mathfrak{l}}_{n-1}$ forms a simple subring of $\bar{\mathfrak{A}}$. If, for $j=1, \cdots, n-1$, we take $\bar{\mathfrak{l}}_j$ as an $\mathfrak{A}$-left space, we obtain the representation of $\mathfrak{A}$ as the total matric algebra of degree $n-1$. Also, if we take $\mathfrak{l}_n$ as an $\mathfrak{A}$-left space, we obtain the representation $A\rightarrow\sigma(A)$ of degree 1.

References

1. L. M. Weiner, The algebra of semimagic squares, this MONTHLY, vol. 62, pp. 237–239.

2. Emil Artin, C. J. Nesbitt, and R. M. Thrall, Rings with Minimum Condition, Univ. Michigan Publ. Math. No. 1, 1946.

3. L. E. Dickson, Algebren und ihre Zahlentheorie, Leipzig, 1929, p. 136.

THE SOLUTION OF LINEAR VECTOR DIFFERENTIAL EQUATIONS CONTAINING GYROSCOPIC TERMS

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1. Introduction. The purpose of this paper is to show how the solution of vector differential equations of the type governing the motion of the bob of a Foucault's pendulum and the motion of an electron in an electrostatic and a uniform magnetic field can be obtained deductively in vector form.

In his *Vectorial Mechanics*, Milne [2] determined the motion of the bob by a process of trial, seeking solutions in terms of rotating vectors. In a recent note to the *Mathematical Gazette*, Easthope [1] showed how the equation could be reduced to the standard simple-harmonic form by referring the motion to suitable rotating axes. The hint for this procedure was given by Milne (*loc. cit.*) and carried out in succeeding applications [3, 4].

Milne did not make the final step of integration, by which the position vector is given as a function of the time, but synthesized [5] it from the already known Cartesian components of rotating vectors. It is shown below that the final result can be obtained deductively by operator methods, after the choice of a suitable reference triad of unit vectors.

Essentially, the problem is a case of finding the solution of a special system of three ordinary linear differential equations in three dependent variables, the components of a vector function $\mathbf{r}$ of the scalar independent variable t; the special character is determined by the presence of a vector-product term, such as is frequently encountered in mechanics. These equations are written in vector-tensor form, in which the derivatives occur as scalar polynomial functions of the operator $D \equiv d/dt$. The differential equations then have the formal appearance of an algebraic system of tensor relations, which suggests an application of the well known theory of characteristic vectors. It turns out that there are three distinct eigenvalues, two of which are complex and conjugate. It is shown that the corresponding eigenvectors (one real and two conjugate complex) are mutually orthogonal. The tensor differential relations are thus reduced to three ordinary differential equations, separately involving the components of $\mathbf{r}$ in the directions of the eigenvectors. After these have been solved by standard operator methods, the solution can be reconstituted into a form independent of this complex reference triad of vectors. The paper concludes with an application of the method to the physical problem that stimulated the investigation.

2. The general inhomogeneous equation. The equation to be considered has the form

$$f(D)\mathbf{r} + 2\boldsymbol{\omega} \times g(D)\mathbf{r} = \mathbf{F}(t), \tag{1}$$

where f and g are polynomial functions having real coefficients, $D \equiv d/dt$, $\boldsymbol{\omega} = \omega\mathbf{k}$ is a constant vector having the direction of the unit vector $\mathbf{k}$, $\mathbf{F}$ a prescribed real vector function of t and $\mathbf{r}$ a function of t, to be determined.

2.1 The complementary function. The second term on the left side of (1), written as the vector product of an axial and a polar vector, may, by introducing the unit tensor $\mathbf{U}$, also be written as the inner product of a proper tensor and the polar vector. Then the homogeneous equation becomes

$$f(D)\mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{U}g(D)\cdot\mathbf{r} = 0, \tag{2}$$

whose solution may be facilitated by seeking directions in which the second term is parallel to the first, *i.e.* the eigenvectors $\mathbf{e}_r$, $r=1, 2, 3$, of the tensor operator $2\boldsymbol{\omega}\times\mathbf{U}g(D)$, corresponding to eigenvalues λ_r. The vector equation may then be resolved into three scalar equations which separately determine the components of $\mathbf{r}$ in the directions of the eigenvectors.

Since $\mathbf{r}=\mathbf{U}\cdot\mathbf{r}$, the latter are the solutions of the equation

$$\left| 2\boldsymbol{\omega} \times \mathbf{U}g - \mathbf{U}\lambda \right| = 0. \tag{3}$$

In terms of Cartesian components of $\boldsymbol{\omega}$ this becomes

$$\begin{vmatrix} -\lambda & -2\omega_3 g & 2\omega_2 g \\ 2\omega_3 g & -\lambda & -2\omega_1 g \\ -2\omega_2 g & 2\omega_1 g & -\lambda \end{vmatrix} = 0,$$

and the corresponding components of the eigenvectors are then proportional to the cofactors of, say, the first row of this determinant, *viz.*

$$(\lambda_r^2 + 4\omega_1^2 g^2,\ 4\omega_1\omega_2 g^2 + 2\lambda_r\omega_3 g,\ 4\omega_3\omega_1 g^2 - 2\lambda_r\omega_2 g).$$

The cubic in λ yields the eigenvalues $\lambda_r=0$, $\pm 2i\omega g$, and the corresponding eigenvectors are proportional to

$$(\omega_1, \omega_2, \omega_3),\ (\omega_1^2 - \omega^2,\ \omega_1\omega_2 \pm i\omega\omega_3,\ \omega_3\omega_1 \mp i\omega\omega_2).$$

The first vector is parallel to $\mathbf{k}$ and the other two are complex and conjugate; they are all mutually orthogonal, in the sense that the three unit vectors $\mathbf{e}_r$ parallel to these directions are such that

$$\mathbf{e}_r\cdot\mathbf{e}_s^* = \mathbf{e}_r^*\cdot\mathbf{e}_s = \delta_{rs}, \tag{4}$$

where an asterisk denotes a complex conjugate and δ_{rs} is the usual Kronecker symbol.

If we choose the Cartesian axes so that

$$\mathbf{k} = (0, 0, 1), \tag{5}$$

the eigenvectors have the simple form*

* The significance of their association with the Cartesian 1- and 2-axes will become apparent below.

$$\mathbf{e}_1 = (1, i, 0)/\sqrt{2}, \qquad \mathbf{e}_2 = (1, -i, 0)/\sqrt{2}, \qquad \mathbf{e}_3 = (0, 0, 1) = \mathbf{k}, \tag{6}$$

corresponding to the eigenvalues

$$\lambda_1 = -2i\omega g, \qquad \lambda_2 = 2i\omega g, \qquad \lambda_3 = 0. \tag{7}$$

Any vector $\mathbf{V}$ can be expressed in terms of components† $\overline{V}_r$ such that

$$\overline{V} = \mathbf{V}_r\mathbf{e}_r, \tag{8}$$

where summation is implied by the repeated subscript. Then, by virtue of (4),

$$\overline{V}_r = \mathbf{V}\cdot\overset{*}{\mathbf{e}}_r. \tag{9}$$

Using (8) and (9), scalar and vector products and the various derivatives of the vector calculus can be expressed in terms of these components. The two component vectors $\mathbf{V}_\parallel$ and $\mathbf{V}_\perp$, parallel and perpendicular to $\mathbf{k}$, are then

$$\mathbf{V}_\parallel = \overline{V}_3\mathbf{e}_3, \qquad \mathbf{V}_\perp = \overline{V}_1\mathbf{e}_1 + \overline{V}_2\mathbf{e}_2. \tag{10}$$

When the Cartesian axes are chosen as in (5), so that $\mathbf{V} = (V_1, V_2, V_3)$, the components $\overline{V}_r$ take the simple form

$$\overline{V}_1 = (V_1 - iV_2)/\sqrt{2}, \qquad \overline{V}_2 = (V_1 + iV_2)/\sqrt{2}, \qquad \overline{V}_3 = V_3. \tag{11}$$

If now the Cartesian 1- and 2-axes are regarded as real and imaginary axes on an Argand plane, $\sqrt{2}\,\overline{V}_2$ represents the vector $\mathbf{V}_\perp$ and $\sqrt{2}\,\overline{V}_1$ its reflection in the 1-axis. Such combinations have long been used in the contexts of dynamics and magneto-optics, and more recently [6], as explicit components in the magneto-ionic theory.

Since the tensor operator may now be written

$$2\omega \times \mathbf{U}g = \sum_\alpha \lambda_\alpha \mathbf{e}_\alpha \overset{*}{\mathbf{e}}_\alpha, \tag{12}$$

the homogeneous equation (2) becomes, for $\mathbf{r} = \bar{x}_r\mathbf{e}_r$,

$$\{f(D) - 2i\omega g(D)\}\bar{x}_1\mathbf{e}_1 + \{f(D) + 2i\omega g(D)\}\bar{x}_2\mathbf{e}_2 + f(D)\bar{x}_3\mathbf{e}_3 = 0. \tag{13}$$

Moreover, the unit vectors $\mathbf{e}_r$ are independent, so that (13) gives three separate scalar differential equations which determine $\bar{x}_1$, $\bar{x}_2$, to as many arbitrary constants as the order of the higher of the polynomials $f(D)$, $g(D)$, and $\bar{x}_3$ to as many as the order of $f(D)$. If such solutions are denoted by $\bar{x}_r = u_r(t)$, then the complementary function of the equation (1) is

$$\mathbf{r} = \mathbf{u}(t) = u_r(t)\mathbf{e}_r. \tag{14}$$

† Since the reciprocal set of vectors $\mathbf{e}^1 = \mathbf{e}_2$, $\mathbf{e}^2 = \mathbf{e}_1$, $\mathbf{e}^3 = \mathbf{e}_3$, there is no point in distinguishing between covariant and contravariant components.

It is possible, however, to reconstitute the right side of (14) into a form that is independent of any reference triad of vectors. Any of the linearly independent particular solutions of the homogeneous equation in $\bar{x}_1$ can be written in the form $x_s - iy_s$ where $x_s(t)$, $y_s(t)$ are real. There will then be a corresponding particular solution of the equation in $\bar{x}_2$, *viz.* $x_s + iy_s$,* and thus

$$\begin{aligned} \mathbf{u}_\perp(t) &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \\ &= A_{1s}(x_s - iy_s)\mathbf{e}_1 + A_{2s}(x_s + iy_s)\mathbf{e}_2, \end{aligned} \tag{15}$$

where the A_{1s}, A_{2s} are the constants of integration, and s ranges from 1 to the order of the equation. Now, using (6), it is not difficult to show that

$$\mathbf{e}_3 \times \mathbf{e}_1 = -i\mathbf{e}_1, \qquad \mathbf{e}_3 \times \mathbf{e}_2 = i\mathbf{e}_2. \tag{16}$$

Hence (15) may be written as the sum

$$\mathbf{u}_\perp(t) = \mathbf{A}_s x_s + \mathbf{k} \times \mathbf{A}_s y_s, \tag{17}$$

where

$$\mathbf{A}_s = A_{1s}\mathbf{e}_1 + A_{2s}\mathbf{e}_2$$

are constant vectors in the plane perpendicular to $\mathbf{k}$, determined by the initial conditions. Thus the real and imaginary parts of u_1 or u_2 provide Cartesian components of $\mathbf{u}$ in the plane perpendicular to $\mathbf{k}$. It will be seen in Section 3 that in the problem of Foucault's pendulum these solutions take the form of uniformly rotating vectors of constant amplitude, as given by Milne's trial solutions [2].

The third Cartesian component may be written as

$$\mathbf{u}_\parallel(t) = u_3\mathbf{e}_3 = C_s z_s \mathbf{k}, \tag{18}$$

where the z_s are linearly independent solutions of the homogeneous equation in $\bar{x}_3$ and the C_s are constants of integration. Thus the complementary function of the equation (1) becomes

$$\mathbf{u} = \mathbf{A}_s x_s + \mathbf{k} \times \mathbf{A}_s y_s + \mathbf{k} C_s z_s, \tag{19}$$

a sum containing the requisite number of scalar constants of integration.

2.2 The particular integral. By resolving $\mathbf{F}(t)$ into components, as in (8) and (9), the inhomogeneous equation (1) yields the three scalar equations

$$\begin{aligned} \{f(D) - 2i\omega g(D)\}\bar{x}_1 &= \overline{F}_1, \\ \{f(D) + 2i\omega g(D)\}\bar{x}_2 &= \overline{F}_2, \\ f(D)\bar{x}_3 &= \overline{F}_3. \end{aligned} \tag{20}$$

If particular integrals of these are $\bar{x}_r = \phi_r(t)$,

$$\mathbf{r} = \boldsymbol{\phi}(t) = \phi_r(t)\mathbf{e}_r \tag{21}$$

is a particular integral of (1), whose general solution can be written

* This follows because the operations $f(D)$, $g(D)$ on real functions result in real functions.

(22) $$\mathbf{r} = \mathbf{u}(t) + \boldsymbol{\phi}(t).$$

Again, the right side of (21) can be reconstituted independently of the triad $\mathbf{e}_r$. Since $\overline{F}_1$, $\overline{F}_2$ are complex conjugates, it follows that if $\sqrt{2}\phi_1 = \xi - i\eta$ then $\sqrt{2}\phi_2 = \xi + i\eta$, so that, by (16)

(23) $$\begin{aligned} \boldsymbol{\phi}_\perp &= \phi_1\mathbf{e}_1 + \phi_2\mathbf{e}_2 \\ &= \xi\mathbf{i} + \eta\mathbf{k} \times \mathbf{i}, \end{aligned}$$

where

$$\mathbf{i} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2},$$

is an arbitrary unit vector perpendicular to $\mathbf{k}$; $\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}$ have respectively the directions of the real and imaginary axes of the Argand plane referred to above. Then, if $\zeta(t) \equiv \phi_3(t)$,

(24) $$\boldsymbol{\phi}_\parallel = \zeta\mathbf{k}$$

and

(25) $$\boldsymbol{\phi} = \xi\mathbf{i} + \eta\mathbf{k} \times \mathbf{i} + \zeta\mathbf{k},$$

is a particular integral of (1). The general solution is then given by (22) with (19) and (25).

3. Example. The undamped motion of a Foucault's pendulum bob, and that of an elastically bound electron in a uniform magnetic field, are covered by taking

(26) $$f(D) \equiv D^2 + n^2, \qquad g(D) \equiv D.$$

Then it can be seen from (13) that the function $u_1(t)$ is the general solution of the equation

(27) $$\{D - i\omega + i\sqrt{(n^2 + \omega^2)}\}\{D - i\omega - i\sqrt{(n^2 + \omega^2)}\}\bar{x}_1 = 0,$$

viz.

(28) $$u_1 = e^{i\omega t}(A_1 e^{-i\sqrt{(n^2+\omega^2)}\,t} + B_1 e^{i\sqrt{(n^2+\omega^2)}\,t}),$$

where A_1 and B_1 are arbitrary constants.* It can be seen also from the above interpretation of the vector components that the two terms in (28) represent vectors of constant amplitude rotating about the direction of $\mathbf{k}$ with constant angular velocities in a right-handed and a left-handed sense, respectively. The locus of the resultant is a Lissajous figure traced out on a plane perpendicular to $\mathbf{k}$. Referred to a frame of reference which itself rotates with angular velocity $-\boldsymbol{\omega}$, the angular velocities of the two vectors are $\pm\sqrt{(n^2+\omega^2)}$ and the locus is an ellipse.

* In the problem of Foucault's pendulum it is proper to neglect ω^2 as compared to n^2 in (28) and in the succeeding formulas.

Similarly,

$$u_2 = e^{-i\omega t}(A_2 e^{i\sqrt{(n^2+\omega^2)}\,t} + B_2 e^{-i\sqrt{(n^2+\omega^2)}\,t}), \tag{29}$$

represents two vectors of constant amplitude rotating respectively with the same angular velocities as in (28), giving again an ellipse when referred to a frame rotating with angular velocity $-\boldsymbol{\omega}$. Finally,

$$u_3 = C \cos nt + D \sin nt \tag{30}$$

represents a linear oscillation of period $2\pi/n$ along the direction of $\mathbf{k}$.

Now the separate terms of (28) or (29) give

$$\begin{aligned} x_1 &= \cos\left\{\sqrt{(n^2+\omega^2)} - \omega\right\}t, & y_1 &= \sin\left\{\sqrt{(n^2+\omega^2)} - \omega\right\}t, \\ x_2 &= \cos\left\{\sqrt{(n^2+\omega^2)} + \omega\right\}t, & y_2 &= -\sin\left\{\sqrt{(n^2+\omega^2)} + \omega\right\}t. \end{aligned} \tag{31}$$

Hence, by (18), (19), and (30) the complementary function may be written

$$\begin{aligned} \mathbf{u}(t) = {} & \mathbf{A}\cos\left\{\sqrt{(n^2+\omega^2)} - \omega\right\}t + \mathbf{k}\times\mathbf{A}\sin\left\{\sqrt{(n^2+\omega^2)} - \omega\right\}t \\ & + \mathbf{B}\cos\left\{\sqrt{(n^2+\omega^2)} + \omega\right\}t - \mathbf{k}\times\mathbf{B}\sin\left\{\sqrt{(n^2+\omega^2)} + \omega\right\}t \\ & + \mathbf{k}(C\cos nt + D\sin nt), \end{aligned} \tag{32}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary constant vectors perpendicular to $\mathbf{k}$, and C and D are arbitrary scalar constants. The first two pairs of terms each represent vectors of constant amplitudes A, B, again rotating about $\mathbf{k}$ with angular velocities $\pm\sqrt{(n^2+\omega^2)} - \omega$ respectively.

In the present case, the component $\phi_1(t)$ of the particular integral is, as usual, given by

$$2i\sqrt{(n^2+\omega^2)}\,\phi_1 = \left[\frac{1}{D - i\omega - i\sqrt{(n^2+\omega^2)}} - \frac{1}{D - i\omega + i\sqrt{(n^2+\omega^2)}}\right]\overline{F}_1,$$

whence

$$\phi_1 = \frac{1}{\sqrt{(n^2+\omega^2)}}\int^t \overline{F}_1(\tau)e^{i\omega(t-\tau)}\sin\left\{\sqrt{(n^2+\omega^2)}(t-\tau)\right\}d\tau. \tag{33}$$

Similarly,

$$\phi_2 = \frac{1}{\sqrt{(n^2+\omega^2)}}\int^t \overline{F}_2(\tau)e^{-i\omega(t-\tau)}\sin\left\{\sqrt{(n^2+\omega^2)}(t-\tau)\right\}d\tau, \tag{34}$$

and

$$\phi_3 = \frac{1}{n}\int^t \overline{F}_3(\tau)\sin n(t-\tau)\,d\tau. \tag{35}$$

The required particular integral is now obtained by substitution from these

results into (21). This can be expressed in a form independent of any coordinate system, directly as in (15) and (17), or from (25). The result is

$$\phi(t) = \frac{1}{\sqrt{(n^2+\omega^2)}}\int^t \left\{\mathbf{F}_\perp(\tau)\cos\omega(t-\tau) - \mathbf{k}\times\mathbf{F}_\perp(\tau)\sin\omega(t-\tau)\right\} \tag{36}$$
$$\times \sin\left\{\sqrt{(n^2+\omega^2)}(t-\tau)\right\}d\tau + \frac{1}{n}\int^t F_\parallel(\tau)\sin n(t-\tau)d\tau.$$

Finally, the constants $\mathbf{A}$, $\mathbf{B}$, C, D can be evaluated for the most general initial conditions,

$$\mathbf{r} = a\mathbf{i} + b\mathbf{k}, \qquad D\mathbf{r} = u\mathbf{j} + v\mathbf{k}, \tag{37}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors perpendicular to $\mathbf{k}$. The result is

$$\begin{aligned} \mathbf{A} &= \frac{a\{\omega + \sqrt{(n^2+\omega^2)}\}\mathbf{i} + u\mathbf{j}\times\mathbf{k}}{2\sqrt{(n^2+\omega^2)}}, \\ \mathbf{B} &= -\frac{a\{\omega - \sqrt{(n^2+\omega^2)}\}\mathbf{i} + u\mathbf{j}\times\mathbf{k}}{2\sqrt{(n^2+\omega^2)}}, \\ C &= b, \qquad D = v/n, \end{aligned} \tag{38}$$

with the lower limit 0 in the integrals on the right side of (36). The general solution is then given by (22) with (32) and (36).

References

1. C. E. Easthope, Note on Foucault's pendulum, Math. Gaz., vol. 38, 1954, pp. 206–207.
2. E. A. Milne, Vectorial Mechanics, Methuen, 1948, Sect. 287, p. 248.
3. ———, *ibid.*, Sect. 292, p. 253.
4. ———, *ibid.*, Sect. 402, p. 333.
5. ———, *ibid.*, Sect. 255, p. 220.
6. K. C. Westfold, The wave equations for electromagnetic radiation in an ionized medium in a magnetic field, Australian J. Sci. Research, Ser. A, vol. 2, 1949, pp. 169–183.

CORRECTION

W. E. Bleick, *Fourier analysis of engine unbalance by contour integration*, this MONTHLY, vol. 63, 1956, pp. 466–472. Equation (17), p. 469 should read "$b_1 = 4[E(\alpha^2) - (1-\alpha^2)K(\alpha^2)]/\pi\alpha$."

MATHEMATICAL NOTES

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A GENERALIZATION OF THE SINE FUNCTION

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One generalization of the elementary circular and hyperbolic functions consists in finding fundamental sets of solutions of the equations $d^ny/dx^n \pm y = 0$, having properties analogous to those of the elementary functions. These functions are variously referred to as higher-order sine functions, Olivier functions, or Villarceau functions. Their history begins with the work of V. Riccati [1] and Wronski [2]. Günther [3] gives an excellent survey of the relevant literature to 1880, and Poli [4] provides a more recent summary. A thorough bibliography will appear in a forthcoming note [5]. For accessible accounts of these functions the reader is referred to papers by Humbert [6], Ward [7], Dramba [8], Oniga [9], and Silberstein [10].

Brodetsky [11] has made a generalization in a different direction by the following considerations relative to the second order equation $d^2y/dx^2 = I(x)y$. Let functions $C(x, \xi)$ and $S(x, \xi)$ be defined such that if $y(x)$ is a solution of the equation then $y(x+\xi) = C(x, \xi)y(x) + S(x, \xi)y'(x)$. The C and S functions have certain properties analogous to those of the elementary circular and hyperbolic functions. In the present note Brodetsky's analysis is extended to the nth order equation

$$d^ny/dx^n = I(x)y, \tag{1}$$

where $I(x)$ is a given continuous function defined on an interval (a, b). The new functions introduced in this way include as special cases the above two generalizations.

Define functions $C_j \equiv C_j(x, \xi)$, $(j = 1, \cdots, n)$, such that if $y(x)$ is a solution of (1) then $y(x+\xi) = \sum_{j=1}^{n} C_j y^{(j-1)}(x)$, where $y^{(i)}(x) \equiv d^iy/dx^i$, $y^{(0)}(x) \equiv y(x)$. Since

$$\frac{\partial y(x+\xi)}{\partial \xi} = \sum_{j=1}^{n} \frac{\partial C_j}{\partial \xi} y^{(j-1)}(x),$$

$$\frac{\partial y(x+\xi)}{\partial x} = \sum_{j=1}^{n} \frac{\partial C_j}{\partial x} y^{(j-1)}(x) + \sum_{j=1}^{n} C_j y^{(j)}(x),$$

and $\partial y(x+\xi)/\partial x = \partial y(x+\xi)/\partial \xi$ it follows that

$$\left(\frac{\partial C_1}{\partial \xi} - \frac{\partial C_1}{\partial x} - I(x)C_n\right)y(x) + \sum_{j=2}^{n}\left(\frac{\partial C_j}{\partial \xi} - \frac{\partial C_j}{\partial x} - C_{j-1}\right)y^{(j-1)}(x) = 0.$$

This relation must hold for every linearly independent solution of (1), whence

$$\frac{\partial C_1}{\partial \xi} - \frac{\partial C_1}{\partial x} = I(x) \quad C_n, \qquad \frac{\partial C_j}{\partial \xi} - \frac{\partial C_j}{\partial x} = C_{j-1} \quad (j = 2, \cdots, n). \tag{2}$$

The relation of the functions $C_j(x, \xi)$ to the differential equation (1) is obtained by differentiating $y(x+\xi)$ n times with respect to ξ. Thus

$$\begin{aligned}\frac{\partial^n(y+\xi)}{\partial \xi^n} &= \sum_{j=1}^{n} \frac{\partial^n C_j}{\partial \xi^n} y^{(j-1)}(x) \\ &= I(x+\xi)y(x+\xi) = I(x+\xi)\sum_{j=1}^{n} C_j y^{(j-1)}(x),\end{aligned}$$

whence

$$\partial^n C_j/\partial \xi^n = I(x+\xi)C_j \qquad (j = 1, \cdots, n). \tag{3}$$

By substituting $\xi = 0$ in $y(x+\xi)$ and its derivatives with respect to ξ, we obtain

$$\frac{\partial^i C_j(x, 0)}{\partial \xi^i} = \begin{Bmatrix} 1, & i = j-1 \\ 0, & i \neq j-1 \end{Bmatrix} \qquad (i = 0, \cdots, n-1; j = 1, \cdots, n), \tag{4}$$

which are the analogues of $\cos 0 = 1$, $\sin 0 = 0$, *etc.* Since $\partial^n C_i/\partial \xi^n = I(x+\xi)C_i$, $\partial^n C_j/\partial \xi^n = I(x+\xi)C_j$, it follows that

$$\left(\frac{\partial^n C_i}{\partial \xi^n}\right)C_j = \left(\frac{\partial^n C_j}{\partial \xi^n}\right)C_i \qquad (i, j = 1, \cdots, n; i \neq j).$$

In Brodetsky's terminology, the functions C_j are the principal integrals of equation (1) in the variable ξ with shifted origin. Hence the solution of (1) with argument $x_0+\xi$ is obtained from the principal integrals C_j of (1) in ξ with origin at $x = x_0$ by multiplying them respectively by the values of $y, y', \cdots, y^{(n-1)}$ at $x = x_0$, and forming the sum $\sum_{j=1}^{n} C_j(x_0, \xi)y^{(j-1)}(x_0)$.

An addition formula is obtained by expanding $y(x+\xi+\eta)$ in two ways. Thus

$$y(x+\xi+\eta) = \sum_{j=1}^{n} C_j(x, \xi+\eta)y^{(j-1)}(x),$$

$$\begin{aligned}y(x+\xi+\eta) &= \sum_{j=1}^{n} C_j(x+\xi, \eta)y^{(j-1)}(x+\xi) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} C_j(x+\xi, \eta)\frac{\partial^{j-1}C_k(x, \xi)}{\partial \xi^{j-1}} y^{(k-1)}(x),\end{aligned}$$

whence

$$C_j(x, \xi + \eta) = \sum_{i=1}^{n} \left(\frac{\partial^{i-1} C_j(x, \xi)}{\partial \xi^{i-1}}\right) C_i(x + \xi, \eta) \qquad (j = 1, \cdots, n). \tag{5}$$

If $I(x) = \text{constant}$ (in particular, $I(x) = \pm 1$), the functions C_j do not depend on x, and (5) reduces to the addition formula for the Olivier functions.

If a Taylor series expansion is assumed for $I(x)$, the following series representations of $C_j(x, \xi)$ are obtained:

$$C_j(x, \xi) = \sum_{k=0}^{\infty} A_{kj}(x) \frac{\xi^k}{k!} \qquad (j = 1, \cdots, n), \tag{6}$$

where

$$A_{kj}(x) = \begin{Bmatrix} 0, & k \neq j - 1 \\ 1, & k = j - 1 \end{Bmatrix} \qquad (k = 0, \cdots, n - 1; j = 1, \cdots, n).$$

The recursion relations for A_{kj}, $(k \geqq n)$, are given by

$$A_{n+s,j} = I^{(s)} A_{0j} + s I^{(s-1)} A_{1j} + [s(s-1)/2!] I^{(s-2)} A_{2j} + \cdots + I A_{sj}$$
$$(j = 1, \cdots, n; s = 0, 1, \cdots),$$

where $A_{kj} \equiv A_{kj}(x)$, $I^{(r)} \equiv d^r I/dx^r$. A detailed analysis based on the Taylor expansion for the case $n = 2$ will be found in Brodetsky's paper.

References

1. V. Riccati, Soc. Jesu, Opusculorum ad res physicas & mathematicas pertinentium, tomus primus, Bononiae, 1757, pp. 108–110.

2. H. Wronski, Oeuvres Mathématiques, Librairie Scientifique J. Hermann, 1925, vol. 1 (first published in 1811), pp. 191–195.

3. S. Günther, Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen, 1881, pp. 393–413.

4. L. Poli, Sur les sinus d'ordre supérieur, Cahiers Rhodaniens, vol. 1, 1949.

5. H. Kaufman, A bibliographical note on higher-order sine functions, to appear in Scripta Math.

6. P. Humbert, Remarques bibliographiques sur les sinus d'ordre supérieur, Atti della Pontificia Accademia delle Scienze, Nuovi Lincei, vol. 82, 1928–29, pp. 308–312.

7. L. E. Ward, Some functions analogous to trigonometric functions, this MONTHLY, vol. 34, 1927, pp. 301–303.

8. C. Dramba, Sur une extension de la notion de fonction analytique, C. R. Acad. Sci. Paris, vol. 228, 1949, pp. 1626–1628.

9. T. Oniga, Sur une généralisation des fonctions circulaires et hyperboliques, C. R. Acad. Sci. Paris, vol. 227, 1950, pp. 1138–1140.

10. L. Silberstein, Differentially cyclical sets of functions, Phil. Mag., Seventh Series, vol. 33, 1942, pp. 457–461.

11. S. Brodetsky, On the linear differential equation of the second order, Proc. Edinburgh Math. Soc., vol. 34, 1915–16, pp. 45–60.

A NOTE ON NEIGHBORING JORDAN CURVES

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One of the problems in conformal mapping is to find conditions on two Jordan curves C and C' so that the difference between the functions which map the unit circle conformally onto the interiors of the two curves will be small in some sense. One of the conditions sometimes assumed is that each curve lies within a certain neighborhood of the other. Under certain additional conditions this assumption is redundant, because if C' lies within an ϵ neighborhood of C, it can be proved that C lies within an ϵ_1 neighborhood of C', where $\epsilon_1 = O(\epsilon)$ as $\epsilon \to 0$. We give an example of a theorem of this type.

Let C and C' be Jordan curves in the complex z plane satisfying the conditions:

(H_1) The interior of C' contains the origin $z=0$.

(H_2) The interior of C contains the circle $|z| = b > 0$.

(H_3) If z_1 and z_2 are on C, then one of the arcs of C connecting z_1 and z_2 lies within a circle of diameter $d = c|z_1 - z_2|$, where c is a fixed constant (necessarily >1).

We now show that if C' lies within an ϵ neighborhood of C, where $\epsilon < b/2c$, then C lies within an ϵ_1 neighborhood of C' if $\epsilon_1 > (2c+1)\epsilon$.

An arc $\overline{z_1 z_2}$ of C which has the property assumed in (H_3) will be called a minimal arc joining z_1 and z_2. If D is the diameter of C and $|z_1 - z_2| < D/2c$, then only one of the two arcs joining z_1 and z_2 can be minimal.

Let $z_1', z_2', \cdots, z_n' = z_1'$ be points of C', ordered according to the orientation of C' and chosen so that each arc $\overline{z_k' z_{k+1}'}$ $(k=1, 2, \cdots, n-1)$ is contained in a circle of diameter $\delta < b/c - 2\epsilon$. For each point z_k' let z_k be a corresponding point of C such that $|z_k' - z_k| < \epsilon$ and $z_n = z_1$. Let γ_k be the minimal arc of C from z_k to z_{k+1} and let Γ be the curve formed by joining the arcs $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ end to end. Considered as point sets, Γ is a subset of C. The variation in arg z over Γ, denoted by $\Delta_\Gamma \arg z$, is a multiple of 2π since the initial and terminal points of Γ coincide.

Let $z = z(t)$ and $z = z'(t)$, $0 \leqq t \leqq 1$, be continuous parametric representations of Γ and C' respectively, with $z_k = z(t_k)$, $z_k' = z'(t_k')$, $0 = t_1 \leqq t_2 \leqq \cdots \leqq t_n = 1$, $0 = t_1' \leqq t_2' \leqq \cdots \leqq t_n' = 1$. Let $a(t)$ and $a'(t)$ be values of $\arg z(t)$ and $\arg z'(t)$ respectively, each continuous for $0 \leqq t \leqq 1$ and chosen so that $|a(0) - a'(0)| < \pi$. This can be done since $|z_1| > b$ and $|z_1 - z_1'| < \epsilon < b/2$ so that the line segment from z_1 to z_1' cannot contain the origin.

Let us assume that $|a(t_k) - a'(t_k')| < \pi$ for some k $(0 \leqq k < n)$. The arcs $\overline{z_k z_{k+1}}$ and $\overline{z_k' z_{k+1}'}$ are both contained in a circle S of radius $c(2\epsilon + \delta)$ about z_k as center, and S does not contain the origin since $c(2\epsilon + \delta) < b$. Let $A(z)$ be a value of arg z for $z \in S$ defined so as to be continuous in S and so that $A(z_k) = a(t_k)$. Since $|A(z_k) - A(z_k')| < \pi$, $|a(t_k) - a'(t_k')| < \pi$, and $A(z_k') - a'(t_k')$ is a multiple of 2π,

$A(z_k')=a'(t_k')$, and from continuity $A(z_{k+1})=a(t_{k+1})$, $A(z_{k+1}')=a'(t_{k+1}')$. Hence, $|a(t_{k+1})-a'(t_{k+1}')|<\pi$.

By induction $|a(t_k)-a'(t_k')|<\pi$ for $1\leqq k\leqq n$. Hence $|\Delta_\Gamma \arg z-\Delta_{c'} \arg z|$ is less than 2π and therefore zero, so $\Delta_\Gamma \arg z=\Delta_{c'} \arg z=2\pi$ and Γ cannot be a proper subset of C. Any point of C is also on Γ and lies within a distance $c(2\epsilon+\delta)$ of one of the points z_k, hence within a distance $c(2\epsilon+\delta)+\epsilon$ of a point of C'. By choosing δ sufficiently small this latter distance can be made less than ϵ_1 if $\epsilon_1>(2c+1)\epsilon$.

EXTENSIONS OF THE LAW OF THE MEAN

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An extension of the law of the mean was published by Karamata [2] and the following year Vučković [4] gave an extension of Karamata's result. Utz [3] pointed out that both authors stated a stronger conclusion than can be secured. The purpose of this note is to show that Vučković's result may be obtained directly from a theorem of Kametani [1]. Karamata's result follows as a special case.

Theorem (*Vučković*). *If the functions f and g are continuous for $a\leqq x\leqq b$ and if the right and left hand derivatives f'_+, f'_-, g'_+, g'_- exist on $a<x<b$, then there exist numbers x_0, p, q with $a<x_0<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ for which* $[pf'_+(x_0)+qf'_-(x_0)]\cdot[g(b)-g(a)]=[pg'_+(x_0)+qg'_-(x_0)][f(b)-f(a)]$.

Proof. Let $a\leqq x\leqq x'\leqq b$, let I be the closed interval $[x, x']$, let $\phi(I)=f(x')-f(x)$ and let $\sigma(I)=g(x')-g(x)$. Then ϕ and σ are additive continuous functions of an interval on the closed interval $I_0=[a, b]$. Hence [1, p. 2] there exist a point x_0 and a descending sequence of intervals $\{I_n\}=\{[a_n, b_n]\}$ with the following properties:

(1) $I_0\supset I_1\supset I_2\cdots\ni x_0$,

(2) $\lim_{n\to\infty}(b_n-a_n)=0$,

(3) x_0 is an interior point of each I_n,

(4) $[f(b_n)-f(a_n)][g(b)-g(a)]=[g(b_n)-g(a_n)][f(b)-f(a)]$ $(n=1, 2, \cdots)$.

Since $a_n<x_0<b_n$ $(n=1, 2, \cdots)$, $b_n-a_n\neq 0$ and we can divide each side of (4) by b_n-a_n which gives

$$\left[\frac{f(b_n)-f(a_n)}{b_n-a_n}\right][g(b)-g(a)]=\left[\frac{g(b_n)-g(a_n)}{b_n-a_n}\right][f(b)-f(a)] \qquad (n=1,2,\cdots).$$

This may be written

$$\begin{aligned}&\left[\frac{f(b_n)-f(x_0)}{b_n-x_0}\,\frac{b_n-x_0}{b_n-a_n}+\frac{f(x_0)-f(a_n)}{x_0-a_n}\,\frac{x_0-a_n}{b_n-a_n}\right][g(b)-g(a)]\\ &\quad=\left[\frac{g(b_n)-g(x_0)}{b_n-x_0}\,\frac{b_n-x_0}{b_n-a_n}+\frac{g(x_0)-g(a_n)}{x_0-a_n}\,\frac{x_0-a_n}{b_n-a_n}\right][f(b)-f(a)].\end{aligned} \tag{1}$$

Since $0<(b_n-x_0)/(b_n-a_n)<1$ $(n=1, 2, \cdots)$, there exist a number p, $0\leqq p\leqq 1$, and a subsequence $\{I_{n'}\}$ of the sequence $\{I_n\}$ such that $\lim_{n'\to\infty}(b_{n'}-x_0)/(b_{n'}-a_{n'})=p$. Also, $\lim_{n'\to\infty}(x_0-a_{n'})/(b_{n'}-a_{n'})=q$ and $p+q=1$. Hence replacing n by n' in (1) and then taking the limit of both sides as $n'\to\infty$, we obtain $[pf'_+(x_0)+qf'_-(x_0)][g(b)-g(a)]=[pg'_+(x_0)+qg'_-(x_0)][f(b)-f(a)]$ with $p\geqq 0$, $q\geqq 0$, $p+q=1$, which completes the proof.

If we set $g(x)=x$ in the above theorem, we obtain $[pf'_+(x_0)+qf'_-(x_0)][b-a]=f(b)-f(a)$, which is Karamata's result.

References

1. S. Kametani, A new formulation of mean value theorem, Nat. Sci. Rep., Ochanomizu University, vol. 1, 1951, pp. 1–5.

2. J. Karamata, Sur la formule des accroissements finis, Srpska. Akad. Nauka. Zb. Rad., vol. 7, Matematički Inst., vol. 1, 1951, pp. 119–124 (Serbo-Croatian, French Summary).

3. W. R. Utz, Mean value theorems for functions with left and right derivatives, this MONTHLY, vol. 62, 1955, pp. 178–179.

4. V. Vučković, Quelques extensions des theoremes de moyenne, Srpska Akad. Nauka. Zb. Rad., vol. 18, Matematički Inst., vol. 2, 1952, pp. 159–166 (Serbo-Croatian, French Summary).

A DETERMINANT

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Put

$$D_n = \left|\binom{n+1}{2s-r}\right| \qquad (r, s = 0, 1, \cdots, n), \tag{1}$$

a determinant of order $n+1$. The interesting result

$$D_n = 2^{n(n+1)/2} \tag{2}$$

was stated without proof by Nesbitt [1, p. 316]. Two proofs of (2) have been given by Niblett [2].

It may be of interest to point out that this result is a special case of a more general result that can be proved rather easily. We observe first that

$$D_n = D_n', \tag{3}$$

where

$$D_n' = \left|\binom{n+1+r}{2s}\right| \qquad (r, s = 0, 1, \cdots, n). \tag{4}$$

Indeed if we put

$$E_n = \left|\binom{r}{s}\right| \qquad (r, s = 0, 1, \cdots, n),$$

then, by the multiplication formula for determinants,

$$D_n E_n = \left| c_{rs} \right|, \qquad c_{rs} = \sum_{k=0}^{n} \binom{n+1}{2s-k}\binom{r}{k} = \binom{n+1+r}{2s}.$$

Since $E_n = 1$, (3) follows at once.

Generalizing (4), we let

$$\Delta_n(x) = \left| \binom{x+r}{2s} \right| \qquad (r, s = 0, 1, \cdots, n). \tag{5}$$

If we subtract the $(n-1)$th row from the nth, the $(n-2)$th from the $(n-1)$th, and so on, we find that $\Delta_n(x) = \Delta_n'(x)$, where

$$\Delta_n'(x) = \left| \binom{x+r}{2s+1} \right| \qquad (r, s = 0, 1, \cdots, n-1).$$

Since

$$\binom{x+r}{2s+1} = \frac{x+r}{2s+1}\binom{x+r-1}{2s},$$

we may multiply each element of the sth column by $2s+1$ and then divide each element of the rth row by $x+r$, and then we have

$$\Delta_n'(x) = \frac{x(x+1)\cdots(x+n-1)}{1\cdot 3\cdots(2n-1)} \left| \binom{x+r-1}{2s} \right|,$$

where $r, s = 0, 1, \cdots, n-1$. Using (5) this gives the recursion

$$\Delta_n(x) = \frac{x(x+1)\cdots(x+n-1)}{1\cdot 3\cdots(2n-1)} \Delta_{n-1}(x-1). \tag{6}$$

Since $\Delta_0(x) = 1$, repeated application of (6) yields the explicit formula

$$\Delta_n(x) = \frac{(x)_n(x-1)_{n-1}\cdots(x-n+1)_1}{1^n\, 3^{n-1}\cdot 5^{n-2}\cdots(2n-1)}, \tag{7}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$; this result may be exhibited in the following form. For n odd, $n = 2m-1$,

$$C\Delta_n(x) = x^m \prod_{k=1}^{m-1} \left\{x^2 - (2k-1)^2\right\}^{m-k}\left\{x^2 - (2k)\right\}^{m-k};$$

for n even, $n = 2m$,

$$C\Delta_n(x) = \prod_{k=0}^{m-1} \left\{x^2 - (2k)^2\right\}^{m-k}\left\{x^2 - (2k+1)^2\right\}^{m-k},$$

where in either case $C = 1^n 3^{n-1} 5^{n-2} \cdots (2n-1)$.

To get (2) rapidly we remark that

$$D_n = \Delta_n(n+1). \tag{8}$$

Thus substitution of (8) in (6) leads to

$$D_n = \frac{(n+1)(n+2)\cdots 2n}{1\cdot 3\cdots(2n-1)} D_{n-1} = \frac{(2n!)}{n!}\,\frac{2\cdot 4\cdots 2n}{(2n)!} D_{n-1} = 2^n D_n.$$

Since $D_0 = 1$, (2) follows at once.

We note also that (7) implies

$$\left|\binom{n}{2s-r}\right| = \left|\binom{n+r}{2s}\right| = 2^{n(n-1)/2}, \tag{9}$$

$$\left|\binom{k}{2s-r}\right| = \left|\binom{k+r}{2s}\right| = 0 \qquad (-n+1 \leqq k \leqq n-1), \tag{10}$$

where in each case $r, s = 0, 1, \cdots, n$.

If we multiply $\Delta_n(k)$ by the Vandermonde determinant, $V_n = |a_s^{2r}|$ $(r, s = 0, \cdots, n)$, we get for $0 \leqq k \leqq n+1$

$$\Delta_n(k) V_n = 2^{-n-1} \left| (1+a_s)^{k+r} + (1-a_s)^{k+r} \right| \qquad (r, s = 0, \cdots, n).$$

Consequently, it follows from (8), (9), and (10) that

$$\left| (1+a_s)^{k+r} + (1-a_s)^{k+r} \right| = \begin{cases} 2^{n+1+k(k-1)/2} \prod_{n \geqq i > j \geqq 0} (a_i^2 - a_j^2) & (k = n, n+1) \\ 0 & (0 \leqq k \leqq n-1). \end{cases}$$

References

1. T. Muir, Contributions to the history of determinants, 1900–1920, London and Glasgow, 1930.

2. J. D. Niblett, A theorem of Nesbitt, this MONTHLY, vol. 59, 1952, pp. 171–174.

A NOTE ON AUTOMETRIZED BOOLEAN ALGEBRAS

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An "autometrized Boolean algebra" has been defined by Ellis [1] as a Boolean algebra B in which the "distance" of two elements x and y is taken to be their symmetric difference (denoted $d(x, y) = x \oplus y$); this "distance" function satisfies formally the axioms for a metric. In this note we shall give simple and natural proofs for the theorems of [2], concerning the "motions" of B. (A motion is a one to one map of B onto itself preserving "distance.")

We first prove a lemma, from which the other results follow immediately. (The notation is that of [1] and [2].)

LEMMA. *The motions of B are the functions $f(x)=x\oplus a$, where a may be any (constant) element of B.*

Proof. Let f be any motion. Then by definition, $d(x, y)=d(f(x), f(y))$ or $x\oplus y=f(x)\oplus f(y)$. Putting $y=0$ we get $f(x)=x\oplus a$, where we have let $a=f(0)$. Conversely, for each $a\in B$, $f(x)=x\oplus a$ is a motion, since $x\oplus y=x\oplus y\oplus(a\oplus a)=(x\oplus a)\oplus(y\oplus a)=f(x)\oplus f(y)$. We now apply this result to prove:

THEOREM 1 of [2]. *If f is a motion of B, $f(x')=f'(x)$.*

Proof. Let $f(x)=x\oplus a$. Then $f'(x)=(x\oplus a)'=1\oplus x\oplus a=x'\oplus a=f(x')$.

THEOREM 2 of [2]. *The group of motions of B is isomorphic with the additive group $G(B)$ of the associated Boolean ring.*

Proof. Consider the correspondence $c: f(x)=x\oplus a\leftrightarrow a\in G(B)$. The product of two motions f and g is defined as the composition $f(g)$. But if $g(x)=x\oplus b\leftrightarrow b$, then $f(g(x))=x\oplus b\oplus a\leftrightarrow b\oplus a$. Hence c is a homomorphism; since (also from the lemma) c is one to one and onto, Theorem 2 is proved.

The use of this lemma also allows very simple proofs to be given for Theorems 3.1 and 3.2 of [1].

References

1. David Ellis, Autometrized Boolean Algebras I, Canadian J. Math., vol. 3, 1951, pp. 87–93.
2. David Ellis, Autometrized Boolean Algebras II, Canadian J. Math., vol. 3, 1951, pp. 145–147.

NOTES ON MATRIX THEORY—IX

RICHARD BELLMAN, The RAND Corporation, Santa Monica, California

1. Introduction. In a recent note [1], we showed that the inequality

$$|\lambda A+(1-\lambda)B| \geqq |A|^{\lambda}|B|^{1-\lambda}, \tag{1}$$

valid for positive definite matrices A and B, for $0\leqq\lambda\leqq 1$, was a simple consequence of Hölder's inequality and the identity

$$\int_{0}^{\infty} e^{-(x,Cx)}\prod_{i} dx_i = \sqrt{\pi^n}/|C|^{1/2}, \tag{2}$$

for C a positive definite matrix of order n.

In this note we wish to use a more recondite identity, a generalization of an integral of Ingham and Siegel, due to A. Selberg, to derive an extensive generalization of (1), namely,

THEOREM. *Let A and B be two positive definite matrices of order n, and let $C=\lambda A+(1-\lambda)B$, for $0\leqq\lambda\leqq 1$. For each $j=1, 2, \cdots, n$, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first $(j-1)$ rows and columns, (in particular, $A^{(1)}=A$). Let $B^{(j)}$, $C^{(j)}$, have similar meanings. If $k_1, k_2, \cdots, k_n$ are*

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

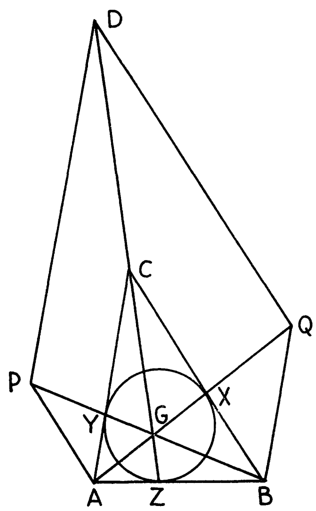
NOTE ON THE GERGONNE POINT OF A TRIANGLE

LAURA GUGGENBUHL, Hunter College, New York

The Gergonne point of a triangle is defined as the point of intersection of the lines from the vertices to the points of contact of the inscribed circle.

An interesting though rarely quoted problem of Joseph Diez Gergonne says [1]: "If through the vertices A and B of a triangle ACB two lines AP, BQ of arbitrary length are drawn in the direction of C, AP parallel to BC, BQ parallel to AC, and if the lines PD and QD are drawn respectively parallel to BQ and AP, meeting in D, then the lines AQ, BP, and CD are concurrent."

The theorem is proved by analytic geometry, with the origin at the vertex C of the triangle, the X-axis on the side CA, and the Y-axis on the side CB.



In passing one may note that if AP and BQ are both taken equal to AB, the point of intersection is the incenter of the triangle; if AP is taken equal to BC, and BQ equal to AC, the point is the median point; and if P is taken as the point at which BY cuts the parallel through A, and Q the point at which AX cuts the parallel through B, where X and Y are points of contact of the incircle, the point of concurrence is obviously the Gergonne point of the triangle.

A curious commentary regarding the origin of the name of the point is suggested by the above problem. The Gergonne point is usually associated with the Nagel point, namely the point of intersection of the lines from the vertices

of a triangle to the internal points of contact of the three escribed circles. In this connection, a reader is frequently referred to Volume 19 of *Nouvelles Annales de Mathématiques* [2]. However, in this source the points, though nameless, are merely included in a list of statements correctly attributed to Nagel [3] and no mention is made of Gergonne.

Nowhere in Gergonne's publications has the author found any reference to the point. In fact, Gergonne did relatively little work in that part of modern geometry which is connected with the notable points and lines of a triangle; and in truth, that part of modern geometry did not really come into its own until some years after Gergonne died. The name is secure in mathematical literature by 1883 [4], but its origin is indeed obscure. It seems certain that the name was given to the point as a memorial, some time after Gergonne's death in 1859. Just why Gergonne was memorialized by this point rather than by any one of a score of others, is truly a provocative question. Perhaps the answer lies in some student's cherished note book; in a budget of innocent errors; or perhaps in some even more likely source.

The author wishes to thank Professor Altschiller Court; M. Goormaghtigh; the referee of this article, Professor Struik; and Professor J. J. Burckhardt of the University of Zurich for several helpful suggestions. Among other things, they have pointed out that the theorem, concerning the lines which join the vertices of a triangle to the points of contact of the incircle, appeared as early as 1678 (on page 38) of Ceva's book, *De Lineis rectis se invicem secantibus statica constructio.*

References

1. Annales de Mathématiques, vol. 15, 1824–1825, p. 88.
2. Théorèmes sur les cercles qui touchent les côtés d'un triangle, Nouvelles Annales de Mathématiques, vol. 19, 1860, p. 354.
3. Christian Nagel, Untersuchungen über die wichtigsten zum Dreieck gehörigen Kreise, Leipzig, 1836, p. 32.
4. J. S. Mackay, The triangle and its six scribed circles, Proc. Edinburgh Math. Soc., vol. 1, 1883, p. 8.

THE IMAGINARY NUMBER PROBLEM

A. W. McGaughey, Bradley University

Too many students of college algebra have the preconceived idea that the so called "imaginary" numbers are imaginary, hence see little value in studying them. Even though the instructor mentions that functions of complex variables are very useful in determining constant potential surfaces and streamlines of flow as well as giving other examples of their uses in advanced mathematics, the student is seldom convinced that he is learning about something of a practical nature, or even knows what is being talked about.

I have been unable to document it, but I believe that, in one of Wentworth's algebra texts which was published about the turn of the century, one can find

the following problems, at least three very similar ones, which should help the algebra student to appreciate the value of complex numbers:

1. Two clocks were striking the hour. It was observed that the difference in their number of strokes was six and also that the number of strokes which one made was twice the square of the number of strokes which the other made. How many strokes did each clock make?

2. On a rather chilly morning a man read both his and his neighbor's thermometer. He noticed that one read six degrees higher than the other and that one reading was twice the square of the other reading. What were the temperature readings?

3. Two men left a certain place and walked in straight lines to their destinations which were six miles apart. If the distance which one of them walked was twice the square of the distance which the other man walked, how far did each man walk?

Each of the problems leads to the two sets of equations:

$$y = x + 6, \qquad y = 2x^2 \quad \text{and} \quad y = x + 6, \qquad x = 2y^2;$$

which have the solutions:

$$\begin{aligned} &x = 2, \quad y = 8; && x = -3/2, \quad y = 9/2; \\ &x = (-23 + i\sqrt{47})/4, && y = (1 + i\sqrt{47})/4; \\ &x = (-23 - i\sqrt{47})/4, && y = (1 - i\sqrt{47})/4. \end{aligned}$$

Only the first pair of values satisfy the first problem, the first two pairs satisfy the second problem; whereas, all four pairs can be interpreted to satisfy the third problem. For this interpretation we can use the values of x and y as the coordinates in the complex plane of the destinations of the men, assuming they left the origin. We find that they are six miles apart and that one man walked six miles, and the other man walked the square root of three miles.

In 3, the distances are x and $2x^2$, $3/2 \leq x \leq 2$, at angle $\cos^{-1}(x + 1/4x - 9/x^3)$. Ed.

A SIMPLE PROBLEM IN CYLINDRICAL COORDINATES

F. B. HILDEBRAND, Massachusetts Institute of Technology

An elementary calculus class is asked to employ double integration, using circular cylindrical coordinates, for the determination of the volume bounded above by a nappe of the cone $x^2+y^2=z^2$, below by the plane $z=0$, and laterally by the cylinder $x^2+y^2=x$. Many of the students set up the integral as

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^2 dr d\theta \quad \text{or} \quad 2\int_0^{\pi/2} \int_0^{\cos\theta} r^2 dr d\theta$$

and obtain a numerical answer which checks that given in the back of the book. Others, however, are led to the formulation

$$V = \int_0^{\pi} \int_0^{\cos\theta} r^2 dr d\theta,$$

and are less fortunate in the outcome of their calculation.

When the members of the class discuss this formulation, one student points out that here the variable r is not always positive, and hence that the equation of the nappe of the cone must be taken as $z=|r|$, so that, if the range $(0, \pi)$ in θ is insisted upon, the integrand should be written as $|r|r$. He proposes the formulation

$$V=\int_0^{\pi/2}\int_0^{\cos\theta} r^2drd\theta-\int_{\pi/2}^{\pi}\int_0^{\cos\theta} r^2drd\theta.$$

Calculation yields the correct result and all is well until another student insists that the element of plane area should be taken as $|r|drd\theta$, so that the integrand truly should be written as $|r|^2=r^2$ and the two errors appear to annul each other, leading again to the formulation under criticism.

While the remaining source of error eventually is discovered, to the satisfaction of the instructor, the varied reactions of the students afterward cause the instructor to wonder whether (1) the same discussion should be instigated in other classes or (2) such discussions should be averted in the future by a categorical statement that the range in θ should be so chosen that r is non-negative (presumably with a remark that the treatment of cases in which this is not possible is "beyond the scope of the course") or (3) there is a preferable third alternative.

A CURIOUS FORMULA FOR DISTANCE*

RAYMOND REDHEFFER, University of California, Los Angeles

The following minimax principle gives an interesting exercise in differential calculus:

THEOREM I. *Let R be a convex region in (x, y, z) space containing the points P and Q. Then the distance from P to Q is*

$$d(P,Q)=\sup_f \inf_{(x,y,z)} \frac{f(P)-f(Q)}{\sqrt{f_x^2+f_y^2+f_z^2}},$$

where the sup *is taken over all non-constant functions differentiable in R. It does not matter whether the* inf *is taken over all $(x, y, z)\in R$, or only over (x, y, z) on the line segment joining P and Q.*

For proof we use the mean-value theorem to obtain

$$f(P)-f(Q)=(x_1-x_0)f_x+(y_1-y_0)f_y+(z_1-z_0)f_z,$$

where $P=(x_1, y_1, z_1)$, $Q=(x_0, y_0, z_0)$. By the Schwarz inequality we have

* This paper was prepared under the sponsorship of the Office of Naval Research and the Office of Ordnance Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$|f(P)-f(Q)| \leqq H\, d(P, Q)$, where $H=\sqrt{f_x^2+f_y^2+f_z^2}$. Hence at some point of the line PQ, $d(P, Q) \geqq |f(P)-f(Q)|/H$ so that surely $d(P, Q) \geqq \inf [f(P)-f(Q)]/H$.

On the other hand, the choice

$$f(x, y, z) = (x_1 - x_0)x + (y_1 - y_0)y + (z_1 - z_0)z$$

yields $[f(P)-f(Q)]/H \equiv d(P, Q)$ as we see by a short calculation. This establishes the theorem.

The result may be applied in various ways, of which the following serves as an illustration:

THEOREM II. *Let the region R be a sphere of radius r, and let M be a positive constant. Then*

$$\inf_{f} \sup_{(x,y,z)} (f_x^2 + f_y^2 + f_z^2) = (M/r)^2,$$

where the inf *is taken over all differentiable functions f in the closed sphere which satisfy $f \geqq M$ on the boundary and $f=0$ at the center. It does not matter whether we require $f \geqq M$ at one boundary point or $f \geqq M$ at all boundary points.*

With the origin at the center, Theorem I asserts that $\inf [f(P)-f(0)]/H \leqq r$ if we choose P on the boundary, and hence $\sup H^2 \geqq (M/r)^2$. On the other hand the choice $f^2=(x^2+y^2+z^2)M^2/r^2$ would give equality, as we see from $ff_x = (M^2/r^2)x$ and from symmetry. This function is not admissible, but is easily approximated by one which is, and the desired result follows. (In two dimensions the reader may imagine a cone with a rounded tip at the origin. An analytical discussion is readily given, but it seems idle to belabor the matter.)

THE FUNDAMENTAL THEOREM OF ARITHMETIC

ROY DUBISCH, Fresno State College

When the student first meets the fundamental theorem of arithmetic he is likely to be puzzled by the "obviousness" of it and, at this level, it does little good to talk about the failure of the theorem for algebraic integers.

The following simple example, then, may be of value. We consider the set of rational numbers, m/n, where m and n are positive integers and $(m, n)=1$. A rational prime is defined as a rational number m/n such that m is an integral prime and n is either an integral prime or 1. Thus it is seen that (1) all integral primes are rational primes; and (2) the definition of an integral prime as an integer p such that $p \neq 1$ and $p=ab$ implies either a or b equal to 1 holds for rational primes. (Note that statements like $(3/5)=(3/10)(2/1)$ are not permissible, since $(3/10)(2/1)=(6/10)$ and $2 \neq 1$.) But, however, the fundamental theorem of arithmetic does not hold for rational primes since, for example, $(6/35)=(3/7)(2/5)=(3/5)(2/7)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1256. *Proposed by W. B. Andreasen, Lockheed Aircraft Corporation*

Discuss the error involved in the following approximate trisection of a circular arc AB. On chord AB locate C such that $BC=BA/3$ and D such that $CD=7AB/6$. With D as center and DC as radius describe an arc to cut arc AB in the approximate trisection point E.

E 1257. *Proposed by N. A. Court, University of Oklahoma*

(1) The medial triangle of each of the four triangles formed by the sides of a complete quadrilateral (q) taken three at a time is homological to the diagonal trilateral of (q).

(2) The four axes of the four homologies coincide.

E 1258. *Proposed by Aaron Herschfeld, Canisius College*

Prove that a necessary and sufficient condition for the rationality of

$$R = \sqrt[3]{a + \sqrt[3]{a + \cdots}},$$

where a is a positive integer, is that $a=N(N+1)(N+2)$, the product of three consecutive integers. In that case find R.

E 1259. *Proposed by C. D. Olds, San Jose State College*

A building a feet square has a walk $b<a$ feet wide around it. Two persons are on the walk; find the chance that they can see each other.

E 1260. *Proposed by Viktors Linis, University of Ottawa*

Evaluate $\int_0^{\pi/2} \cot\theta \ln \sec\theta d\theta$.

SOLUTIONS

The Red and White Cows

E 1226 [1956, 491]. *Proposed by J. V. Pennington, Houston, Texas*

A rancher bought a white cow, and in the following year a red one. Each succeeding year he duplicated his purchases of the preceding two years, buying the same number of cows, of the same colors and in the same order. Thus, in the third year, he bought a white and then a red cow; in the fourth year, a red,

then a white, and then a red cow; and so on. What was the color of the nth cow?

Solution by T. F. Mulcrone, Loyola University. If we associate with a white cow the number 1 and with a red cow the number 2, then the sequence defined in the problem becomes

$$\{a_n\} = 1;\ 2;\ 1,\ 2;\ 2,\ 1,\ 2;\ 1,\ 2,\ 2,\ 1,\ 2;\ \cdots,$$

whose nth term $a_n = [kn] - [k(n-1)]$, where $k = (\sqrt{5}+1)/2$ and $[x]$ is the greatest integer not exceeding x (see problem 4247 [1948, pp. 588–592]). Thus the nth cow is white or red according as a_n is 1 or 2.

Also solved by J. P. Ballantine, A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, Azriel Rosenfeld, E. D. Schell, and the proposer. Late solutions by Hazel E. Evans, Celestine O'Callaghan, and D. S. Passman.

The proposer gave also the equivalent statement, "The nth cow is red or white according as the fractional part of $n(\sqrt{5}+1)/2$ is or is not less than $(\sqrt{5}-1)/2$."

A Property of *e*

E 1227 [1956, 491]. *Proposed by R. L. Helmbold, Carnegie Institute of Technology*

Find all values of $a \geqq 1$ such that $a^x \geqq x^a$ holds for all values $x \geqq 0$.

Solution by D. S. Greenstein, Radio Corporation of America, Camden, N. J. The restriction $a \geqq 1$ may be replaced by $a \geqq 0$. Taking logarithms and dividing by the positive quantity ax, we must have $(\ln x)/x \leqq (\ln a)/a$. Since $(\ln x)/x$ attains its maximum uniquely at $x = e$, it follows that a must equal e.

Also solved by Michael Goldberg, Emil Grosswald, B. A. Hausmann, A. S. Hendler, P. B. Johnson, J. B. Johnston, Sidney Kravitz, Joe Lipman, D. C. B. Marsh, J. W. Mettler, J. B. Muskat, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, Anina Schub, A. V. Sylwester, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by I. M. Isaacs and D. S. Passman.

Editorial Note. Following are some MONTHLY references related to E 1227: [1916, 233–237], [1921, 141–143], [1931, 444–447], [1936, 229–230], E 640 [1945, 278], E 853 [1949, 555], E 1144 [1955, 446]. The problem is dealt with in Euler's *Introductio in Analysin Infinitorum*, II, p. 294. As the proposer remarked, E 1227 settles without computation the old problem, "Which is greater, e^π or π^e?"

An Affine Invariant

E 1228 [1956, 491]. *Proposed by Viktors Linis, University of Ottawa*

Let $n(P)$ be the number of distinct lines L through a point P such that L divides the area of a given triangle in two equal parts. Show that the locus of all points P with $n(P) \geqq 2$ is a region the ratio of whose area to the area of the given triangle is an absolute constant.

Solution by the proposer. The theorem is a consequence of the facts that $n(P)$ and the ratio of areas are affine invariants and that any triangle is affinely equivalent to an equilateral triangle.

For an equilateral triangle the required region is bounded by three arcs of hyperbolas, each of which has two sides of the triangle for asymptotes, and

pairs of which are tangent to one another at the midpoints of the medians. Taking the side of the equilateral triangle as the unit length, the rectangular equation of one of the hyperbolas referred to one side and one vertex of the triangle as x-axis and origin is $y(\sqrt{3}x-y)=3/16$. The area of the region can be obtained by simple integration as $\sqrt{3}$ $(\ln 8-2)/16$, and hence the required constant is $c=(\ln 8-2)/4\approx 0.0198$.

Also solved by Michael Goldberg, J. B. Johnston, and M. S. Klamkin.

Editorial Note. For an allied problem see E 774 [1950, 484].

Configurations of Four Coplanar Points

E 1229 [1956, 492]. *Proposed by M. P. Drazin, Trinity College, Cambridge, England*

Given any point O in the plane of a triangle $\Delta\equiv ABC$, let the sides a, b, c of Δ subtend angles A', B', C' at O, and let the distances from O to the vertices of Δ be a', b', c'. Show that the triangle with sides aa', bb', cc' has angles $A'-A$, $B'-B$, $C'-C$, and find the sextic polynomial relation connecting a, b, c, a', b', c'.

Solution by the proposer. (i) Take, for definiteness, the case in which O lies inside Δ, and construct, externally to Δ, triangle PCB directly similar to triangle OAB. Then $\measuredangle OBP=\measuredangle ABC$ and $BP/BO=BC/BA$, so that triangle OBP is similar to the given triangle ABC, corresponding sides being in the ratio b'/c. Consequently $\measuredangle BOP=\measuredangle BAC$ and $OP=bb'/c$, while also $PC=aa'/c$. Thus triangle OCP has sides aa'/c, bb'/c, c' and angle $A'-A$ at O, whence the first assertion follows by symmetry.

As an application, we deduce, on taking $a'=b'=c'=R$, the well known result that, given any circle ABC having its center O on the same side of BC as A, then the angle subtended by BC at O is twice that subtended at A.

(ii) Since $A'+B'+C'=2\pi$, we have

$$1+2\cos A'\cos B'\cos C' = \cos^2 A'+\cos^2 B'+\cos^2 C'.$$

On substituting $(b'^2+c'^2-a^2)/2b'c'$ for $\cos A'$, and similar expressions for $\cos B'$ and $\cos C'$, we obtain

$$\begin{aligned}4a'^2b'^2c'^2 &+ (b'^2+c'^2-a^2)(c'^2+a'^2-b^2)(a'^2+b'^2-c^2)\\ &= a'^2(b'^2+c'^2-a^2)^2+b'^2(c'^2+a'^2-b^2)^2+c'^2(a'^2+b'^2-c^2)^2,\end{aligned}$$

or, after a little simplification,

$$a^2b^2c^2+\Sigma\, a^4a'^2+\Sigma a^2(a'^2-b'^2)(a'^2-c'^2)=\Sigma b^2c^2(b'^2+c'^2)+\Sigma(b^2+c^2)b'^2c'^2.$$

A more symmetric form of this can be obtained by treating O, A, B, C on equal footing and replacing a, b, c, a', b', c' by d_{23}, d_{31}, d_{12}, d_{01}, d_{02}, d_{03}.

Partially solved by D. C. B. Marsh, who obtained the sextic relation.

The nth Integer Prime to 30

E 1230 [1956, 492]. *Proposed by Vassili Daiev, Sea Cliff, Long Island*

Find the nth term of the sequence of ordered positive integers prime to 30.

Solution by J. B. Muskat, Allston, Mass. Since $\phi(30)=8$, let $n=8q+r$, $0 \leqq r < 8$. Define $f(r)$ as follows: $f(0)=-1$, $f(1)=1$, $f(2)=7$, $f(3)=11$, $f(4)=13$, $f(5)=17$, $f(6)=19$, $f(7)=23$. Then the nth term a_n is $30q+f(r)$.

Also solved by D. A. Breault and Underwood Dudley (jointly), Leonard Carlitz, Monte Dernham, Hazel E. Evans, A. R. Hyde, P. B. Johnson, M. S. Klamkin, Joe Lipman, D. C. B. Marsh, G. E. Meador, Herbert Nadler, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, and the proposer. Some of these solutions were not entirely correct. Late solution by D. S. Passman.

The answer can be given in many forms. Thus Marsh gave the formula

$$a_n = 30[n/8] - 1 + 2r + 4[r/2] + 2[r/3] - 4[r/4] + 2[r/5] - 6[r/6] + 2[r/7],$$

where r is the remainder after n is divided by 8, and $[x]$ is the greatest integer not exceeding x; Ogilvy gave the formula

$$a_n = 2[(5n + 1 \pm 1)/4] + 2[(5n + 1 \pm 1)/8] - 1,$$

where the plus or minus is chosen according as the residue of n modulo 8 does not or does exceed 3; Ringenberg gave the formula

$$a_n = 30p + 15 + (-1)^q(k^2 - k + 2),$$

where

$$p = [(n-1)/8], \qquad q = [(n+3)/4], \qquad k = |n - 8p - 9/2| + 1/2;$$

and the proposer gave the formula

$$\begin{aligned} 4a_n = 15n - \{&11\langle(n-1)/8\rangle + 2\langle(n-2)/8\rangle + \langle(n-3)/8\rangle \\ &+ 8\langle(n-4)/8\rangle + 7\langle(n-5)/8\rangle + 14\langle(n-6)/8\rangle \\ &+ 13\langle(n-7)/8\rangle + 4\langle(n-8)/8\rangle\}, \end{aligned}$$

where $\langle x\rangle$ is 1 or 0 according as x is or is not an integer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4270 [1957, 49]. *Correction.* The number should be 4720.

4728. *Proposed by R. P. Boas, Jr., Northwestern University*

A. M. Rodov has propounded a proof that if $f(x)$ is continuous and the first of the following integrals converges, then the second diverges

$$\int_1^\infty f(x)dx, \qquad \int_1^\infty x^{-2}\{f(x)\}^{-1}dx.$$

(a) Construct a counter-example. (b) More generally, show that if $g(x)$ and $\phi(x)$ are positive and $\int_1^\infty \phi(x)dx$ diverges, then at least one of

$$\int_1^\infty \phi(x)g(x)dx \quad \text{and} \quad \int_1^\infty \{\phi(x)/g(x)\}dx$$

diverges.

4729. *Proposed by Paul Erdös, The University, Birmingham, England*

Let b_k, $k=1, 2, \cdots$, be any sequence of non-negative integers such that $\limsup b_k^{1/k} < 2$. Assume further that $\sum_{k=1}^n b_k \to \infty$, $\liminf \sum_{k=1}^n b_k/n = 0$. Prove that $\sum_{k=1}^\infty b_k/2^k$ is irrational.

4730. *Proposed by E. J. F. Primrose, University College, Leicester, England*

If a finite set of points in complex 3-dimensional space has the property that the line joining any two points of the set passes through a third point of the set, must all points of the set be coplanar?

4731. *Proposed by D. S. Stoller, Los Angeles, Calif.*

Consider an imperfect sorting process acting on a very large number of items, each of which belongs in one and only one of k bins. Let Q_i represent the probability that if an item belongs in bin i, it is sorted into bin i, and if it does not belong in bin i, it is sorted into some other unspecified bin. Find Q, the probability that an item is sorted into the correct bin.

4732. *Proposed by D. J. Newman, A VCO Research Division, Lawrence, Mass.*

Show that, in the ring of polynomials with integer coefficients, $(J(x))$, there are ideals which require arbitrarily many generators.

SOLUTIONS

Summation of Inverse Tangents

4678 [1956, 191]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Hartford (Conn.) Graduate Center*

Evaluate $\tan^{-1} 1 - \tan^{-1}(1/3) + \tan^{-1}(1/5) - \cdots$.

I. *Solution by Kovina Milosevich, Mathematical Institute, Skopje, Yugoslavia.* Starting with the known expansion

$$\frac{\pi}{4\cosh(\pi x/2)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n-1)^2 + x^2}$$

and integrating, we obtain $\tan^{-1} e^{\pi x/2} - \pi/4 = \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1}\{x/(2n-1)\}$. The required result is the special case which results upon letting $x \to 1$ and employing Abel's theorem. This gives

$$\tan^{-1} e^{\pi/2} - \pi/4 = \tan^{-1}(\tanh(\pi/4)) = \tfrac{1}{2}\tan^{-1}(\sinh(\pi/2)) = \tfrac{1}{2}gd(\pi/2).$$

II. *Solution by M. A. Rashid, Panjab University, Lahore, Pakistan.* De Moivre's theorem shows that, if

$$(1 + ib_1/a_1)(1 + ib_2/a_2) \cdots \to A + iB,$$

then

$$\tan^{-1}(b_1/a_1) + \tan^{-1}(b_2/a_2) + \cdots \to \tan^{-1}(B/A).$$

We have, therefore, $(1+ix)(1-ix/3)(1+ix/5)\cdots$

$$= \frac{Z}{\pi/4}\cdot\frac{1 - Z/\pi}{1 - (\pi/4)/\pi}\cdot\frac{1 + Z/\pi}{1 + (\pi/4)/\pi}\cdots,$$

$$= \frac{(Z)(1 - Z^2/\pi^2)(1 - Z^2/2^2\pi^2)\cdots}{(\pi/4)(1 - (\pi/4)^2/\pi^2)(1 - (\pi/4)^2/2^2\pi^2)\cdots},$$

$$= \frac{\sin Z}{\sin(\pi/4)} = \frac{\sin\{(1 + ix)\pi/4\}}{\sin(\pi/4)} = \cosh(\pi x/4) + i\sinh(\pi x/4),$$

where we have put Z for $(1+ix)\pi/4$. Hence

$$\tan^{-1} x - \tan^{-1}(x/3) + \tan^{-1}(x/5) - \cdots = \tan^{-1}(\tanh(\pi x/4)),$$

with the special case $x=1$ for the proposed series.

Also solved by A. D. Anderson, Leon Bankoff, Leonard Carlitz, A. E. Danese, R. B. Deal,

H. E. Fettis, G. B. Findley, N. J. Fine, H. E. Goheen, Peter Henrici, L. I. Lowell, Justin MacCarthy, A. J. Macintyre, M. Morduchow, F. D. Parker, Paul Payette, D. A. Robinson, D. C. Russell, Daniel Shanks, Leon Steinberg, Ernst Trost, Chih-yi Wang, R. E. Wild, J. E. Wilkins, Jr., Louise A. Wolf, David Zeitlin, and the proposer.

Editorial Note. Several readers pointed out that the formula was stated by Ramanujan, *Collected Papers*, p. 42. It also appears as exercise 16, p. 370, in Hobson, *Treatise on Plane Trigonometry*, 6th ed., 1925.

Ten Concyclic Orthopoles

4679 [1956, 191]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

Solution by Chih-yi Wang, University of Minnesota. We make use of the following known

THEOREM. *If a line meets the circumcircle of a triangle, the Simson lines of the points of intersection with the circle meet in the orthopole of the line for the triangle.* (See Court, *College Geometry*, 2nd ed., p. 289.)

Let the circumcircle $A_1A_2A_3A_4A_5$ be the unit circle, and the coordinates of A_i be $(\cos\theta_i, \sin\theta_i)$, $i=1, 2, \cdots, 5$. For definiteness let us find the coordinates of Ω_{12}. The equations of the Simson lines of A_1 and of A_2 are given by

$$y - \tfrac{1}{2}(\sin\theta_j + \sin\theta_3 + \sin\theta_4) = \tfrac{1}{2}\sin(\theta_3 + \theta_4 - \theta_j)$$
$$= \tan\tfrac{1}{2}(\theta_3 + \theta_4 + \theta_5 - \theta_j)\left[x - \tfrac{1}{2}(\cos\theta_3 + \cos\theta_4 + \cos\theta_j) + \tfrac{1}{2}\cos(\theta_3 + \theta_4 - \theta_j)\right],$$

for $j=1, 2$. By solving the simultaneous equations we obtain

$$\Omega_{12} = (\alpha + \tfrac{1}{2}\cos(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2),\ \ \beta + \tfrac{1}{2}\sin(\theta_3 + \theta_4 + \theta_5 - \theta_1 - \theta_2)),$$

where $\alpha=\frac{1}{2}\sum\cos\theta_k$, $\beta=\frac{1}{2}\sum\sin\theta_k$, $k=1, 2, \cdots, 5$. Since α and β are symmetric functions, by interchanging the subscripts we see readily that the ten points Ω_{ij} all lie on the circle of radius $\frac{1}{2}$ with center (α, β).

Also solved by J. W. Clawson, R. Goormaghtigh, O. J. Ramler, Sister M. Stephanie, and the proposer.

Editorial Note. Goormaghtigh gave this theorem in *Mathesis*, 1939, p. 312. Ramler gives an extension to the cyclic heptagon. If Ω_{ijk} denotes the Kantor point of a triangle $A_iA_jA_k$ with respect to the quadrangle formed by the remaining four vertices, then the thirty-five points Ω_{ijk} all lie on a circle one-half as large as the circumcircle of the heptagon.

Another Generalization of Clairaut's Differential Equation

4680 [1956, 191]. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

Solve the following generalization of Clairaut's differential equation

$$y - xy' + \frac{x^2y''}{2!} - \cdots + (-1)^{n-1}\frac{x^{n-1}y^{(n-1)}}{(n-1)!} + (-1)^n \frac{x^nF(y^{(n)})}{n!} = G(y^{(n)}).$$

Solution by the proposer. After differentiating the given equation and replacing $y^{(n)}$ by r, we can rewrite it as

$$dx^n/dr - x^nF'(r)H(r) = (-1)^{n-1}G'(r)n!H(r), \quad \text{where} \quad H(r) = (r - F(r))^{-1}.$$

The standard solution of this first order linear equation is

$$x^n = \exp\left\{\int F'(r)H(r)dr\right\} \int (-1)^{n-1}G'(r)n!H(r) \exp\left\{-\int F'(r)H(r)\,dr\right\} dr,$$

or $x=\phi(r)$. Now $(d^ny)/(dx^n)=r$, whence

$$y = \int \cdots \int r(dx)^n = \int \phi'(r) \int \phi'(r) \cdots \int r\phi'(r)(dr)^n.$$

This last equation and the equation $x=\phi(r)$ constitute the parametric form of the solution.

It is to be noted that there is no singular solution unless $r=F(r)$ in which case the equation reduces to that treated by Witty, this MONTHLY, 1952, pp. 100–102. See also the proposer's note, this MONTHLY, 1953, pp. 97–99.

Also solved by Muneer A. Rashid and Chih-yi Wang.

A Summation Problem

4682 [1956, 191]. *Proposed by R. C. Lyness, Preston, England*

(a) Prove that when the series

$$1 + \sum_{r=1}^{\infty}\binom{r\alpha}{r-1}\frac{x^r}{r}$$

is convergent, its sum, y, satisfies $y=1+xy^\alpha$.

(b) Prove also that

$$\sum_{r=1}^{\infty}\binom{r\alpha+\beta-1}{r-1}\frac{x^r}{r} = \frac{y^\beta - 1}{\beta}.$$

I. *Solution by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.* Case (a) is a special case of (b) which, in turn, is an application of Lagrange's reversion formula (See Bromwich, *Infinite Series*, p. 158): If $y=xf(y)$, then $g(y)=\sum_0^\infty p_ny^n$, where np_n is the coefficient of y^{-1} in the expansion of $g'(y)/x^n$.

Here $g(y)=(y^\beta-1)/\beta$, and $f(y)=y^{\alpha+1}/(y-1)$. It follows that the coefficient of y^{-1} in the expansion of $y^{\beta-1}y^{\alpha n}(y-1)^{-n}$ is

$$(-1)^n\binom{n-1-\alpha n-\beta}{n-1} = \binom{n\alpha+\beta-1}{n-1},$$

by the binomial theorem. Thus

$$\frac{y^{\beta}-1}{\beta}=\sum_{n=1}^{\infty}\binom{n\alpha+\beta-1}{n-1}\frac{x^{n}}{n}.$$

II. *Solution by Chih-yi Wang, University of Minnesota.* This is a simple and interesting application of relation (7) in Gould's paper, *Some generalizations of Vandermonde's convolution*, this MONTHLY, 1956, pp. 84–91. Following Gould's notation we have, since $x=(y-1)/y^{\alpha}$,

$$\sum_{r=1}^{\infty}\binom{r\alpha+\beta-1}{r-1}\frac{x^{r}}{r}=\frac{1}{\beta}\left[\sum_{r=0}^{\infty}A_{r}(\alpha,\beta)x^{r}-1\right]=\frac{1}{\beta}(y^{\beta}-1).$$

This is (b) and $\beta=1$ gives (a). Gould notes that these results are valid if $|x| < |(\alpha-1)^{\alpha-1}/\alpha^{\alpha}|$.

Also solved by H. W. Gould and Nathaniel Grossman.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Automatic Digital Computers. By M. V. Wilkes, F. R. S., Wiley, New York, 1956, x+305 pages, \$7.00.

This book is a good survey of the digital computing field. Being a survey, it will probably not satisfy each specialist in his own line, but it will give him a good understanding of the problems arising in other lines. As the author states, the book will not enable a competent computer-programmer to design automatic digital computers, nor will it enable a computer-designer to become a competent computer-programmer.

The book contains excellent sections on the history of automatic computers, the design of many of the present computers, the philosophy of coding, and the philosophy of computer design. The author goes quite deeply into the electronic circuitry of computers and into coding techniques in order to delineate the still unresolved controveries, such as serial against parallel design and automatic machine compiling of programs against direct coding in machine language.

The typography is clear and the illustrations both profuse and of good quality. One of the most valuable contributions is the comprehensive annotated bibliography.

This book should certainly be in the library of every large-scale computing facility and will be of real interest to all who are associated in any way with the digital computing field.

JOHN E. MAXFIELD
Naval Ordnance Test Station
China Lake, Calif.

Mathematics of Finance. By Robert and Helen Cissell. Houghton Mifflin, Boston, 1956. ix+198+88 pages, $4.50.

The nine chapters in this attractive looking textbook contain the material usually covered in a one-semester elementary course in the mathematics of investment. The chapter headings are: Simple Interest and Bank Discount, Compound Interest and Discount, Ordinary Annuities, Other Annuities Certain, Amortization and Sinking Funds, Bonds, Depreciation, Life Annuities, Life Insurance. There is no review of topics such as logarithms and progressions. The book contains a good selection of problems; answers to the odd-numbered problems are given.

This textbook has several excellent features. The many well-drawn time diagrams should prove especially helpful to the student. The authors point out common errors that students frequently make in setting up equations; these advance warnings should help to prevent the occurrence of some of the usual mistakes.

The various figures and charts in the book serve to stimulate interest in the subject. There are reproductions of notes, drafts, bonds, bond tables, and an F. H. A. amortization schedule; graphs illustrate the premium paid and the protection given under various life insurance plans. Practical bits of information are given in the hope that the student will become a wise borrower and investor; he is informed on topics such as sources of credit and the interest rates charged, the problems involved in financing the purchase of a home, and the wisdom of purchasing various types of life insurance.

One feature will be especially appreciated by students. An index of forty-one formulas is printed inside the back cover. These formulas are classified as to type (simple interest, ordinary annuities, etc.), and opposite each formula there is listed its use, followed by the page of the text on which the formula was introduced.

The tables deserve favorable comment. The fact that they are printed on yellow paper in very clear black ink makes them easy to locate and read. The logarithm tables are six-place tables, and the compound interest and annuity tables are ten-place tables. The arrangement of these ten-place tables is one not usually found in mathematics of finance books. (Cissells' tables were reproduced from the Compound Interest Tables of the Financial Publishing Company). The reviewer found that she was able to locate values much more rapidly in Cissells' tables than she could in the tables that she was accustomed to using.

Adverse criticism of the book is directed at the style of writing of the authors.

Excessive use is made of the first person plural. Explanations are phrased in the terminology that an instructor might use if he were talking quite casually to his class, but not in the terminology that he would use if he knew his words were to be recorded in print. A sentence on page 119 will illustrate this point: "As long as you owe several thousand dollars, you will be paying a lot of interest every year even though you have a relatively low rate."

In their attempts to oversimplify definitions and the presentation of new material, the authors have sometimes made statements which lack clarity and which do not have the precise meaning that was intended. For example, in their chapter on compound interest and discount, the authors have a section headed *Use of Logarithms*. In this section they state the following: "Logarithms can be used to solve problems when compound interest tables are not available or when the time or rate is not in the table. . . . If time is determined by logarithms and simple interest is to be used for the remaining fraction of a period, the amount should be determined for the integral number of periods. Then the simple interest formula can be used to get the additional time needed for this sum to equal the final amount." This is the complete quotation on that particular use of logarithms, and there is given no example which might clarify the authors' intended meaning.

If one is considering this textbook for possible classroom adoption, he should study carefully both its good points and its failings before making a decision.

VIOLET HACHMEISTER LARNEY
State University of New York
College for Teachers, Albany

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The National Science Foundation has announced the following Summer Institutes for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director listed below. Unless otherwise indicated, the Director is located at the same institution as the institute.

University of Colorado, June 24 to July 20: College teachers only. Professor R. E. Langer, University of Wisconsin, Madison.

University of Kansas, June 10 to August 2: High school and college teachers. Professor G. B. Price.

High school teachers only:

Columbia University, Teachers College, July 8 to August 16: Professor H. F. Fehr.

Indiana University, June 17 to August 9: Mrs. Marie S. Wilcox.

Miami University, June 17 to July 26: Professor T. C. Holyoke.

Montana State College, June 22 to August 23: Professor J. W. Hurst.

Polytechnic Institute of Puerto Rico, June 1 to July 27: Professor Mariano Garcia, College of Agriculture and Mechanical Arts, University of Puerto Rico, Mayaguez, P. R.

Teachers College at Oneonta, New York (for Junior High School Teachers), July 1 to August 9: Professor Vera Sanford.

State University of Iowa, June 17 to July 27: Professor L. A. Knowler.

University of Buffalo, July 8 to August 2: Professor Harriet F. Montague.

University of Chicago, June 24 to August 2: Professor A. L. Putnam.

University of Colorado, June 17 to July 27: Professor B. W. Jones.

University of Massachusetts, July 1 to August 16: Professor R. W. Wagner.

University of Notre Dame, June 21 to August 6: Professor A. E. Ross.

University of Wyoming, June 17 to August 9: Professor W. N. Smith.

PERSONAL ITEMS

St. Louis University: Dr. John Riner, Head of Department of Mathematics, St. Peter's College, has been appointed Assistant Professor; Mrs. Virginia H. Kern, McDonnell Aircraft Company, has been appointed Lecturer.

University of Massachusetts: Miss Elaine Cook, University of West Virginia, Miss Betty M. Navratil, Mathematical Analyst, Collins Radio Company, Cedar Rapids, Iowa, and Mr. D. H. Trahan have been appointed Instructors; Mr. R. E. Libera, Student, American International College, Mr. J. A. Pavelcak, Teacher, Hopkins Academy, Hadley, Massachusetts, and Mrs. Doris Stockton, Graduate Student, Brown University, have been appointed part-time Instructors; Dr. D. E. Moser has been promoted to Assistant Professor.

Dr. A. W. Adler has been appointed Instructor at Princeton University.

Dr. H. I. Ansoff, Rand Corporation, has accepted a position as Development Planning Specialist with the Lockheed Aircraft Corporation, Burbank, California.

Mr. J. D. Armstrong, Bolles School, is employed as a mathematician and computer at the Army Ballistic Missile Agency, Redstone Arsenal, Huntsville, Alabama.

Mr. G. E. Barlow, Jr., Assistant County Agent, Gate City, Virginia, has a position as an extension agricultural engineer at Iowa State College.

Dr. L. C. Barrett, University of Utah, has been appointed Associate Professor at Arizona State College.

Mr. H. H. Berry, Senior Research Engineer, Crosley Division, AVCO Manufacturing Company, Cincinnati, Ohio, is Associate Mathematician, Armour Research Foundation, Illinois Institute of Technology.

Miss H. Christine Boyd, Graduate Assistant, University of Mississippi, has been appointed Assistant Professor at Louisiana Polytechnic Institute.

Mr. C. M. Braden, Institute of Technology, University of Minnesota, has been appointed Assistant Professor at Macalester College.

Associate Professor J. L. Brenner, State College of Washington, has accepted a position as a mathematician with the Stanford Research Institute.

Associate Professor Eleazer Bromberg, New York University, has been promoted to Professor.

Mr. H. H. Brown, Franklin Institute, has accepted a position as a mathematician on the Technical Staff, Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor H. E. Campbell, Emory University, has been appointed Assistant Professor at Michigan State University.

Dr. W. C. Carter, Raytheon Manufacturing Company, has accepted a position as

Manager, Systems Programming Department, Datamatic Corporation, Newton Highlands, Massachusetts.

Professor Lamberto Cesari, Purdue University, was Visiting Lecturer at the Universities of Sao Paulo and Rio de Janeiro during the summer of 1956.

Mr. E. H. Connell, Consolidated-Vultee Aircraft Corporation, has accepted a position as a senior scientist with Lockheed Aircraft Corporation, Sunnyvale, California.

Mr. C. L. Davis, Mathematician, Curtiss-Wright Corporation, Clifton, New Jersey, has a position as a mathematician programmer with General Motors Research Staff, Detroit, Michigan.

Dr. R. L. Davis, University of Michigan, has been appointed Assistant Professor at the University of Virginia.

Assistant Professor W. E. Deskins, Ohio State University, has been appointed Assistant Professor at Michigan State University.

Professor O. L. Dustheimer, Ohio Northern University, has been appointed Professor at Youngstown University.

Professor H. S. Everett, University of Chicago, is on leave of absence and has been appointed Visiting Professor at Pennsylvania State University.

Mr. C. C. Faith, Michigan State University, has been promoted to Assistant Professor.

Mr. C. K. Fendall, Graduate Student, Reed College, has a position as an associate engineer for Boeing Airplane Company, Seattle, Washington.

Professor Emeritus L. R. Ford was presented with a collection of manuscripts dedicated to him on the occasion of his seventieth birthday, October 28, 1956. The papers were collected by Professors Karl Menger and Gordon Pall. The authors of the papers are: W. R. Abel and L. M. Blumenthal, H. T. Davis, W. L. Duren, Jr., H. J. Ettlinger and J. L. Cornette, G. C. Evans, L. R. Ford, Jr., J. W. Green, R. E. Langer, G. W. Mackey, Karl Menger, Gordon Pall, Tibor Radó, W. T. Scott, H. S. Wall, and G. T. Whyburn.

Assistant Professor Joel Franklin, University of Washington, has accepted a position as senior mathematician with the ElectroData Division, Burroughs Corporation, Pasadena, California.

Miss Joyce B. Friedman, ACF Electronics, has accepted a position as a mathematician with Technical Operations, Inc., Washington, D. C.

Mr. S. W. Golomb, Harvard University, is employed as a senior research mathematician at the Jet Propulsion Laboratory, California Institute of Technology.

Mr. R. M. Gordon, National Cash Register Company, has accepted a position as sales technical specialist with the ElectroData Division, Burroughs Corporation, Pasadena, California.

Mr. L. R. Harper, Jr., has been appointed Instructor at the University of Minnesota.

Assistant Professor Frank Hawthorne, Hofstra College, has been appointed Supervisor of Mathematics, New York State Education Department.

Mr. G. A. Heuer has been appointed Instructor at Concordia College.

Mr. Bernard Jacobson, Michigan State University, has been appointed Assistant Professor at Franklin and Marshall College.

Mr. M. L. Keedy has been appointed Instructor at the University of Nebraska.

Assistant Professor Paolo Lanzano, St. Louis University, is employed by Douglas Aircraft Company, Santa Monica, California.

Mr. C. W. Leininger, University of Texas, has been appointed Assistant Professor at Arlington State College.

Mr. A. J. Leino has accepted a position as a research engineer at Consolidated-Vultee Aircraft Corporation, San Diego, California.

Associate Professor L. H. Loomis, Harvard University, has been promoted to Professor.

Professor T. A. Love, Head, Department of Mathematics, Tennessee Agricultural and Industrial State College, has been appointed Professor and Chairman of the Department of Mathematics, Fisk University.

Mr. R. K. McConnell, Jr., New York University, has been appointed Assistant Professor at the University of Rhode Island.

Dr. D. G. Miller, Research Associate, Chemistry Department, Brookhaven National Laboratory, Upton, New York, has a position as a chemist at the University of California Radiation Laboratory, Livermore, California.

Dr. Josephine Mitchell, General Electric Company, has accepted a position as a research mathematician with Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Mr. J. T. Morse has accepted a position as a research mathematician with the Carter Oil Company, Tulsa, Oklahoma.

Mr. H. W. Moyer has a position as an associate engineer with Douglas Aircraft Company, Holloman Air Force Base, New Mexico.

Mr. H. L. Newman, Manager, TV Advanced Development, Sylvania Electric Products, Buffalo, New York, is now engineering specialist for Sylvania at Mountain View, California.

Associate Professor M. M. Ohmer, Southwestern Louisiana Institute, has been promoted to Professor.

Dr. J. B. O'Toole, Philco Corporation, has accepted a position as a research physicist with the Hughes Research and Development Laboratories, Culver City, California.

Miss Elaine B. Pavelka, Graduate Student, Northwestern University, is a teacher at Leyden Community High School, Franklin Park, Illinois.

Dr. Mary H. Payne, Columbia University, is employed as a research engineer at Fairchild Guided Missiles Division, Wyandanch, New York.

Dr. C. L. Perry, Jr., U.S. Naval Postgraduate School, has been appointed head of the mathematics group at Stanford Research Institute, Menlo Park, California.

Dr. Ronald Pyke, University of Washington, has been appointed a research associate at Stanford University.

Dr. L. B. Rall, Oregon State College, is employed as a mathematician by the Shell Development Company, Emeryville, California.

Dr. W. P. Reid, Staff Member, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, has been appointed Associate Professor at Michigan State University.

Dr. T. J. Rivlin, New York University, has accepted a position as senior mathematical analyst with Fairchild Engine and Airplane Corporation, Long Island, New York.

Dr. H. L. Rolf, Vanderbilt University, has been appointed Assistant Professor at Baylor University.

Miss Ruth L. Royer, Chico State College, has been promoted to Assistant Professor.

Dr. Stewart Schlesinger, Staff Member, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, is now Manager, Digital Computing Group, Aeronutronic Systems, Glendale, California.

Assistant Professor N. J. Schoonmaker, University of Massachusetts, has been appointed Head, Department of Mathematics, University of Vermont.

Mr. Cecil Schwartz, Senior Power Plant Engineer, Glenn L. Martin Company, Baltimore, Maryland, is a thermodynamics engineer for Marquardt Aircraft Company, Van Nuys, California.

Assistant Professor R. D. Sheffield, University of Mississippi, has a position as a senior nuclear engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. Bernard Sherman, University of California at Los Angeles, has accepted a position as research mathematician with Westinghouse Research Laboratories, Pittsburgh, Pennsylvania.

Associate Professor R. L. Shively, Manchester College, has been appointed Assistant Professor at Western Reserve University.

Dr. Alton H. Smith, University of Southern California, has accepted a position as a mathematician in the Computer Systems Division, Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. Malcolm Smith, Cook Research Laboratories, is now a senior staff member with Motorola, Riverside, California.

Dr. Ella Marth-Snader, District of Columbia Teachers College, has been appointed Professor at Chicago Teachers College.

Mr. W. A. Soper, Jr., Associate Engineer, Westinghouse Electric Corporation, Baltimore, Maryland, has accepted a position as a senior member of the Technical Staff, Airborne Systems Laboratory, Radio Corporation of America, Waltham, Massachusetts.

Dr. R. D. Stalley, Fresno State College, has been appointed Assistant Professor at Oregon State College.

Dr. J. M. Stark, Massachusetts Institute of Technology, has been appointed Professor at Lamar State College of Technology.

Associate Professor C. W. Topp, Fenn College, has been promoted to Professor.

Assistant Professor H. G. Tucker, University of Oregon, has been appointed Assistant Professor at the University of California, Riverside.

Assistant Professor H. E. Weissler, St. Mary's University, Texas, has a position as an assistant technical director, Falstaff Brewing Corporation, St. Louis, Missouri.

Miss Cecilia T. Welna, University of Massachusetts, has been appointed Instructor at Hillyer College.

Associate Professor J. L. Zemmer, University of Missouri, is on leave of absence and has been appointed a visiting fellow at Yale University.

Mr. H. C. Boardman, Director of Research, Chicago Bridge and Iron Company, Illinois, died on August 6, 1956. He was a member of the Association for thirteen years.

Professor Emeritus W. H. Kirchner, University of Minnesota, died on October 8, 1956. He was a member of the Association for thirty-four years.

Professor G. H. MacCullough, Worcester Polytechnic Institute, died on October 15, 1956. He was a member of the Association for seven years.

Mr. P. A. Piza, San Juan, Puerto Rico, died on November 5, 1956. He was a member of the Association for ten years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTIETH ANNUAL MEETING OF THE ASSOCIATION

The fortieth annual meeting of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on Saturday, December 29, 1956, in conjunction with the annual meetings of the American Mathematical Society and the Association for Symbolic Logic. There were registered 623 persons, including 410 members of the Association.

Sessions of the Association were held on Saturday morning and afternoon in the Upper Strong Auditorium of the University of Rochester. Professor Selby Robinson presided at the morning session and President W. L. Duren presided at the annual business meeting and at the afternoon session. The Program Committee for the meeting consisted of R. V. Churchill, Chairman; M. Gweneth Humphreys, and Selby Robinson.

FIRST SESSION OF THE ASSOCIATION

"Mathematics for the Undergraduate: Some Brave Experiments and Cogent Lessons," by Professor A. E. Ross, University of Notre Dame.

Instruction by Television or Films

"At the University of Washington," by Professor C. B. Allendoerfer, University of Washington.

"At Washington University, St. Louis," by Professor R. R. Middlemiss, Washington University.

"Activities at Other Places," by Professor P. S. Jones, University of Michigan.

"Retiring Presidential Address: Maintaining Communication," by Professor E. J. McShane, University of Virginia.

SECOND SESSION OF THE ASSOCIATION

Annual Business Meeting and Award of the Chauvenet Prize.

"Linear and Quadratic Programming," by Professor A. W. Tucker, Princeton University.

"Prime Numbers," by Professor J. B. Rosser, Cornell University.

"A Unifying Concept in n-dimensional Geometry," by Professor Walter Prenowitz, Brooklyn College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Friday afternoon in the Welles-Brown Room of the Rush Rhees Library, with seventeen members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Duren of the following Nominating Committee for 1957: H. W. Brinkmann, Chairman; R. L. Jeffery, and T. L. Wade.

Professor B. W. Jones of the University of Colorado was elected Second Vice-President for 1957–1958.

The New Jersey Section was officially constituted and Dean A. E. Meder, Jr., of Rutgers University was elected as Governor from the New Jersey Section to serve until June 30, 1958.

The Board voted to accept with an expression of gratitude the following grants: (a) from the National Science Foundation $1,200 for a planning meeting of the Committee on Films for Classroom Instruction; (b) from the National Science Foundation $55,200 for a continuation during 1957–1959 of the Program of Visiting Lecturers; and (c) from the Ford Foundation $150,000 for the continued support of the Committee on the Undergraduate Program in Mathematics.

It was voted to hold the Annual Meeting normally scheduled for December 1958 during the latter part of January 1959.

The Editor was authorized to publish four 96-page issues of the MONTHLY during 1957.

On the recommendation of the Committee to Study the Activities of the Association, the Board voted to request the Policy Committee for Mathematics to arrange as soon as possible a meeting of representatives of all mathematical organizations looking toward the establishment of an American Institute of Mathematics.

On the recommendation of the Committee to Recommend an Association Headquarters and to Nominate a Secretary-Treasurer, the Board voted to re-elect Professor H. M. Gehman for another five-year term (1958–1962) as Secretary-Treasurer of the Association. The Board also directed the Executive and Finance Committees to arrange that the University of Buffalo be compensated for a portion of Professor Gehman's salary so as to permit a corresponding reduction in his academic duties, that he be personally compensated for services during the summer months, that similar financial arrangements be made for an Associate Secretary, and that the Association pay rent for the quarters provided by the University of Buffalo.

It was voted to waive the usual initiation fee in the case of high school students (as well as college undergraduates) provided that the membership application blank is accompanied by the proper certification of status.

The following motion prepared by Professor Saunders MacLane was adopted:

Carl Allendoerfer is now approaching the end of his five year term of service as Editor-in-Chief of the American Mathematical Monthly. His service has been marked by a lively and progressive editorial policy and practice. He has cajoled authors, prodded referees and encouraged associate editors; he has read proof, dictated letters, and met innumerable problems, all to the end of producing a

stimulating journal for the members of the Association. The Board of Governors of the Mathematical Association of America takes this occasion to express by resolution its heartfelt appreciation of these his services. In the spirit of the MONTHLY articles: "What is X?" we can answer the question "What is a good editor?" by "Carl Allendoerfer."

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting was held on Saturday, December 29, 1956 in the Upper Strong Auditorium of the University of Rochester, Rochester, New York. President W. L. Duren presided.

The Secretary announced the results of the balloting for officers in which 1673 votes were cast: Professor G. B. Price of the University of Kansas was elected President for the two-year term 1957–1958, and Professors H. M. Bacon of Stanford University and J. R. Mayor of the University of Wisconsin were elected Governors for the three-year term 1957–1959.

The membership of the Association on December 26 was 6491, a gain of 424 since the beginning of 1956.

The three amendments to the By-Laws printed in the Program of the Rochester Meeting were adopted by the Association.

Reports were made by the chairmen of the following committees: B. W. Jones for the Committee on Visiting Lecturers; E. J. McShane for the Committee on the Undergraduate Program in Mathematics; G. B. Price for the Committee to Study the Activities of the Association; C. B. Allendoerfer for the Committee to Recommend an Association Headquarters and to Nominate a Secretary-Treasurer.

AWARD OF THE CHAUVENET PRIZE

The 1956 Chauvenet Prize was awarded to Professor R. H. Bruck of the University of Wisconsin for his paper entitled *"Recent Advances in the Foundations of Euclidean Plane Geometry"* published in Slaught Paper No. 4 (this MONTHLY, vol. 62 (1955) no. 7, part II, pp. 2–17). This award carries with it a cash prize of $100.

The 1956 Chauvenet Prize is awarded for a noteworthy expository paper published in English during the three-year period 1953–1955 by a member of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. This is the eleventh award of the Chauvenet Prize since its institution by the Association in 1925.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Thursday, December 27, through Saturday. The annual Gibbs Lecture, entitled "Mathematics and the future of science" was given by Professor M. H. Stone of the

University of Chicago. An invited address was delivered by Professor D. C. Spencer of Princeton University.

The Association for Symbolic Logic met on Thursday, December 27, at which time an invited address was given by Dr. G. Kreisel of the Institute for Advanced Study.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Walter Rudin, Chairman; E. H. Batho, Dorothy L. Bernstein, H. M. Gehman, N. G. Gunderson, Hewitt Kenyon, R. W. MacDowell, R. D. Schafer.

Registration headquarters were located in the Women's Residence Hall of the University of Rochester. Dormitory accommodations and meals were also provided in the Women's Residence Hall. Tea was served on Thursday afternoon at the Faculty Club. A conducted tour of the George Eastman House Museum of Photographic Art was held on Friday afternoon. The Employment Register and the Book Exhibit were on display in the basement of the Women's Residence Hall.

A banquet was held on Friday evening at which Professor J. F. Randolph acted as toastmaster. Mr. Sol Linowitz, a trustee of the University of Rochester, welcomed the visiting mathematicians on behalf of the University. Responses were given by President Richard Brauer of the American Mathematical Society, President W. L. Duren of the Mathematical Association of America, and President S. C. Kleene of the Association for Symbolic Logic. Professor M. H. Stone told of the activities of the International Mathematical Union. Professor R. L. Jeffery presented a resolution of thanks to our hosts at the University of Rochester for having planned so well for the comfort and pleasure of the visiting mathematicians.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 26, 1956. Professor Truman Wester, Chairman of the Section, presided. There were 93 persons in attendance, including 51 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor O. P. Sanders, University of Arkansas; Vice-Chairman, Professor W. A. Rutledge, University

of Tulsa; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma. At the business meeting, the Section voted to participate in the 1958 M.A.A. sponsored High School Mathematics Contest. The group voted to continue meeting in conjunction with the Oklahoma Education Association in the fall and to hold a separate spring meeting for research papers

The following papers were presented:

1. Principal address: *Teacher education and the college entrance examination board commission on mathematics*, by Professor Henry Van Engen; Editor, *Mathematics Teacher*, Iowa State Teachers College.

2. p (mod 30) *and its contents*, by Professor G. E. Meador, Oklahoma City University.

The representation of p(mod 30) is discussed as a special case of p(mod M) where p and M are relatively prime. Thus p(mod 30) equals $30q$ plus p and consists of eight arithmetic sequences whose first terms are; 1, 7, 11, 13, 17, 19, 23, and 29, with 30 the common difference in each case. Another representation of p(mod 30) is indicated by these eight numbers located on a circle which has been divided into thirty units. The extent to which circular addition and circular multiplication may be applied is discussed. Finally p(mod 30) is shown to contain all primes except 2, 3, and 5, and every product of two or more of these numbers is a composite number belonging to the group.

3. *The supplementary training program for high school science teachers*, by Professor J. H. Zant, Oklahoma Agricultural and Mechanical College.

The purpose of this supported program is to up-grade science and mathematics teaching by requiring these 50 high school teachers to take a 9-months program of graduate courses in the fields of biology, chemistry, mathematics, physics and engineering. It is hoped by this means to enable them to inspire and motivate more students to go to college and study science and engineering. Special courses, designed to accomplish the above purpose, have been organized. All are courses in science and are taught by productive, research scientists from the College staff, augmented by scientists from other institutions. Such programs will be continued next year and the number of institutions sponsoring them will be increased to approximately fifteen.

4. *Certain rationality relationships in triangles*, by Professor H. W. Linscheid, Southwestern State College.

Rationality of the three sides of a triangle and of its circumdiameter is a sufficient condition for the rationality of the trigonometric functions of the angles of the triangle, the length of the inradius, the lengths of the exradii, and the area of the triangle. A parametric representation is given for each of these quantities. A parametric representation is given also for the sides of a triangle containing an angle of 60° or 120° and for which the sides are represented by rational numbers.

5. Luncheon Address: *Modernizing the secondary school mathematics curriculum*, by Professor Henry Van Engen, Iowa State Teachers College.

6. Discussion: *What to do about college freshmen not prepared for mathematics courses on the college level*, led by Professor L. W. Johnson, Oklahoma Agricultural and Mechanical College.

The discussion leader introduced the proposals:

A. That Oklahoma colleges cease teaching high school courses such as solid geometry, elementary and intermediate algebra and similar remedial courses during the regular term and that remedial mathematics be taught only during the summer term when college plants and staffs are not overtaxed.

It was pointed out that this would provide better utilization of both professional staff and

space and that it would operate to improve the economic status of both college and high school mathematics teachers by expanding the opportunity for 12-months employment.

It was argued that the inconvenience to both parents and graduating high school seniors would operate to return the problem to the high school, where it properly belongs; also, that, "The shock to parents in learning that their children are not adequately prepared for college should occur while the children are in the local community, not later when they are away in college." It is hoped that the policy might eventually cause high schools to provide their own remedial instruction.

B. That the major colleges band together in this action, and in devising an examination to be given in the high schools during either the junior or senior year to let students who are so weak they must take remedial mathematics know about it in time to make proper preparation before attempting to enter a college or university in September.

It was recognized that some sort of High School "Exit" Examinations would have to be inaugurated to determine who would be required to attend college the summer before entering college in the fall. The feasibility of such examinations was discussed.

C. That high schools be encouraged to offer remedial courses to 12th grade students desiring them before they leave high school.

The discussion was lively. There was not complete agreement that the proposals were workable, but the consensus of opinion was that they had much merit and should be given serious consideration.

R. V. ANDREE, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The organizational meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers University, New Brunswick on November 3, 1956. Dean A. E. Meder, Jr., of Rutgers University presided over the morning session. Professor D. R. Davis of the New Jersey State Teachers College at Montclair presided over the afternoon session. In the unavoidable absence of the Secretary-Treasurer minutes for the latter part of the afternoon session were recorded by Mr. R. S. Lockhart, Madison High School, Madison. There were 71 persons present including 61 members of the Association.

At the business meeting By-Laws prepared by a committee under the chairmanship of Professor C. A. Nelson of Douglass College, Rutgers University, were adopted, subject to minor changes to be made by the Executive Committee. The following officers were elected: Chairman, Dean A. E. Meder, Jr., Rutgers University; Secretary-Treasurer, Professor I. L. Battin, Drew University; Members at Large of the Executive Committee: for one year, (Senior Member, Chairman of the Program Committee), Professor D. R. Davis, New Jersey State Teachers College at Montclair; for two years, Professor S. S. Wilks, Princeton University; for three years, Dr. H. O. Pollak of the Mathematical Research Department, Bell Telephone Company Laboratories, Murray Hill. The Executive Committee was authorized to make arrangements for another annual meeting in the fall of 1957. The Executive Committee was also authorized to investigate the possibility of holding a joint meeting with the Association of Mathematics Teachers of New Jersey, the time, possibly, to be in the spring.

Professor D. R. Davis served as Chairman of the ad hoc Program Committee. The following papers were presented:

1. *Mathematics in communication*, by Dr. Brockway McMillan, Assistant Director of Systems Engineering, Bell Telephone Laboratories, Murray Hill. (By invitation).

The following diagram illustrates the important role which mathematics has played in the

past, and is now playing, in the field of communication. Statistical generalization is required in passing from A to B. Statistics are important in all topics.

	CIRCUITS	TRAFFIC
TRANSMISSION	*A* Linear Graphs Differential Equations LaPlace Transform Functions of a Complex Variable	*B* Random Processes Fourier Integral
SWITCHING	*C* Linear Graphs Boolean Algebra Combinatory Analysis	*D* Random Walk (Markov Processes)

2. *Mathematics for digital computers*, by Mr. G. W. Kays, International Business Machines Corporation. (By invitation).

Electronic computers answer the need of the scientist and industrialist for extensive and rapid calculations. The high school student who continues his studies in mathematics will find many opportunities for a career in this rapidly expanding field. Aeronautics, market analysis, and production control are only a few of the fields that use this powerful tool. Examples of the type of mathematics applications shown included the extraction of a square root, the roots of a polynomial by synthetic division, the simultaneous solution of linear equations with elements of linear programming. The philosophies and techniques of elementary mathematics as they apply to the computer field were emphasized.

3. *Boolean algebra and its role in switching theory*, by Professor A. H. Diamond, Stevens Institute of Technology. (By invitation).

A Boolean algebra of two elements can be applied to the design of relay networks by interpreting the expressions of the algebra appropriately, *e.g.* "$a+bc'$" is interpreted to mean "relay a is closed or relay b is closed and relay c is open." A network of given design determines a formula which has the value 1 or 0, when certain relays are closed and others open, according as the circuit as a whole is closed or open. A fundamental problem is that of determining formulas corresponding to a minimum number of relay contacts for a given design.

I. L. Battin, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The second annual meeting of the Northeastern Section of the Mathematical Association of America was held at the University of Connecticut in Storrs, Connecticut, on November 24, 1956. Professor Howard Eves, Chairman of the Section, presided at the morning session and Professor Stanley Bezuszka, S.J., Vice-Chairman of the Section, presided at the afternoon session. Seventy-one persons attended the meeting. Of these, sixty were members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Stanley Bezuszka, S.J., Boston College; Vice-Chairman, Professor D. E. Richmond, Williams College; Secretary-Treasurer, Professor R. E. Johnson, Smith College. It was voted that the Chairman appoint a committee to study the question of sponsoring the High School Contest within the territory of the Section, said committee to present a recommendation at the next annual meeting of the Section.

The following papers were presented by invitation:

1. *An integral transform related to heat conduction*, by Professor D. V. Widder, Harvard University.

A temperature function is a solution $u(x, t)$ of the partial differential equation $\delta^2u/\delta x^2=\delta u/\delta t$. The source solution is $k(x, t)=(4\pi t)^{-1/2}\exp(-x^2/4t)$ for $t>0$. The chief purpose of the present study is to investigate the transform $f(t)=\int_0^\infty k(y, t)\phi(y)dy$. If this integral converges absolutely for $t>0$, we say that $f(t)\in A$. If $D^{-1/2}f(t)\in A$, where the Riemann-Liouville definition for the fractional integral $D^{-1/2}$ is intended, we say that $f(t)\in B$. A typical conclusion is that a temperature function has an absolutely convergent Poisson integral representation $u(x, t)=\int_{-\infty}^{\infty}k(x-y, t)\phi(y)dy$ if and only if $u(0, t)\in A$ and $u_x(0, t)\in B$.

2. *A property of the binomial coefficients*, by Professor F. W. Perkins, Dartmouth College.

This paper contains a discussion of the following theorem, based on a classical result due to Legendre and more recent work of Dickson: Let $n_Kn_{K-1}\cdots n_1n_0$ be the representation of a nonnegative integer n in a number system with a positive prime p as base. Then the number of those terms in the binomial expansion of $(x_0+x_1)^n$ whose coefficients are not divisible by p is $\prod_{k=0}^{K}(1+n_k)$. This may be generalized for the multinomial expansion of $(x_0+x_1+\cdots+x_m)^n$, where the number of coefficients not divisible by the prime p is $\prod_{k=0}^{K} C_m^{m+n_k}k$. A number of corollaries are discussed.

3. *The Cauchy integral theorem and the Poincaré-Bendixson theory*, by Professor F. M. Stewart, Brown University.

One proof of the Cauchy integral theorem begins by showing that a function analytic in a convex region has an indefinite integral. The invariance of the integral under deformations of the path of integration is then established by an easy direct computation. The same methods can be used to study the index of a curve with respect to a vector field. This simplifies the use of the index in proving results about the integral curves of $x'=f_1(x, y)$, $y'=f_2(x, y)$, while avoiding the use of the Jordan curve theorem in places where it is neither needed nor wholly adequate.

4. *Some algebraic aspects of elementary dimensional analysis*, by Professor D. E. Christie, Bowdoin College.

The symbolism of elementary dimensional analysis is considered as a free group on a finite but arbitrary number of generators. When a large number of generators is used, it is shown that ordinary empirical results induce endomorphisms of the group. An extended system distinguishing vectors from scalars is given a simple matric representation. Such examples are proposed as a means of motivating the teaching of algebra to undergraduate physics students.

5. *Euclides somewhat vindicatus*, by Professor I. H. Rose, University of Massachusetts.

A first approximation to a set of axioms for plane Euclidean geometry, exhibiting certain desirable pedagogical and logical properties, is presented. The axioms stem mainly from work of Fréchet and Hilbert, but embody also several much-maligned ideas of Euclid.

6. *Probability studies in the seventeenth century*, by Professor Oystein Ore, Yale University.

The point of departure is the well-known discussion of probability problems in 1654 between Blaise Pascal and Antoine Gombaud, sieur de Baussay, often grossly misrepresented as the "gambler" de Méré. The personal relations between the two men as well as the historical nature of their problems is elucidated in connection with the history of the development of the probability theory in the 17th century.

R. E. JOHNSON, *Secretary*

OFFICERS AND COMMITTEES AS OF JANUARY 1, 1957

OFFICERS

President, G. B. PRICE, University of Kansas (1957–1958)
First Vice-President, R. V. CHURCHILL, University of Michigan (1956–1957)
Second Vice-President, B. W. JONES, University of Colorado (1957–1958)
Editor, R. D. JAMES, University of British Columbia (1957–1961)
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo (1953–1957)
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo (1953–1957)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

SAUNDERS MACLANE, University of Chicago (1953–1958)
E. J. MCSHANE, University of Virginia (1955–1960)
W. L. DUREN, JR., University of Virginia (1957–1962)

Governors at Large

H. W. BRINKMANN, Swarthmore College (1955–1957)
M. A. ZORN, Indiana University (1955–1957)
A. S. HOUSEHOLDER, Oak Ridge National Laboratory (1956–1958)
M. F. SMILEY, State University of Iowa (1956–1958)
H. M. BACON, Stanford University (1957–1959)
J. R. MAYOR, University of Wisconsin (1957–1959)

Sectional Governors (July 1, 1954–June 30, 1957)

Allegheny Mountain, MORRIS OSTROFSKY, Westinghouse Electric Corporation
Indiana, P. D. EDWARDS, Ball State Teachers College
Kentucky, H. H. DOWNING, University of Kentucky
Metropolitan New York, R. M. FOSTER, Polytechnic Institute of Brooklyn
Nebraska, M. A. BASOCO, University of Nebraska
Northern California, W. H. MYERS, San Jose State College
Oklahoma, L. W. JOHNSON, Oklahoma Agricultural and Mechanical College
Rocky Mountain, C. A. HUTCHINSON, University of Colorado
Wisconsin, R. C. HUFFER, Beloit College

Sectional Governors (July 1, 1955–June 30, 1958)

Kansas, C. B. READ, University of Wichita
Missouri, F. F. HELTON, Central College
New Jersey, A. E. MEDER, JR., Rutgers University
Northeastern, G. B. THOMAS, JR., Massachusetts Institute of Technology
Ohio, L. L. LOWENSTEIN, Kent State University
Pacific Northwest, IVAN NIVEN, University of Oregon
Southeastern, F. W. KOKOMOOR, University of Florida
Southwestern, M. S. HENDRICKSON, University of New Mexico
Upper New York State, J. F. RANDOLPH, University of Rochester

Sectional Governors (July 1, 1956–June 30, 1959)

Illinois, E. C. KIEFER, Millikin University
Iowa, BERNARD VINOGRADE, Iowa State College
Louisiana-Mississippi, Z. L. LOFLIN, Southwestern Louisiana Institute

Maryland-Dist. of Col.-Virginia, O. J. RAMLER, Catholic University of America
Michigan, B. M. STEWART, Michigan State University
Minnesota, G. K. KALISCH, University of Minnesota
Philadelphia, N. J. FINE, University of Pennsylvania
Southern California, P. H. DAUS, University of California at Los Angeles
Texas, C. R. SHERER, Texas Christian University

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1956–1959), J. F. RANDOLPH (1954–1957), H. M. GEHMAN, *ex officio.*

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

TIBOR RADO, *Chairman* (1954–1959), P. R. HALMOS (1952–1957), SAMUEL EILENBERG (1953–1958), E. E. FLOYD (1956–1961), W. R. SCOTT (1957–1959).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

MARK KAC, *Chairman* (1955–1957), S. C. KLEENE (1956–1958), R. P. DILWORTH (1957–1959).

JOINT COMMITTEE ON PLACES OF MEETINGS

C. B. MORREY, JR., *Chairman* (1955–1957), R. M. THRALL (1956–1958), R. D. SCHAFER (1957–1959).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

L. M. KELLY, *Chairman* (July 1954–June 1957), R. E. GREENWOOD (July 1956–June 1959), LEO MOSER (July 1955–June 1958); L. E. BUSH, *Director* (July 1953–June 1958).

COMMITTEE ON SECTIONS

C. H. FRICK (1955–1958), ROY DUBISCH (1957–1960), EDITH R. SCHNECKENBURGER, *ex officio.*

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

R. D. SCHAFER, *Chairman* (1956–1958), H. S. M. COXETER (1955–1957), F. B. WRIGHT (1957–1959), R. D. JAMES, *ex officio.*

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

J. S. FRAME, *Chairman*, H. M. BACON, RAY BERKOWITZ, W. M. HIRSCH, MORRIS OSTROFSKY, G. W. PATTERSON, J. A. WARD.

COMMITTEE ON FILMS FOR CLASSROOM INSTRUCTION

P. S. JONES, *Chairman*, H. H. CAMPAIGNE, MARGUERITE LEHR.

COMMITTEE ON HIGH SCHOOL CONTESTS

W. H. FAGERSTROM, *Chairman.* (Other members to be appointed later).

COMMITTEE TO PREPARE A THIRD EDITION OF "PROFESSIONAL OPPORTUNITIES IN MATHEMATICS"

F. J. WEYL, *Chairman*, L. W. COHEN, A. S. HOUSEHOLDER, C. C. HURD, C. R. SHERER, T. H. SOUTHARD.

COMMITTEE TO STUDY THE ACTIVITIES OF THE ASSOCIATION

G. B. PRICE, *Chairman*, C. W. CURTIS, DAVID GALE, ROTHWELL STEPHENS, R. R. STOLL, PATRICK SUPPES, G. B. THOMAS, JR.

Committee on the Undergraduate Program in Mathematics

E. J. McShane, *Chairman*, W. L. Duren, Jr., J. G. Kemeny, G. B. Price, A. L. Putnam, A. W. Tucker, R. L. Davis, *Executive Secretary*.

Committee on Visiting Lecturers

B. W. Jones, *Chairman*, G. B. Huff, D. E. Richmond, R. A. Rosenbaum.

Nominating Committee for 1957

H. W. Brinkmann, *Chairman*, T. L. Wade, R. L. Jeffery.

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
D. E. Richmond (1955–1957), G. B. Price, *ex officio*, H. M. Gehman, *ex officio*.
On the National Research Council:
H. M. Gehman (July 1, 1956–June 30, 1959).
On the Council of the American Association for the Advancement of Science:
H. J. Ettlinger (1956–1957), M. R. Hestenes (1957–1958).
On the American Council on Education:
G. B. Price, *ex officio*, H. M. Gehman, *ex officio*.
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
P. S. Jones (1957–1959).
On the Committee on Definitions of Electrical Terms:
S. A. Schelkunoff.
On a Committee of the N.C.T.M. on a Vocational Pamphlet:
A. L. Putnam.
On the Committee on Mathematical Training of Social Scientists:
A. W. Tucker.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF JANUARY 1, 1957

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1950. For information about preceding years, consult the American Mathematical Monthly for March 1955.)

President

E. R. Hedrick	1916	E. T. Bell	1931–1932
Florian Cajori	1917	Arnold Dresden	1933–1934
E. V. Huntington	1918	D. R. Curtiss	1935–1936
H. E. Slaught	1919	A. J. Kempner	1937–1938
D. E. Smith	1920	W. B. Carver	1939–1940
G. A. Miller	1921	R. W. Brink	1941–1942
R. C. Archibald	1922	W. D. Cairns	1943–1944
R. D. Carmichael	1923	C. C. MacDuffee	1945–1946
H. L. Rietz	1924	L. R. Ford	1947–1948
J. L. Coolidge	1925	R. E. Langer	1949–1950
Dunham Jackson	1926	Saunders MacLane	1951–1952
W. B. Ford	1927–1928	E. J. McShane	1953–1954
J. W. Young	1929–1930	W. L. Duren, Jr.	1955–1956

Vice-President

N. H. McCoy	1949–1950	W. L. Duren, Jr.	1953–1954
L. M. Graves	1950–1951	H. S. M. Coxeter	1954–1955
Jewell H. Bushey	1951–1952	G. B. Price	1955–1956
F. L. Griffin	1952–1953		

Secretary-Treasurer

W. D. Cairns	1916–1942	W. B. Carver	1943–1947

Editor

C. V. Newsom	1947–1951	C. B. Allendoerfer	1952–1956

Governor (arranged alphabetically)

C. R. Adams	1948–1950	Ralph Hull	1946–1948, 1954–1956
E. B. Allen	1949–1952	S. B. Jackson	1953–1956
C. B. Allendoerfer	1949–1951	C. G. Jaeger	1950–1953
W. L. Ayres	1948–1950	R. D. James	1952–1955
R. H. Bardell	1948–1951	B. W. Jones	1951–1953
I. A. Barnett	1952–1955	P. S. Jones	1953–1956
C. F. Barr	1951–1954	M. S. Knebelman	1949–1952
T. A. Bickerstaff	1950–1953	D. H. Lehmer	1947–1949, 1952–1954
J. W. Bradshaw	1947–1950	A. J. Lewis	1948–1951
H. E. Bray	1947–1950	F. A. Lewis	1952–1955
J. C. Brixey	1951–1954	K. O. May	1953–1956
B. H. Brown	1952–1955	R. J. Michel	1952–1955
M. C. Brown	1948–1951	E. B. Miller	1953–1956
L. E. Bush	1947–1950	F. H. Miller	1948–1951
S. S. Cairns	1953–1955	W. E. Milne	1952–1954
R. H. Cameron	1950–1953	F. R. Morris	1948–1951
C. C. Camp	1948–1951	C. W. Munshower	1952–1955
G. R. Clements	1950–1953	C. O. Oakley	1953–1956
T. F. Cope	1951–1954	E. N. Oberg	1950–1951, 1954–1956
N. A. Court	1945–1947, 1948–1951	H. P. Pettit	1951–1954
H. S. M. Coxeter	1945–1947, 1951–1953	J. C. Polley	1951–1954
W. M. Davis	1951–1953	G. B. Price	1952–1955
H. L. Dorwart	1948–1951	G. E. Raynor	1950–1953
W. L. Duren, Jr.	1947–1950	F. A. Rickey	1953–1956
J. M. Earl	1951–1954	E. B. Roessler	1951–1954
G. M. Ewing	1949–1952	J. B. Rosenbach	1951
Tomlinson Fort	1949–1952	R. G. Sanger	1949–1952
J. S. Frame	1950–1953	I. S. Sokolnikoff	1947–1950
Philip Franklin	1954–1956	C. E. Springer	1954–1956
A. E. Gault	1950–1953	E. P. Starke	1947–1950
R. E. Gilman	1949–1952	F. H. Steen	1951–1954
R. F. Graesser	1952–1955	H. P. Thielman	1947–1950
D. W. Hall	1947–1950	C. W. Trigg	1953–1956
E. H. Hanson	1950–1953	A. W. Tucker	1953–1955
E. R. Heineman	1953–1956	Earl Walden	1949–1952
M. R. Hestenes	1950–1952	R. J. Walker	1949–1951
E. H. C. Hildebrandt	1947–1950	Marie J. Weiss	1950–1952
D. L. Holl	1953–1954	F. B. Wiley	1949–1952
Aughtum S. Howard	1951–1954		

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to January 1, 1957)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers

of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

ARTICLE VII—DUES

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.
ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.
INDIANA, May 4, 1957.
IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.
KANSAS, University of Kansas, Lawrence, April 13, 1957.
KENTUCKY, Berea College, Berea, April 27, 1957.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.
METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.
MICHIGAN, Wayne State University, Detroit, March 23, 1957.
MINNESOTA, Carleton College, Northfield, May 11, 1957.
MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.
NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.
NEW JERSEY, Fall, 1957.
NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 28, 1957.
NORTHERN CALIFORNIA
OHIO, University of Cincinnati, April 20, 1957.
OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA, November 28, 1957.
ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.
SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.
SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.
SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.
TEXAS, University of Houston, Houston, April, 1957.
UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.
WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. The following five numbers have already been published:

1. *Fourier Series; The Genesis and Evolution of a Theory* by R. E. Langer, v+86 pages.
2. *Outline of the History of Mathematics* (6th edition) by R. C. Archibald. iv+114 pages.
3. *Proceedings of the Symposium on Special Topics in Applied Mathematics.* iv+73 pages.
4. *Contributions to Geometry.* iv+75 pages.
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Copies at one dollar each postpaid may be ordered from: Professor Harry M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

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CONTENTS

APRIL 1957

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Ralph D. James, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, University of Buffalo, Buffalo 14, N. Y.

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COMMUTATIVITY IN FINITE MATRICES*

OLGA TAUSSKY, National Bureau of Standards

1. Introduction. The real and complex numbers have the properties

$$ab = ba \text{ for all } a \text{ and } b \tag{1}$$

and

$$\text{for every } a \neq 0 \text{ there is an inverse } a^{-1}, \tag{2}$$

which implies

$$ab = 0 \text{ only if } a = 0 \text{ or } b = 0. \tag{2'}$$

A classical theorem of Frobenius† states that there are no other hypercomplex systems which have these properties. If we want to consider more general systems, we must give up at least one of these properties. For example, if we do not insist on the commutative law (1), the quaternions of Hamilton become acceptable, but no others. The set of $n\times n$ matrices $A, B, \cdots$, with complex elements can be regarded as a hypercomplex system with n^2 base elements and in this system, in general, $AB\neq BA$. In general, also, there is no inverse A^{-1} even if $A\neq 0$ and, furthermore, AB can be zero without $A=0$ or $B=0$. The two axioms may even be violated simultaneously, so that when $AB=0$ we may still have $BA\neq 0$. A simple example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As soon as $AB\neq BA$ for a pair A, B, various other things go wrong too. Of these, we mention three:

(a) If $A_n=(a_{ik}^{(n)})$ and we define $\lim_{n\to\infty} A_n$ to be $(\lim_{n\to\infty} a_{ik}^{(n)})$, we may define the exponential of a matrix X by the formal power series $e^X = \sum_{i=0}^{\infty} X^i/i!$. Then, in general, $e^Ae^B\neq e^{A+B}$, but when $AB=BA$, we always have $e^Ae^B=e^{A+B}$.

Example: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^2 = 0, \quad B^2 = 0, \quad (A+B)^2 = I,$$

* Invited address at the meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 5, 1956.

The preparation of this paper was supported (in part) by the Office of Naval Research.

† References listed by sections will be found at the end of the paper.

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

but

$$e^{A+B} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}.$$

(b) Let $\lim_{n\to\infty} A^n = X$, and $\lim_{n\to\infty} B^n = Y$. Then $\lim_{n\to\infty} (AB)^n$ does not exist, in general, when $AB \neq BA$, while for $AB = BA$ we have $(AB)^n = A^n B^n$ for all $n \geq 1$, so that $\lim_{n\to\infty} (AB)^n = \lim_{n\to\infty} A^n \lim_{n\to\infty} B^n = XY$.

Examples: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so that $X = 0$, $Y = 0$, but

$$\lim_{n\to\infty} (AB)^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq XY.$$

If

$$A = \begin{pmatrix} 1/2 & 2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 0 \\ 2 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 4 + 1/4 & 0 \\ 0 & 0 \end{pmatrix},$$

and $X = 0$, $Y = 0$, but $\lim_{n\to\infty} (AB)^n$ does not exist.

(c) Let the eigenvalues of A be $\alpha_1, \cdots, \alpha_n$ and those of B, $\beta_1, \cdots, \beta_n$. When $AB \neq BA$, the eigenvalues $\gamma_1, \cdots, \gamma_n$ of AB are, in general, not $\alpha_1\beta_1, \cdots, \alpha_n\beta_n$ for any pairing. When $AB = BA$, however, the eigenvalues of any polynomial $p(A, B)$ are $p(\alpha_i, \beta_i)$, by another classical theorem of Frobenius.

Example: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 0$, while $\gamma_1 = 1$, $\gamma_2 = 0$.

The main object of this paper is a study of the replacement of $AB = BA$ in (c) by a weaker rule.

2. Commutators. Two theorems of McCoy and Drazin that are stated in Section 3 involve the notion of the commutator of two matrices A and B. We call $AB - BA$ the commutator of A and B and denote it by (A, B). Since $(A, B) = 0$ when $AB = BA$, the commutator may be said to "measure" how much AB differs from BA. In the theory of groups, if $AB \neq BA$, then $A^{-1}B^{-1}AB$ is called the commutator of A and B. Since, for matrices, A^{-1} and B^{-1} do not always exist, but addition and multiplication are defined, we take instead, $AB - BA$. We can also study commutators of higher order, for example, $(A, (A, B)) = A(AB - BA) - (AB - BA)A$ or $(B, (A, B))$.

The following remarks, about a special case of commutators, while not immediately relevant to our topic, indicate an interesting area of study.

Consider the commutators in which the last factor is A^*, the others all A, where A^* is the transposed and conjugate complex matrix of A. We write $(A, A^*) = AA^* - A^*A = C_2$, $A(A, A^*) - (A, A^*)A = C_3$, *etc.* Matrices for which $C_2 = (A, A^*) = 0$, the so-called normal matrices, play an important role as a natural generalization of hermitian matrices.

Suppose we have a matrix for which $C_2 \neq 0$, we might ask whether or not $C_3 = 0$. It turns out that if $C_2 \neq 0$ then $C_3 \neq 0$. Further if, $C_2 \neq 0$ then $(C_2, C_3) \neq 0$. Let us next consider C_4. We find that $C_4 = 0$ is possible even if $C_2 \neq 0$, $C_3 \neq 0$.

Example: If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then

$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and we have

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad C_4 = 0, \quad (C_2, C_3) = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

However, if in the case $n = 2$, we have $C_2 \neq 0$, $C_3 \neq 0$, $C_4 \neq 0$, then $C_n \neq 0$ for any n. For $n = 3$ the behavior is different as is shown by the following examples:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_4 \neq 0, \quad C_5 = 0; \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_5 \neq 0, \quad C_6 = 0.$$

3. Two theorems of McCoy and Drazin. McCoy proved that if $(A, B)\neq 0$, but $(A, (A, B))=(B, (A, B))=0$, then, although $AB\neq BA$, all eigenvalues of the product AB are of the form $\alpha_i\beta_i$ for a certain pairing and, more generally, all polynomials $p(A, B)$ have eigenvalues $p(\alpha_i, \beta_i)$ for the same pairing.

Later, M. P. Drazin generalized McCoy's theorem by showing that, if all commutators of A and B of a fixed order k (*i.e.*, involving k brackets) vanish, then all polynomials $p(A, B)$ have eigenvalues $p(\alpha_i, \beta_i)$.

Apparently there are none or at most very few theorems known which say: "such and such a statement is true if and only if $AB=BA$"; it is always "if $AB=BA$ then . . . ". Hence we are tempted to replace $AB=BA$ in various ways by weaker hypotheses. One way is to assume that instead of $AB-BA=0$ some of the higher commutators vanish. Another one is suggested by McCoy and Drazin's theorems: It is known that $AB=BA$ implies that all $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$, but the converse is not true. Consider, for example,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here $AB\neq BA$, but it can be shown (*cf.* Condition 4.1 below) that all $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$. Further, no commutator of the form $(A, (A, (A, \cdots (A, (A, B)) \cdots)))$, however long, can vanish.

4. Property *P*. McCoy asked and answered the question: which are the pairs of $n\times n$ matrices A, B, (with complex elements) for which all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$? Each of the following two conditions is necessary and sufficient.

Condition 4.1. *There exists a matrix S, such that $S^{-1}AS$, $S^{-1}BS$ are both left triangular, or both right triangular.*

Condition 4.2. *All matrices $f(A, B)(AB-BA)$ are nilpotent, where f is an arbitrary polynomial in A and B, and nilpotency of a matrix X means that $X^r=0$ for a certain integer r.*

A special case of McCoy's theorem was pointed out recently by H. Schneider, namely, if $AB=0$ and $BA\neq 0$. Then $AB-BA=-BA$ and $(-BA)^2=0$, further $A(BA)=0$, $(B(BA))^2=0$. Hence, by Condition 4.2, such a pair qualifies, and from Condition 4.1 it follows that a matrix S exists such that $S^{-1}AS$, $S^{-1}BS$ are both triangular.

Another example of pairs of matrices with property P was used by A. Brauer. Let A be any $n\times n$ matrix and v one of its eigenvectors. Let B be an $n\times n$ matrix of rank 1 whose columns are all multiples of v. Then $n-1$ of the eigenvalues of $A+B$ coincide with $n-1$ eigenvalues of A. One reason for this is that B has $n-1$ eigenvalues zero. Further it can be shown that A and B have property P. For, let SAS^{-1} be triangular with the eigenvalue corresponding to v in the upper left corner. Then v goes over into Sv which is the vector $(1, 0, \cdots, 0)$. Hence SBS^{-1} is triangular too, with $n-1$ zeros in the main diagonal.

5. Property L. We obtain a larger class of matrices if we do not assume that all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$, but only that a subset has this property. We may assume, for example, that all $\lambda A + \mu B$ have as eigenvalues $\lambda\alpha_i + \mu\beta_i$, where λ, μ are arbitrary, complex numbers. Such a pair A, B is said to have property L. We now obtain a larger class of matrices if $n \geq 3$ (for $n = 2$ the classes coincide). For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

have eigenvalues, 1, 2, 3, and -2, -2, 1. However, AB does not have the eigenvalues 1 or -2. On the other hand, it can be shown that the pair A, B has property L. Another example is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case all matrices $\lambda A + \mu B$ have all eigenvalues 0, but

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has some eigenvalues $\neq 0$.

If, however, A and B are both normal, then it is true that $AB = BA$ if we assume only that $\lambda A + \mu B$ have all eigenvalues $\lambda\alpha_i + \mu\beta_i$. This was proved by Wiegmann and Wielandt. Wielandt proved it from the following more general theorem:

Let A, B be normal and let $\gamma_i(z)$ be the eigenvalues of $A + zB$ and assume that $\sum_{i=1}^{n} |\gamma_i(z)|^2 \geq \sum_{i=1}^{n} |\alpha_i + z\beta_i|^2$ for at least three values of z which are the vertices of a triangle in the z-plane which contains O in the interior. It follows that $AB = BA$.

We, of course, assume much more, namely, that $\gamma_i(z) = \alpha_i + z\beta_i$ for all values of z.

6. Matrices in $A + zB$ with multiple eigenvalues. In the previous section we treated the matrices for which all $\lambda A + \mu B$ have eigenvalues $\lambda\alpha_i + \mu\beta_i$, and again, we can point out a larger class.

This can be done in a way that is easily described for $n = 2$. Here a pencil of matrices $A + zB$ has either all matrices with a double eigenvalue, or exactly two, or exactly one. There is no other possibility. If the pair A, B has property L, then it is very easy to see that either all matrices have a double eigenvalue, or exactly one does. However, the converse of this is true too. For $n > 2$ either all matrices in the pencil have a multiple eigenvalue or at most $n(n-1)$ do. If, however, the pair A, B has property L, then either all matrices in the pencil

$A+zB$ have a multiple eigenvalue or at most $n(n-1)/2$ do. The converse of this is not true, as is seen by the pair

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}1&1&2\\1&1&2\\-1&-1&-2\end{bmatrix}.$$

Here both A and B have a triple eigenvalue 0, but $A+B$ is nonsingular, so that property L does not hold. On the other hand, the matrices A and B are the only ones in the whole pencil $\lambda A+\mu B$ which have a multiple eigenvalue. However, a partial converse is true even for $n>2$.

7. Diagonable pencils. In a discussion of multiple eigenvalues we must consider eigenvectors too. We know, for example, that if all matrices $A+zB$ have the full number of eigenvectors (*i.e.*, if for all finite values of z and $z=\infty$, the matrix $A+zB$ is similar to a diagonal matrix), then $AB=BA$. If, in the whole pencil, even a single matrix is not similar to a diagonal matrix then AB need not be equal to BA. This is shown by the example,

$$A=\begin{pmatrix}0&0\\0&1\end{pmatrix},\qquad B=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

On the other hand, if $AB=BA$, then the pencil $A+zB$ contains either only diagonable matrices, or none, or exactly one. For $n=2$ the case of "none" never happens. For $n=3$ an example of a pencil with no diagonable matrices apart from 0 is given by

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}.$$

A pencil with only one diagonable matrix is given by

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad B=\begin{pmatrix}\beta&\beta_1\\0&\beta\end{pmatrix}.$$

References to 1

1. G. N. Cebotarev, On the solution of the matrix equation $e^B\cdot e^C=e^{B+C}$, Doklady Akad. Nauk SSSR (N.S.), vol. 96, 1954, pp. 1109–1112.

2. M. Fréchet, Les solutions non commutables de l'équation matricielle $e^x\cdot e^y=e^{x+y}$, Rend. Circ. Mat. Palermo (2), vol. 1, 1952, pp. 11–27.

3. M. Fréchet, Rectification, Rend. Circ. Mat. Palermo (2), vol. 2, 1953, pp. 71–72.

4. G. Frobenius, Über vertauschbare Matrizen, S.-B. Berlin. Math. Ges., 1896, pp. 601–14.

5. H. Geiringer, On the solution of systems of linear equations by certain iteration methods, Reissner Anniversary Volume, Contributions to Applied Mechanics, Edwards, Ann Arbor, 1948, pp. 365–393, in particular, p. 379.

6. C. W. Huff, On pairs of matrices (of order two) A, B, satisfying the condition $e^Ae^B\neq e^{A+B}\neq e^Be^A$, Rend. Circ. Mat. Palermo (2) vol. 2, 1953–54, pp. 326–330.

7. A. G. Kakar, Non-commuting solutions of the matrix equation exp $(X+Y)=\exp X \exp Y$, Rend. Circ. Mat. Palermo (2), vol. 2, 1953–54, pp. 331–345.

8. K. Morinaga and T. Nono, On the non-commutative solution of the exponential equation $e^x e^y = e^{x+y}$, J. Sci. Hiroshima Univ. Ser. A, vol. 17, 1954, pp. 345–358.

9. O. Taussky and J. Todd, Infinite powers of matrices, J. London Math. Soc., vol. 17, 1942, pp. 146–151.

References to 2

1. T. Kato and O. Taussky, Commutators of A and A^*, J. Washington Acad. Sci., vol. 46, 1956, pp. 38–40.

References to 3

1. N. H. McCoy, On quasi-commutative matrices, Trans. Amer. Math. Soc., vol. 36, 1934, pp. 327–40.

2. M. P. Drazin, Some generalizations of matrix commutativity, Proc. London Math. Soc., (3), vol. 1, 1951, pp. 222–231.

3. W. E. Roth, On k-commutative matrices, Trans. Amer. Math. Soc., vol. 39, 1936, pp. 483–95.

References to 4

1. A. Brauer, Limits for the characteristic roots of a matrix IV, Duke Math. J., vol. 19, 1952, pp. 75–91.

2. M. P. Drazin, J. W. Dungey, K. W. Gruenberg, Some theorems on commutative matrices, J. London Math. Soc., vol. 26, 1951, pp. 221–228.

3. L. S. Goddard, Note on a matrix theorem of A. Brauer and its extension, Canad. J. Math., vol. 7, 1955, pp. 527–530.

4. L. S. Goddard and H. Schneider, Pairs of matrices with a non-zero commutator, Proc. Cambridge Philos. Soc., vol. 51, 1955, pp. 551–553.

5. J. K. Goldhaber, The homomorphic mapping of certain matrix algebras onto rings of diagonal matrices, Canad. J. Math. vol. 4, 1952, pp. 31–42.

6. J. K. Goldhaber and G. Whaples, On some matrix theorems of Frobenius and McCoy, Canad. J. Math., vol. 5, 1953, pp. 332–335.

7. N. H. McCoy, On the characteristic roots of matrix polynomials, Bull. Amer. Math. Soc., vol. 42, 1936, pp. 592–600.

8. H. Perfect, Methods of constructing certain stochastic matrices, Duke Math J., vol. 20, 1953, pp. 395–404 (see also review in Math. Rev.); II Duke Math. J. vol. 22, 1955, pp. 305–311.

9. H. Schneider, A pair of matrices with property P, this MONTHLY, vol. 62, 1955, pp. 247–249.

References to 5–7

1. T. S. Motzkin and O. Taussky, Pairs of matrices with property L, Trans. Amer. Math. Soc., vol. 73, 1952, pp. 108–114.

2. T. S. Motzkin and O. Taussky, Pairs of matrices with property L, Proc. Nat. Acad. Sci. U.S.A., vol. 39, 1953, pp. 961–963.

3. T. S. Motzkin and O. Taussky, Pairs of matrices with property L (II), Trans. Amer. Math. Soc., vol. 80, 1955, pp. 387–401.

4. H. Wielandt, Pairs of normal matrices with property L, J. Res. Nat. Bur. Standards, vol. 51, 1953, pp. 89–90.

5. N. A. Wiegmann, A note on pairs of normal matrices with property L, Proc. Amer. Math. Soc., vol. 4, 1953, pp. 35–36.

6. I. Kaplansky, Completely continuous normal operators with property L, Pacific J. Math., vol. 3, 1953, pp. 721–4.

THREE HYPERBOLAS ASSOCIATED WITH A TRIANGLE

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1. Preliminaries. a. If a conic (q) is circumscribed about a triangle (T), the triangle (T') formed by the tangents to (q) at the vertices of (T) may be called the *tangential triangle of* (T) *for* (q).

b. THEOREM. *A triangle* (T) *and its tangential triangle* (T') *for a conic* (q) *are in perspective; the center of perspectivity* M *and the axis of perspectivity* m *are pole and polar with respect to* (q) [1].

Moreover, *the point* M *and the line* m *are trilinear pole and polar for each of the two triangles* (T) *and* (T'), as is readily verified.

The point M and the line m will be referred to as the *Lemoine point* and the *Lemoine axis* of (T) for the conic (q).

c. *Note.* Given the triangle (T), and one of the three elements, M, m, (q), the remaining two are determined. This is an immediate consequence of the preceding article.

d. THEOREM. *Given a triangle* (T), *the locus of the trilinear poles, for* (T), *of the lines passing through a fixed point* M *is a conic* (q) *circumscribed about* (T) [2].

The given point M is the Lemoine point of (T) for (q).

e. Conversely, *a conic* (q) *circumscribed about a given triangle* (T) *is the locus of the trilinear poles, for* (T), *of the lines passing through a fixed point, namely, the Lemoine point of* (T) *for* (q).

2. The isotomic transformation.

a. THEOREM. *If a point describes a straight line* u, *its isotomic, conjugate point, for a given triangle* (T), *describes a conic* (q) *circumscribed about* (T) [3, p. 125].

Moreover, *the Lemoine axis of* (T) *for* (q) *is the isotomic (or reciprocal) transversal* u' *of* u *for* (T).

Conic (q) may be called the *isotomic conic* of line u for (T).

b. Conversely, *a conic* (q) *circumscribed about a triangle* (T) *is the isotomic conic of a straight line, namely, of the isotomic conjugate* u, *for* (T), *of the Lemoine axis* u' *of* (T) *for* (q).

c. The two preceding propositions (Sections 2a, 2b) gain in interest when the line u coincides with its own isotomic conjugate transversal. There are four such lines in the plane of a triangle (T), namely, the three sides of the medial triangle $(T')=A'B'C'$ of (T), and the line at infinity of the plane of (T). The trilinear poles, for (T), of these four lines are the respective vertices of the anticomplementary, or more briefly, *an-ry*, triangle $(T'')=A''B''C''$ of (T), and the common centroid G of the three triangles (T), (T'), and (T'').

Of these four lines, the line at infinity i seems to be the only one whose isotomic conic with respect to (T) has been considered [6].

If the Lemoine axis of a triangle (T) for a conic (E) circumscribed about (T) coincides with the line at infinity i, the tangential triangle of (T) for (E) coincides with the an-ry triangle (T'') of (T). Hence, the isotomic conic (E) of i for (T) coincides with the circumscribed minimal (or Steiner) ellipse (E) of (T) [5].

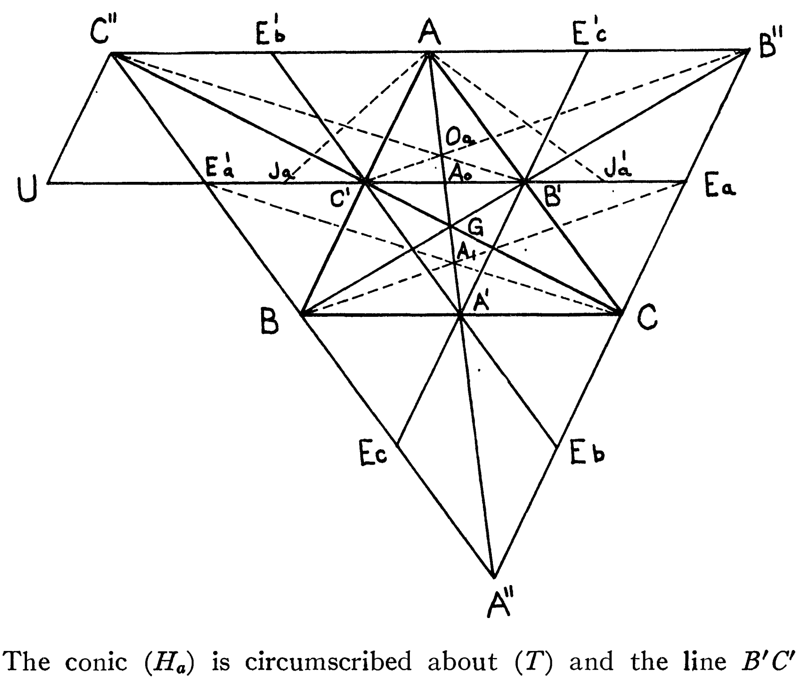
d. Moreover, the Lemoine point of (T) for (E) is the trilinear pole, for (T), of the Lemoine axis i of (T) for (E) (Section 1b), that is, the centroid G of (T). We have thus the following property of (E) (Section 1d): the trilinear poles, for a triangle (T), of the lines passing through the centroid of (T) lie on the circumscribed minimal ellipse of (T).

Observe that the Lemoine point G of (T) for (E) (Section 1b) is also the pole of the Lemoine axis i for the conic (E) (Section 1d). Hence, the point G is the center of the ellipse (E).

In what follows we shall consider the isotomic conics, for (T), of the other three self-conjugate isotomic transversals of (T).

3. A circumscribed hyperbola. a. The isotomic conic (H_a), for (T), of the self-conjugate isotomic transversal $B'C'$ which joins the midpoints B', C' of the sides AC, AB of (T) has the following two properties (Section 2):

(1) *The conic (H_a) is the locus of the isotomic conjugate points, for (T), of the points of $B'C'$.* (2) *(H_a) is the locus of the trilinear poles, for (T), of the lines passing through the vertex A'' of the an-ry triangle (T'') of (T).*



b. The conic (H_a) is circumscribed about (T) and the line $B'C'$ is the

Lemoine axis of (T) for (H_a) (Sections 3a, 2c), that is, the tangents to (H_a) at the vertices of (T) pass through the traces of $B'C'$ on the respectively opposite sides of (T). Hence, the conic (H_a) is tangent to the medians BB', CC', of (T), passing through the vertices B, C, while the tangent to (H_a) at A is the side $B''C''$ of the an-ry triangle (T'') of (T).

We shall refer to the point A as the *principal* point of the conic (H_a).

c. The points B', C' obviously lie inside the circumscribed Steiner ellipse (E) of (T) (Sections 2c, 2d). Hence line $B'C'$ meets (E) in two real points J_a, J_a'. The isotomic point I_a of J_a for (T) lies on the conic (H_a) (Section 3a) and on the line at infinity (Section 2c). The same is true for the isotomic I_a' of J_a', and it follows that (H_a) has two real points at infinity. Hence, the isotomic conic (H_a) for (T) of the self-conjugate isotomic transversal $B'C'$ is a hyperbola.

d. The median AGA' of (T) passes through the center G of the ellipse (E) (Section 2d). Now the chord J_aJ_a' is parallel to the tangent $B''C''$ to (E) at A. Hence the midpoint of J_aJ_a' is the trace A_0, on J_aJ_a', of the median AGA', which point is also the midpoint of the segment $B'C'$. Consequently, lines AJ_a, AJ_a' will meet the parallel BC to $B'C'$ in two points symmetrical with respect to A', that is, lines AJ_a, AJ_a' are a pair of isotomic lines for the angle A of (T). Therefore line AJ_a' passes through the isotomic conjugate point I_a (Section 3c) of the point J_a for the triangle (T). Similarly, line AJ_a passes through the isotomic point I_a' of J_a' for (T). Consequently (Section 3c), the two lines AJ_a, AJ_a' are a pair of asymptotic directions of the hyperbola (H_a).

Note. The points J_a, J_a' may be located, without drawing the ellipse (E), as follows. The tangents to (E) at points A, B meet in point C'' (Section 2c). Hence the polar, for (E), of the midpoint C' of AB passes through C'' and through the point at infinity of AB, that is, the polar of C' for (E) is the parallel to AB through C''. The trace U of that polar on line $B'C'$ is the homologous point of C' in the involution (I) of conjugate points, for ellipse (E), on line $B'C'$. The center of (I) is the point A_0, and points J_a, J_a' are the double elements of (I). Hence $(A_0J_a)^2=(A_0C')(A_0U)$. On the other hand, $C'U=AC''=BC=4(A_0C')$. Therefore, $A_0U=5(A_0C')$ and $AJ_a=AJ_a'=\sqrt{5}(A_0C')$.

e. Let $E_a=(B'C', A''B'')$, $E_a'=(B'C', A''C'')$. Point E_a is the midpoint of CB'', and AB is parallel to $A''B''$. Hence the pencil $B(CE_aB''A)$ is harmonic, and its section $(A'A_1GA)$ by AA'' is likewise harmonic. Thus the point $A_1=(BE_a, AG)$ is the harmonic conjugate of A with respect to the pair of points G, A'.

Now the tangents BB', CC' to (H_a) at points B, C (Section 3b) intersect in point G. Hence BC is the polar of G for (H_a) and, since point A lies on (H_a), point A_1 lies on the hyperbola (H_a).

Observe that point A_1 is the point of intersection of the diagonals BE_a, CE_a' of trapezoid BCE_aE_a'.

f. Point C is the midpoint of $A''B''$ and line AB is parallel to $A''B''$, whence $C'(A''CB''A)$ is a harmonic pencil. It follows that the range $(A''A_0O_aA)$, which

this pencil determines on the transversal AA'', is also harmonic. Thus the point $O_a = (C'B'', AGA'')$ is the harmonic conjugate of A'' for the points A, G.

Now the polar BC, for (H_a), of point G (Section 3e) is parallel to the tangent $B''C''$ of (H_a) at point A (Section 3b). Hence AGA'' is a diametral line of (H_a). On the other hand, lines BA_1E_a, $C'O_aB''$ are parallel, and C' is the midpoint of AB. Therefore, in triangle ABA_1, point O_a is the midpoint of side AA_1. Hence O_a is the center of (H_a) (Section 3e). Consequently,

(1) *The center of the hyperbola* (H_a) *is the harmonic conjugate, with respect to the principal point of* (H_a) *and the centroid of* (T), *of the vertex of* (T'') *which corresponds to the principal point of* (H_a).

(2) *Point* A_1 (Section 3e) *is the diametric opposite of the principal point of* (H_a) *on this conic.*

Observe that the center O_a of (H_a) is the point of intersection of the diagonals $C'B''$, $B'C''$ of trapezoid $B'B''C''C'$.

g. We have (Section 3f), $-1 = (A''GO_aA) = (AO_aGA'')$. If the latter harmonic division is compared with the harmonic division $(A'A_1GA)$ (Section 3e), it is seen that the three pairs of points A', A; G, G; A, A'' are corresponding elements in the homothecy $(G, -1:2)$ which transforms triangle (T) into its an-ry triangle (T''). Now the biratio (that is, the anharmonic ratio) is invariant under a homothecy, hence the fourth pair of points A_1, O_a are also homologous elements under the homothecy. Consequently, the diametric opposite of the principal point of the hyperbola (H_a) and the center of this conic are corresponding points in the homothecy $(G, -1:2)$.

A numerical verification of this proposition may be readily obtained. In triangle (T) we have $AG:AA' = 2:3$, and hence $A_1G:A_1A' = 2:3$ (Section 3e). Thus, if we put $A_1G = 2$, we have successively,

$$A_1A' = 3, \quad GA' = 2 + 3 = 5, \quad AG = (2)(5) = 10,$$
$$AA_1 = 10 + 2 = 12, \quad O_aA_1 = 12:2 = 6, \quad GO_a = 6 - 2 = 4,$$
$$GA_1:GO_a = 2:4 = 1:2.$$

That this last ratio is negative is evident in the figure.

h. (1) *The lines drawn through the center* O_a *of* (H_a) (Section 3f) *parallel to the lines* AJ_a, AJ_a' (Section 3d) *are the asymptotes of the hyperbola. They meet side* BC *of* (T) *in two isotomic points.*

Having located the asymptotes, the axes of (H_a) are obtained by drawing the bisectors of the angles formed by them.

(2) *Let* R_a *be the fourth common point, besides the vertices of* (T), *of* (H_a) *and the circumcircle* (O) *of* (T). *This point is the analog, for* (H_a), *of the Steiner point* R *of the Steiner ellipse* (E) [4]. *The isotomic of* R_a *is the point of intersection of the line* $B'C'$ *with the Longchamps axis of* (T).

(3) *The trilinear polar of* R_a, *for* (T), *joins the Lemoine point of* (T) *to the vertex* A'' *of* (T'').

4. Two other hyperbolas. The hyperbola (H_a) has been obtained as the

isotomic conic of the self-conjugate isotomic transversal $B'C'$ for (T) (Section 3a).

Considering the two remaining, analogous transversals $C'A'$, $A'B'$, two analogous hyperbolas (H_b), (H_c) are obtained. These conics are circumscribed about (T), are tangent at the vertices B, C, respectively, to the sides $C''A''$, $A''B''$ of the an-ry triangle (T'') of (T), and tangent to the medians of (T) at the remaining pairs of vertices of (T).

The centers, asymptotes, *etc.*, of these conics are obtained in the same way as the corresponding elements were obtained for the hyperbola (H_a). The explicit formulation of the corresponding propositions and their proofs offer no new difficulties and are omitted.

In what follows, some relations of the three conics to one another and to the circumscribed Steiner ellipse (E) of (T) will be considered.

5. Three circumscribed hyperbolas. a. The two hyperbolas (H_b), (H_c) pass through the vertex A of (T) and are tangent at that point to the median AA' of (T); hence the two conics have no point in common, besides the vertices of (T), and this is true also for the pairs of hyperbolas (H_c), (H_a); (H_a), (H_b).

Both the hyperbola (H_a) and the Steiner ellipse (E) are circumscribed about (T) and touch side $B''C''$ of the an-ry triangle (T'') of (T) at point A. It follows that these two conics have no other point in common, and a similar statement holds for (E) and each of the two hyperbolas (H_b), (H_c). Consequently, each of the four conics (H_a), (H_b), (H_c), (E) is tangent to the remaining three, and no two of them have a point in common, except the vertices of the triangle (T).

Observe that if (T) is an equilateral triangle, its circumcircle (O) is tangent to the sides of the an-ry triangle (T'') of (T) at the vertices of (T), whence the Steiner ellipse (E) coincides with (O). The three hyperbolas (H_a), (H_b), (H_c) are congruent, and a rotation of the entire figure about an angle of 120°, using the center of (O) for center of rotation, will bring the figure into coincidence with itself. The esthetic qualities of the figure may be worthy of the skill of a draftsman.

b. The triangle $O_aO_bO_c$ formed by the centers O_a, O_b, O_c of the hyperbolas (H_a), (H_b), (H_c) will be referred to as the *central triangle* of the three conics and will be denoted by ω.

The center O_a lies on the segment GA (Section 3f), and we have $GO_a:GA = 2:5$ (Section 3g). Similarly, $GO_b:GB = GO_c:GC = 2:5$. Therefore, the central triangle of the three hyperbolas (H_a), (H_b), (H_c) is homothetic to the basic triangle (T).

The center of the homothecy is the centroid G of (T); hence the point G is also the centroid of ω.

The point G is also the center of the ellipse (E) (Section 2d). Hence the above proposition may be stated as follows:

The center of the Steiner ellipse is the centroid of the central triangle of the three hyperbolas considered.

c. Let $(T_1) = A_1B_1C_1$ denote the triangle formed by the point A_1 (Section 3f) and its analogs B_1, C_1 for the hyperbolas (H_b), (H_c). By a method similar to the one used in connection with triangle ω (Section 5b), we may obtain the following results:

Triangle (T_1) is homothetic to the given triangle (T); the center of homothecy is the centroid G of (T), and the ratio of homothecy is $-1:5$ (Section 3g); *the centroid G of (T) is also the centroid of (T_1).*

d. Point G is the centroid of both triangle ω and triangle (T_1). Moreover, the points A_1, O_a, and, by analogy, the corresponding pairs of points B_1, O_b; C_1, O_c for (H_b), (H_c), are homologous points in the homothecy $(G, -1:2)$ (Section 3f), whence we conclude: The diametric opposites, on the respective conics, of the principal points of the three hyperbolas (H_a), (H_b), (H_c), are the vertices of the medial triangle of the triangle formed by the centers of those three curves.

e. The preceding proposition (Section 5d) gives rise to a considerable number of corollaries. Here is one example.

The line O_bO_c is parallel to BC (Section 5b) and therefore to $B''C''$; the latter line touches (H_a) in the diametric opposite A of the point A_1 common to O_bO_c and (H_a), whence the line O_bO_c is tangent to the hyperbola (H_a).

Analogous considerations apply to the lines O_cO_a, O_aO_b. Consequently, the line of centers of any two of the three hyperbolas considered is tangent to the third hyperbola, the point of contact being the diametric opposite of the principal point of the latter curve.

f. We have (Section 3g) $GO_a:GA = GA_1:GA' = 2:5$, and similarly, for the analogous pairs of points O_b, B; B_1, B' and O_c, C; C_1, C' relative to (H_b) and (H_c). Hence the figure formed by triangles ω and (T_1) corresponds to the figure formed by (T) and its medial triangle $(T') = A'B'C'$ in the homothecy $(G, 2:5)$. Thus, if we construct for triangle ω the conics analogous to the conics (H_a), (H_b), (H_c), (E), of (T), the former set of conics will correspond to the latter set in the homothecy $(G, 2:5)$. In the conics so obtained we may consider the centers of the conics, *etc.*

The same construction may be applied again to the new figure, and so on, *ad infinitum.*

The same thing may be done in the opposite direction. We may construct the figure which corresponds to the figure (T), (T'), and its conics, in the homothecy $(G, 5:2)$. The new hyperbolas will have for their centers the points $A, B, C, \cdots$. This process, too, may be repeated *ad infinitum.*

g. Let R_b, R_c be the fourth points of intersection of (H_b), (H_c) with the circumcircle (O) of (T). We have (Sections 3h (2) and 3h (3)):

The four points R_a, R_b, R_c, R have for their isotomic points, for (T), the traces of the Longchamps axis of (T) on the sides of the medial triangle (T') of (T), and the point at infinity of that axis; the trilinear polars, for (T), of the same four points are the lines which join the Lemoine point of (T) to the vertices of triangle (T'') and to the common centroid G of (T) and (T'').

References

1. L. Cremona, Projective Geometry, p. 239, art. 336.
2. J. J. Mathieu, Étude de géométrie comparée, Nouv. Ann. Math., ser. 2, vol. 4, 1865, p. 407.
3. G. de Longchamps, Étude de géométrie comparée, Nouv. Ann. Math., 1866, p. 118 ff.
4. Nathan Altshiller-Court, College Geometry, 2nd ed., New York, Barnes and Noble, 1952, p. 282, art. 679.
5. J. Steiner, Werke, vol. 2, p. 690.
6. J. A. Third, Two geometric transformations, Proc. Edinburgh Math. Soc., vol. 18, 1900, p. 23.

"IF THIS BE TREASON . . . "

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If I had to name one trait that more than any other is characteristic of professional mathematicians, I should say that it is their willingness, even eagerness, to admit that they are wrong. A sure way to make an impression on the mathematical community is to come forward and declare, "You are doing such-and-such all wrong and you should do it *this* way." Then everybody says, "Yes, how clever you are", and adopts your method. This of course is the way progress is made, but it leads to some curious results. Once upon a time square roots of numbers were found by successive approximations because nobody knew of a better way. Then somebody invented a systematic process and everybody learned it in school. More recently it was realized that very few people ever want to extract square roots of numbers, and besides the traditional process is not really very convenient. So now we are told to teach root extraction, if we teach it at all, by successive approximations. Once upon a time people solved systems of linear equations by elimination. Then somebody invented determinants and Cramer's rule and everybody learned that. Now determinants are regarded as old-fashioned and cumbersome, and it is considered better to solve systems of linear equations by elimination.

We are constantly being told that large parts of the conventional curriculum are both useless and out of date and so might better not be taught. Why teach computation by logarithms when everybody who has to compute uses at least a desk calculator? Why teach the law of tangents when almost nobody ever wants to solve an oblique triangle, and if he does there are more efficient ways? Why teach the conventional theory of equations, and especially why illustrate it with ill-chosen examples that can be handled more efficiently by other methods? As a professional mathematician, I am a sucker for arguments like these. Yet, sometimes I wonder.

There are a few indications that there is a reason for the survival of the traditional curriculum besides the fact that it is traditional. When I was teaching mathematics to future naval officers during the war, I was told that the Navy had found that men who had studied calculus made better line officers than men who had not studied calculus. Nothing is clearer (it was clear even to the Navy) than that a line officer never has the slightest use for calculus. At the most, his duties may require him to look up some numbers in tables and do a little arithmetic with them, or possibly substitute them into formulas. What is the explanation of the paradox?

I think that the answer is supplied by a phenomenon that everybody who teaches mathematics has observed: the students always have to be taught what they should have learned in the preceding course. (We, the teachers, were of course exceptions; it is consequently hard for us to understand the deficiencies of our students.) The average student does not really learn to add fractions in arithmetic class; but by the time he has survived a course in algebra he can add numerical fractions. He does not learn algebra in the algebra course; he learns it in calculus, when he is forced to use it. He does not learn calculus in the calculus course, either; but if he goes on to differential equations he may have a pretty good grasp of elementary calculus when he gets through. And so on through the hierarchy of courses; the most advanced course, naturally, is learned only by teaching it.

This is not just because each previous teacher did such a rotten job. It is because there is not time for enough practice on each new topic; and even if there were, it would be insufferably dull. Anybody who has really learned to interpolate in trigonometric tables can also interpolate in air navigation tables, or in tables of Bessel functions. He should learn, because interpolation is useful. But one cannot drill students on mere interpolation; not enough, anyway. So the students solve oblique triangles in order (among other things) to practice interpolation. One must not admit this to the students, but one may as well realize the facts.

Consequently, I claim that there is a place, and a use, even for nonsense like the solution of quartics by radicals, or Horner's method, or involutes and evolutes, or whatever your particular candidates for oblivion may be. Here are problems that might conceivably have to be solved; perhaps the methods are not the most practical ones; but that is not the point. The point is that in solving the problems the student gets practice in using the necessary mathematical tools, and gets it by doing something that has more motivation than mere drill. This is not the way to train mathematicians, but it is an excellent way to train mathematical technicians. Now we can understand why calculus improves the line officer. He needs to practice very simple kinds of mathematics; he gets this practice in less distasteful form by studying more advanced mathematics.

It is the fashion to deprecate puzzle problems and artificial story problems. I think that there is a place for them too. Problems about mixing chemicals or

sharing work, however unrealistic, give good practice and even have a good deal of popular appeal: witness the frequency with which puzzle problems appear in newspapers, magazines, and the flyers that come with the telephone bill. There was once a story in *The Saturday Evening Post* whose plot turned on the interest aroused by a perfectly preposterous diophantine problem about sailors, coconuts, and a monkey. It is absurd to claim that only "real" applications should be used to illustrate mathematical principles. Most of the real applications are too difficult and/or involve too many side issues. One begins the study of French with simple artificial sentences, not with the philosophical writings of M. Sartre. Similarly one has to begin the study of a branch of mathematics with simple artificial problems.

We may dislike this state of affairs, but as long as it exists we must face it. It would be pleasant to teach only the new and exciting kinds of mathematics; it would be comforting to teach only the really useful kinds. The traditional topics are some of the topics that once were either new and exciting, or useful. They have persisted partly by mere inertia—and that is bad—but partly because they still serve a real purpose, even if it is not their ostensible purpose. Let us keep this in mind when we are revising the curriculum.

NON-LINEAR RECURRENCE RELATIONS FOR CERTAIN CLASSICAL FUNCTIONS

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Let $\{f_n(x)\}$ be a sequence of functions. The combinations $(f_n')^2-f_nf_n''$, $(f_n')^2-f_{n-1}'f_{n+1}'$, and $f_n^2-f_{n-1}f_{n+1}$ have appeared in recent papers. We consider some relationships involving these functions, when $\{f_n(x)\}$ is the sequence of ultraspherical, Hermite, and generalized Laguerre polynomials using Szegö's [1] notation and the Bessel functions of the first kind.

THEOREM 1. *If $f_n(x)$ is a polynomial of degree n, $f_0(x)=1$, $f_1(x)=2\lambda x$, and*

$$(1)\qquad (1-x^2)\left[(f_n')^2-f_{n-1}'f_{n+1}'\right]=n(n+2\lambda)f_n^2-(n+1)(n+2\lambda-1)f_{n-1}f_{n+1}$$

*for each $n\geqq 1$, where λ is not of the form $-(n-1)/2$, $n=0, 1, \cdots$, then $f_n(x)=P_n^{(\lambda)}(x)=P_n(x)$ for $n=0, 1, \cdots$, where $\{P_n(x)\}$ is the sequence of ultraspherical polynomials.**

Proof. It may be verified directly that $f_2(x)=P_2(x)$. Assume that $f_m(x)=P_m(x)$ for $0\leqq m\leqq n$ where $n\geqq 2$. Then

* This theorem is a generalization of a result of Carlitz [2].

(2) $(1 - x^2)[(P_n')^2 - P_{n-1}'f_{n+1}'] = n(n + 2\lambda)P_n^2 - (n + 1)(n + 2\lambda - 1)P_{n-1}f_{n+1}$.

Let $f_{n+1}(x) = P_{n+1}(x) + g(x)$ where $g(x)$ is a polynomial of degree $\leqq n+1$. Substituting in (2) and using the fact [3] that $\{P_n(x)\}$ satisfies (1), we obtain

$$(1 - x^2)P_{n-1}'g' = (n + 1)(n + 2\lambda - 1)P_{n-1}g. \tag{3}$$

By comparing the coefficients of x^{2n} in (3), one sees that the coefficient of x^{n+1} in $g(x)$ is uniquely determined and so $g(x)$ is of degree $\leqq n$. Since $(1 - x^2)P_{n-1}'$ and P_{n-1} have no polynomial factor of degree $\geqq 1$, it follows from (3) that

$$g(x) = C(1 - x^2)P_{n-1}', \qquad g'(x) = C(n + 1)(n + 2\lambda - 1)P_{n-1},$$

where C is a constant. If $C \neq 0$, this gives

$$(1 - x^2)P_{n-1}'' - 2xP_{n-1}' = (n + 1)(n + 2\lambda - 1)P_{n-1},$$

which, together with the known [1] differential equation

$$(1 - x^2)P_{n-1}'' - (2\lambda + 1)xP_{n-1}' + (n - 1)(n + 2\lambda - 1)P_{n-1} = 0,$$

gives a contradiction. Hence $C = 0$, $f_{n+1}(x) = P_{n+1}(x)$ and the theorem is proved.

THEOREM 2. *If* $f_0(x) = 1$, $f_1(x) = 2\lambda x$, *and*

$$\begin{aligned}(4)\quad (1 - x^2)^2[(f_n')^2 - f_nf_n''] \\ &= n(n + 2\lambda)f_n^2 - (n + 1)(n + 2\lambda - 1)f_{n-1}f_{n+1} - x(1 - x^2)f_nf_n',\end{aligned}$$

for each $n \geqq 1$, *where* λ *is not of the form* $-(n-1)/2$, $n = 0, 1, \cdots$, *then* $f_n(x) = P_n^{(\lambda)}(x) = P_n(x)$ *for* $n = 0, 1, \cdots$.

Proof. Since [4] the sequence $\{P_n(x)\}$ satisfies (4), and (4) determines f_{n+1} uniquely in terms of f_{n-1} and f_n, the theorem follows.

THEOREM 3. *If* $f_0(x) = 1$, $f_1(x) = 2\lambda x$, *and*

$$\begin{aligned}(1 - x^2)^2[(f_n')^2 - f_nf_n''] \\ &= n(n + 2\lambda + x^2)f_n^2 - (n + 1)(n + 2\lambda - 1)f_{n-1}f_{n+1} - (n + 2\lambda - 1)xf_{n-1}f_n\end{aligned}$$

for each $n \geqq 1$, *where* λ *is not of the form* $-(n-1)/2$, $n = 0, 1, \cdots$, *then* $f_n(x) = P_n^{(\lambda)}(x) = P_n(x)$ *for* $n = 0, 1, \cdots$.

Proof. Since [1] we have $(1 - x^2)P_n' = -nxP_n + (n + 2\lambda - 1)P_{n-1}$, then, in view of Theorem 2,

$$\begin{aligned}(1 - x^2)^2[(P_n')^2 - P_nP_n''] \\ &= n(n + 2\lambda + x^2)P_n^2 - (n + 1)(n + 2\lambda - 1)P_{n-1}P_{n+1} - (n + 2\lambda - 1)xP_{n-1}P_n.\end{aligned}$$

The remainder of the proof is similar to that in Theorem 2.

THEOREM 4. *If* $f_0(x)=1, f_1(x)=2x$, *and*

$$(f_n')^2 - f_nf_n'' = 2n[f_n^2 - f_{n-1}f_{n+1}] \tag{5}$$

for each $n\geqq 1$, *then* $f_n(x)=H_n(x)$ *for* $n=0, 1, \cdots$, *where* $\{H_n(x)\}$ *is the sequence of Hermite polynomials.*

Proof. Since [5, 6] the sequence $\{H_n(x)\}$ satisfies (5), the proof is similar to that in Theorem 2.

THEOREM 5. *If* $f_0(x)=1, f_1(x)=-x+\alpha+1$, *and*

$$x^2[(f_n')^2 - f_nf_n''] = n(n+\alpha+1)f_n^2 - (n+1)(n+\alpha)f_{n-1}f_{n+1} \tag{6}$$

for each $n\geqq 1$, *where* α *is not a negative integer, then* $f_n(x)=L_n^{(\alpha)}(x)=L_n(x)$ *for* $n=0, 1, \cdots$, *where* $\{L_n(x)\}$ *is the sequence of generalized Laguerre polynomials.*

Proof. Since [4] the sequence $\{L_n(x)\}$ satisfies (6), the proof is similar to that in Theorem 2.

Note that equation (6) may be written in the form [4]

$$x^2[(f_n')^2 - f_nf_n''] + \alpha f_n^2 = (n+1)(n+\alpha)[f_n^2 - f_{n-1}f_{n+1}].$$

Since $(L_n')^2-L_nL_n''>0$ by [6], it follows, as is known [3, 4], that

$$n(n+\alpha+1)L_n^2 - (n+1)(n+\alpha)L_{n-1}L_{n+1} \geqq 0,$$

$$L_n^2 - L_{n-1}L_{n+1} \geqq 0, \qquad \text{if } \alpha \geqq 0.$$

THEOREM 6. *If* $f_n(x)$ *is a polynomial of degree* n, $f_0(x)=1, f_1(x)=-x+\alpha+1$, *and*

$$x[(f_n')^2 - f_{n-1}'f_{n+1}'] = nf_n^2 - (n+1)f_{n-1}f_{n+1} + f_{n-1}f_n \tag{7}$$

for each $n\geqq 1$, *where* α *is a constant, then* $f_n(x)=L_n^{(\alpha)}(x)=L_n(x)$ *for* $n=0, 1, \cdots$.

Proof. We first use [1] the relations

$$xL_n' = nL_n - (n+\alpha)L_{n-1} = (n+1)L_{n+1} - (n+\alpha+1-x)L_n, \tag{8}$$

$$nL_n = (-x+2n+\alpha-1)L_{n-1} - (n+\alpha-1)L_{n-2}, \tag{9}$$

to show that $\{L_n(x)\}$ satisfies (7).

Let $E=x[(L_n')^2-L_{n-1}'L_{n+1}']$. Then, by (8) and (9),

$$\begin{aligned} xE &= [nL_n - (n+\alpha)L_{n-1}]^2 \\ &\quad - [nL_n - (n+\alpha-x)L_{n-1}][(n+1)L_{n+1} - (n+\alpha+1)L_n] \\ &= n(2n+\alpha+1)L_n^2 - [(n+\alpha)(3n+\alpha+1) - (n+\alpha+1)x]L_{n-1}L_n \\ &\quad + (n+\alpha)^2L_{n-1}^2 - n(n+1)L_nL_{n+1} + (n+1)(n+\alpha-x)L_{n-1}L_{n+1}. \end{aligned}$$

We replace $(n+\alpha)L_{n-1}$ by its value obtained from (9) after n is changed to $n+1$. This gives

$$(n+\alpha)E = [(n+1)(n+\alpha+1) - x]L_n^2 + (n+1)^2 L_{n+1}^2 + [n+1][x-(2n+\alpha+2)]L_nL_{n+1}.$$

In the terms with the variable coefficients, namely, $-xL_n^2+(n+1)xL_nL_{n+1}$, we replace the factor xL_n by its value obtained from (9) after n is replaced by $n+1$. We obtain $E=nL_n^2-(n+1)L_{n-1}L_{n+1}+L_{n-1}L_n$, which shows that $\{L_n(x)\}$ satisfies (7).

We are now ready to prove the theorem. It may be verified directly that $f_2=L_2$. Assume that $f_m(x)=L_m(x)$ for $0\leqq m\leqq n$ where $n\geqq 2$.

Then,

$$x[(L_n')^2 - L_{n-1}'f_{n+1}'] = nL_n^2 - (n+1)L_{n-1}f_{n+1} + L_{n-1}L_n. \tag{10}$$

Let $f_{n+1}(x)=L_{n+1}(x)+g(x)$, where $g(x)$ is a polynomial of degree $\leqq n+1$. Substituting in (10) and using the above result that $\{L_n(x)\}$ satisfies (7), we find that $xL_{n-1}'g'=(n+1)L_{n-1}g$; this shows that if Cx^k is the term in g which involves the highest power of x, then $C=0$. Hence $g(x)=0$ and $f_{n+1}=L_{n+1}$.

THEOREM 7. *If* $f_0(x)=J_0(x)$, $f_1(x)=J_1(x)$, *and*

$$x[(f_n')^2 = f_nf_n''] = x[f_n^2 - f_{n-1}f_{n+1}] + f_nf_n' \tag{11}$$

for each $n\geqq 1$, *then* $f_n(x)=J_n(x)$ *for* $n=0, 1, \cdots$, *where* $J_n(x)$ *is the Bessel function of the first kind of order* n.

Proof. Since [4] the sequence $\{J_n(x)\}$ satisfies (11), the proof follows as in Theorem 2.

References

1. G. Szegö, Orthogonal Polynomials, New York, 1939.

2. L. Carlitz, Note on Legendre polynomials, Bull. Calcutta Math. Soc., vol. 46, 1954, pp. 93–95.

3. V. R. Thiruvenkatachar and T. S. Nanjundiah, Inequalities concerning Bessel functions and orthogonal polynomials, Proc. Indian Acad. Sci. Sect. A, vol. 33, 1951, pp. 373–384.

4. H. Skovgaard, On inequalities of the Turán type, Math. Scand., vol. 2, 1954, pp. 65–73.

5. W. A. Al-Salam, On a characterization of some orthogonal functions, this MONTHLY, vol. 64, 1957, pp. 29–32

6. B. N. Mukherjee and T. S. Nanjundiah, On an inequality relating to Laguerre and Hermite polynomials, Math. Student, vol. 19, 1951, pp. 47–48.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

ELEMENTARY PROOFS OF THE COMMUTATIVITY OF p-RINGS

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1. Introduction. A p-ring, first introduced by McCoy and Montgomery [2], is a ring with unit* 1 satisfying $a^p = a$, $pa = 0$ (p prime). The object of this note is to give some elementary proofs of the following

THEOREM 1. *A p-ring is commutative.*†

First proof. Since $x^p = x$ in a p-ring, we immediately obtain (compare Lemma 1 of [1]) $(x^{p-1}yx^{p-1} - x^{p-1}y)^2 = (x^{p-1}yx^{p-1} - yx^{p-1})^2 = 0$, and this implies

$$x^{p-1}y = x^{p-1}yx^{p-1} = yx^{p-1}. \tag{1.1}$$

Since $x^p = x$ and x^{p-1} is in the center, it is apparent at once that

$$(xy - yx)x^{p-1} = x^{p-1}(xy - yx) = xy - yx. \tag{1.2}$$

Since this is true for every x and y, it remains true if x is replaced by $x+i$, where $i = 1, \cdots, p-1$. Since, however, $xy - yx$ is unchanged under this replacement, it follows that

$$(x + i)^{p-1}(xy - yx) = xy - yx \qquad (i = 0, \cdots, p - 1). \tag{1.3}$$

Now, an elementary number-theoretic result states that, since p is prime,

$$(x + 1)(x + 2) \cdots (x + (p - 1)) = x^{p-1} + (p - 1). \tag{1.4}$$

Since $pa = 0$, a combination of (1.3) and (1.4) gives

$$(x + 1)(x + 2) \cdots (x + (p - 1))(xy - yx) = 0. \tag{1.5}$$

Replacing, in (1.5), x by $x+1$ we obtain

$$\begin{aligned} &(x + 2)(x + 3) \cdots (x + (p - 1))x(xy - yx) \\ &\qquad = x(x + 2)(x + 3) \cdots (x + (p - 1))(xy - yx) = 0. \end{aligned} \tag{1.6}$$

Subtracting (1.6) from (1.5), we get

$$1 \cdot (x + 2)(x + 3) \cdots (x + (p - 1))(xy - yx) = 0. \tag{1.7}$$

Similarly, upon replacing x by $x+1$ in (1.7) and subtracting as above, we obtain

* McCoy and Montgomery do not require the existence of a unit.

† Forsythe and McCoy [1] have already given an elementary proof of Theorem 1. In this note, however, the approach is quite different from [1] in being essentially of number-theoretic nature.

$$(1.8) \qquad 2!(x+3)(x+4)\cdots(x+(p-1))(xy-yx)=0.$$

Continuing this process, we ultimately obtain

$$(1.9) \qquad (p-2)!(x+(p-1))(xy-yx)=0.$$

Now, replacing again x by $x+1$ in (1.9), we get

$$(1.10) \qquad (p-2)!x(xy-yx)=0.$$

Subtracting the last two equations, and making use of $(p-1)!=p-1$ in a p-ring, we obtain $(p-1)(xy-yx)=0$. Hence $xy-yx=0$, $xy=yx$, and the theorem is proved.

Second proof. This proof uses some of the results of the first proof. Thus, using (1.3), (1.1) and (1.4), we obtain

$$\begin{aligned} xy-yx &= (xy-yx)^p \\ &= \{x^{p-1}(xy-yx)\}\{(x+1)^{p-1}(xy-yx)\}\cdots\{(x+(p-1))^{p-1}(xy-yx)\} \\ &= \{x(x+1)\cdots(x+(p-1))\}^{p-1}(xy-yx)^p = 0. \end{aligned}$$

Hence $xy=yx$, and the proof is complete.

In conclusion, I wish to express my gratitude to the referee for his valuable suggestions.

References

1. A. Forsythe and N. H. McCoy, On the commutativity of certain rings, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 523–526.

2. N. H. McCoy and D. Montgomery, A representation of generalized Boolean rings, Duke Math. J., vol. 3, 1937, pp. 455–459.

ON THE NOTE "ON THE PROPAGATION OF ERROR BY MULTIPLICATION" BY PERRY AND MORELOCK

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It may be worth pointing out that the note cited in the title [1] represents a familiar manipulation of bivariate probability distributions. If X and Y are independent random variables, uniformly distributed in the intervals $[a-1/2, a+1/2]$ and $[b-1/2, b+1/2]$ respectively, then the distribution of $W=XY-ab$ may readily be found. One example of a textbook discussion of this standard approach is Chapter 10 of [2].

Assume that $1/2 \leqq a < b$. This assumption, together with that of independence, is implicit in the note by Perry and Morelock. The independence assumption, in particular, needs careful examination for many applications.

The joint p.d.f. (probability density function) of X and Y is 1 for $a-1/2 \leqq x \leqq a+1/2$ and $b-1/2 \leqq y \leqq b+1/2$; and it is zero elsewhere. (I use lower case letters for the arguments of a p.d.f. relating to the random variables denoted by the corresponding capitals.) If we let $W=XY-ab$ and $Z=Y$ (to be wholly

explicit) then the Jacobian of the transformation is $1/z$ and the joint p.d.f. of W and Z is $1/z$ in the quadrilateral bounded by the lines $z=b\pm 1/2$, $z=(x+ab)/(a\pm 1/2)$, and zero elsewhere. Integrating out over z, one obtains as the p.d.f. of W:

$$\begin{array}{ll} \ln(w+ab)-\ln[(a-1/2)(b-1/2)] & \text{for } (-2a-2b+1)/4\leqq w\leqq(2a-2b-1)/4, \\ \ln[(b+1/2)/(b-1/2)] & \text{for } (2a-2b-1)/4\leqq w\leqq(-2a+2b-1)/4, \\ -\ln(w+ab)+\ln[(a+1/2)(b+1/2)] & \text{for } (-2a+2b-1)/4\leqq w\leqq(2a+2b+1)/4, \\ 0 & \text{elsewhere.} \end{array}$$

And, from this, the final expressions given by Perry and Morelock may readily be found by integration.

The reader of [1] may be puzzled by the fact that the graph on page 179 of [1] is not a graph of the function $P\{E>K\}$, as suggested there, but rather a graph of the p.d.f. given in the preceding paragraph of this note. Perry and Morelock do not explicitly state the p.d.f.; their $P\{E>K\}$ is the complement of the usual cumulative distribution function. (Their E is my W.)

Finally, the graph of [1] is slightly misleading for two reasons. First, there is inordinate rounding of abscissas; the points labeled -6.8, -1.3, .7, and 7.2 could have been labeled -6.75, -1.25, .75, and 7.25, respectively and exactly. Second, the two slanted straight line segments of the graph are actually slightly curved, the left, concave down, and the right, concave up; these curvatures, although small, would just be noticeable on the scale of the exhibited figure.

The editor is informed that these concavities were present on the graph submitted with the note; apparently they became lost during preparations for printing.

References

1. N. C. Perry and J. C. Morelock, On the propagation of error by multiplication, this MONTHLY, vol. 63, 1956, pp. 177–179.

2. A. M. Mood, Introduction to the Theory of Statistics, McGraw-Hill, 1950.

A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION

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1. Introduction. If $f(x)$ is a complex-valued, continuous function of the real variable x such that for every real x, y

$$f(x)f(y) = f(x+y) \tag{1}$$

then we know that either $f(x)=0$, or there exists a complex number α such that $f(x)$ can be written in the form $e^{\alpha x}$ for all real x. Actually, as is well known, the condition of continuity is too stringent and can be replaced by mere measurability.

However, in some practical applications, whereas continuity may be expected, the defining condition (1) may not be readily verifiable for every possible

pair x, y. The author was faced with such a situation and had the following problem:

$f(x)$ is a complex-valued, continuous function of the real variable x, having the property $\mathcal{P}$ that if, for any real x and any positive integer n, $k_n(x)$ denote the number of distinct points in the set $f(x)$, $\{f(x/2)\}^2$, $\{f(x/3)\}^3, \cdots$, $\{f(x/n)\}^n$, then $k_n(x) \leqq$ some finite k for all x and n.

What can be said about $f(x)$? Can it, for instance, always be written in the form $e^{\alpha x+\beta}$? The author has been unable to obtain the complete answer, but a partial solution is given in the next section.

2. Partial solution of the problem. We shall now restrict ourselves to real-valued functions, and prove the following

THEOREM. *Let $f(x)$ be a real-valued, continuous function of $x \geqq 0$ having the property $\mathcal{P}$ defined in Section 1. Further, let $f(x) > 0$ for some $x \geqq 0$. Then there exists a real number α such that, for all $x \geqq 0$, $f(x)$ can be written in the form $e^{\alpha x}$.*

Proof. Since $f(x) > 0$ for some x, therefore $f(x) > 0$ throughout some closed interval $[a, b]$. It will be assumed that $a > 0$. In this interval, the function $g(x) = [\log f(x)]/x$ is defined and continuous.

For any $x > ab/(b-a)$, there is at least one integer r such that $a \leqq x/r \leqq b$. Let $n(x)$ and $m(x)$ be respectively the smallest and largest such integers. Then for all n such that $n(x) \leqq n \leqq m(x)$, $g(x/n) = n\{\log f(x/n)\}/x$ is defined. From property $\mathcal{P}$ it follows that, however large x may be, there can be at most k distinct points in the set

$$g\{x/n(x)\}, \quad g\{x/[n(x) + 1]\}, \cdots, g\{x/m(x)\}.$$

But we shall show that this implies the constancy of $g(x)$ throughout $[a, b]$.

Since $g(x)$ is continuous in $[a, b]$, therefore, unless it is a constant, there are $k+1$ points $a_0, a_1, \cdots, a_k$ in $[a, b]$ such that $g(a_r) \neq g(a_s)$ if $r \neq s$. It follows that there exists a set of non-overlapping intervals $I_0, I_1, \cdots, I_k$ such that $a_r \in I_r \subset [a, b]$, $r = 0, \cdots, k$, and $g(x_r) \neq g(x_s)$ if $r \neq s$, $x_r \in I_r$ and $x_s \in I_s$.

Now, it is easy to see that by choosing x large enough, we can find positive integers $n_0, n_1, \cdots, n_k$ such that $x/n_r \in I_r$. Hence, the set $g\{x/n(x)\}, \cdots$, $g\{x/m(x)\}$ contains at least $k+1$ distinct points, which was previously seen to be impossible, on account of property $\mathcal{P}$. Consequently, $g(x)$ equals a constant α in $[a, b]$, so that throughout this interval, $f(x) = e^{\alpha x}$.

Further, it is easy to see that $f(x) \neq 0$ for all $x > a$. For, otherwise there is a $c > a$ which is the g.l.b. of all zeros of $f(x)$ to the right of a, and by continuity $f(c) = 0$. On the other hand, if we consider any sequence of points c_n in (a, c) such that $c_n \to c$ as $n \to \infty$, then we know that $f(x) > 0$ in $[a, c_n]$, so that $f(c_n) = e^{\alpha c_n} \to e^{\alpha c}$ as $n \to \infty$. This is impossible. Hence, $f(x)$ has no zero to the right of a, and $f(x) = e^{\alpha x}$ for all $x \geqq a$. In a similar manner, it can be seen that $f(x)$ has no zero in $(0, a)$, and that $f(x) = e^{\alpha x}$ for all $x \geqq 0$.

COROLLARY. *If $f(x)$ is a real-valued, continuous function of the real variable x having the property $\mathcal{P}$, then one of the following alternatives must hold:*

(a) $f(x)\equiv 0$;

(b) $f(x)>0$ *for some* x, *in which case* $f(x)>0$ *for all* x, *and there exist real numbers* α, β *such that* $f(x)$ *can be written in the form* $e^{\alpha x}$ *for* $x\geqq 0$, *and* $e^{\beta x}$ *for* $x\leqq 0$;

(c) $f(x)<0$ *for some* x, *in which case* $f(x)<0$ *for all* x, *and there exist real numbers* α, β *such that* $f(x)$ *can be written in the form* $-e^{\alpha x}$ *for* $x\geqq 0$ *and* $-e^{\beta x}$ *for* $x\leqq 0$.

3. Unsolved problem. Coming back to the problem of a complex-valued, continuous $f(x)$ having the property $\mathcal{P}$, we can express $f(x)$ as the product of two functions $|f(x)|$ and $f(x)/|f(x)|$ having the property $\mathcal{P}$. Of these two, $|f(x)|$ is real-valued and can be taken care of by the previous section.

It therefore remains to consider a continuous function $f(x)$ of unit modulus and having property $\mathcal{P}$. It is conjectured that in this case, the real line can be broken up into intervals in each of which $f(x)$ can be written in the form $\mu e^{i\alpha x}$ where μ is a root of unity and α is a real number. The author has been unable to prove or disprove this conjecture.

REMARKS CONCERNING THE NON-EXISTENCE OF ODD PERFECT NUMBERS

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Let $\sigma(n)$ be the sum of the divisors of the positive integer n. Then n is called a perfect number if $\sigma(n)=2n$. It has long been conjectured that there are no odd perfect numbers. Euler [2] proved that if n is an odd perfect number, then

$$n = p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}, \tag{1}$$

where $p, q_1, q_2, \cdots, q_t$ are distinct primes and $p\equiv 1\equiv\alpha \pmod 4$. Sylvester [6] proved that $t\geqq 4$, and, in fact, that $t\geqq 7$ if $n\not\equiv 0 \pmod 3$. In recent years further necessary conditions have been discovered for the odd integer n to be perfect. In this note we shall show that certain other conditions are necessary.

If the number n, given by (1), is to be perfect, we must have

$$\sigma(p^{\alpha})\prod_{i=1}^{t}\sigma(q_i^{2\beta_i}) = 2p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}.$$

To prove that n is not perfect it is therefore sufficient to show that $\sigma(p^{\alpha})$ or one of the $\sigma(q_i^{2\beta_i})$ has a prime factor which does not divide n. We shall assume throughout this note that $n\not\equiv 0 \pmod 3$.

If $n\equiv 2 \pmod 3$, then $p\equiv 2 \pmod 3$, and $\sigma(p^{\alpha})$, which is divisible by $p+1$, is divisible by 3. Hence, if n is to be perfect we must have $n\equiv 1 \pmod 3$. Now, if this is true, $\sigma(p^{\alpha})\equiv\alpha+1 \pmod 3$, and therefore, if $\alpha+1$ is divisible by 3, n is not perfect.

This can be generalized as follows. Let r be any prime not dividing n, and let p belong to the exponent e modulo r. Then, n is not perfect if $\alpha+1\equiv 0 \pmod{er}$. For, in this case, $\sigma(p^{\alpha})$ is divisible by $p^{e(r-1)}+p^{e(r-2)}+\cdots+p^{e}+1$, which is divisible by r. As an application of this remark, we see that n is not perfect if n is not divisible by 5, $\alpha+1$ is divisible by 5, and $p\equiv 1$ or $4 \pmod 5$.

It was proved by Steuerwald [5] that n, given by (1), is not perfect if $\beta_1=\beta_2=\cdots=\beta_t=1$. We shall extend this by showing that n is not perfect if $\beta_i\equiv 1 \pmod 3$ for $i=1,\cdots,t$. Assume that n is perfect. If, for any i, $q_i\equiv 1 \pmod 3$, $\sigma(q_i^{2\beta_i})$, and therefore n, is divisible by 3. Hence we may assume that $q_i\equiv 2 \pmod 3$ for $i=1,\cdots,t$. Suppose that we have shown that n is not divisible by any prime less than the prime q, and that n is divisible by q. If $q=p$, n is divisible by $(q+1)/2$, which contains a prime factor less than q. Hence, q is one of the q_i, and n is divisible by $\sigma(q^2)=q'$. If q' is composite, it contains a prime factor less than q. Hence, q' is a prime. Since $q'\equiv 1 \pmod 3$, we must have $q'=p$. But then n is divisible by $(q'+1)/2=\frac{1}{2}(q^2+q)+1=q''$. If q'' is composite, it contains a prime factor less than q. Hence, q'' is a prime. Since $q''\equiv 1 \pmod 3$, we must have $q''=p$, which is impossible since $q''=(p+1)/2$. Thus n cannot be divisible by q, and this contradiction completes the proof.

In [3] Kanold gives several necessary conditions for the odd integer n given by

$$n = p^{\alpha} q_1^{2\beta} q_2^{2} \cdots q_t^{2}$$

to be perfect. We have shown that n is not perfect if $\beta\equiv 1 \pmod 3$. We shall now show that we may drop this requirement if we impose a condition on q_1. In particular, we shall show that n is not perfect if $q_1\equiv 2 \pmod 3$. We continue to assume, of course, that $n\not\equiv 0 \pmod 3$. Assume that n is perfect. Then $q_i\equiv 2 \pmod 3$ for $i=2,\cdots,t$. If $i\neq 2$, q_i cannot divide $\sigma(q_2^2)$. This is a result of the fact that $\sigma(q_2^2)=f_3(q_2)$, where $f_3(x)$ is the third cyclotomic polynomial, and if $f_3(x)\equiv 0 \pmod{q_i}$ has a solution, then $q_i\equiv 1 \pmod 3$ [4, p. 164]. Thus, $\sigma(q_2^2)=p^m$, and by a lemma due to A. Brauer [1], $m=1$. The same is true of $\sigma(q_i^2)$ for $i=3,\cdots,t$. Herein lies a contradiction. For, $t>3$, and so, even though $q_2\neq q_3$, we have $\sigma(q_2^2)=\sigma(q_3^2)$.

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References

1. A. Brauer, On the non-existence of odd perfect numbers of form $p^{\alpha}q_1^2q_2^2\cdots q_{t-1}^2q_t^4$, Bull. Amer. Math. Soc. vol. 49, 1943, pp. 712–718.

2. L. Euler, Commentationes arithmeticae collectae, vol. 2, Tractatus de numerorum doctrina, 1849, p. 514; Opera postuma, vol. 1, 1862, pp. 14–15.

3. H. J. Kanold, Kreisteilungspolynome und ungerade vollkommene Zahlen, Ber. Math.-Tagung Tübingen 1946 (1947), pp. 84–87.

4. T. Nagell, Introduction to Number Theory, New York, 1951.

5. R. Steuerwald, Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss., 1937, pp. 68–72.

6. J. J. Sylvester, Collected Mathematical Papers, vol. 4, Cambridge University Press, 1912, pp. 611–614.

THE CENTROID OF ANALYTIC MAPPINGS

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Let $f(z) = u + iv = \sum_{n=1}^{\infty} a_n z^n$ be regular for $|z| = |x+iy| \leqq R$. It is well known that the area of the domain D corresponding to $|z| \leqq r < R$ (multiply-covered parts being counted multiply) is given [2] by

$$A = \iint_{|z|\leq r} dudv = \iint \left| \frac{\partial(u, v)}{\partial(x, y)} \right| dxdy = \int_0^r \int_0^{2\pi} | f'(re^{i\theta}) |^2 r dr d\theta$$
$$= \pi \sum_{n=1}^{\infty} n | a_n |^2 r^{2n}.$$

Let $(\bar{u}, \bar{v})$ represent the centroid of the domain D. Following the usual definition (multiply-covered areas may be avoided if, for example, one restricts oneself to schlicht functions [1]), we have

$$A\bar{u} = M_v = \iint ududv; \qquad A\bar{v} = M_u = \iint vdudv; \qquad |z| \leqq r.$$

Writing $u = (f + \bar{f})/2$, $v = (f - \bar{f})/2i$, we obtain

$$M_v = \frac{1}{2} \int_0^r \int_0^{2\pi} [f(re^{i\theta}) + \bar{f}(re^{i\theta})] | f'(re^{i\theta}) |^2 r dr d\theta,$$

$$M_u = \frac{1}{2i} \int_0^r \int_0^{2\pi} [f(re^{i\theta}) - \bar{f}(re^{i\theta})] | f'(re^{i\theta}) |^2 r dr d\theta.$$

Substituting

$$f(re^{i\theta}) = a_1 re^{i\theta} + a_2 r^2 e^{2i\theta} + \cdots,$$
$$f'(re^{i\theta}) = 2a_2 re^{i\theta} + 3a_3 r^2 e^{2i\theta} + \cdots,$$
$$\bar{f}(re^{i\theta}) = \bar{a}_1 re^{-i\theta} + \bar{a}_2 r^2 e^{-2i\theta} + \cdots,$$

and carrying out the integrations, we obtain

$$M_v = \frac{\pi}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j(a_i a_j \bar{a}_{i+j} + \bar{a}_i \bar{a}_j a_{i+j}) r^{2(i+j)},$$

$$M_u = \frac{\pi}{2i} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j(a_i a_j \bar{a}_{i+j} - \bar{a}_i \bar{a}_j a_{i+j}) r^{2(i+j)}.$$

These are equivalent to

$$M_v = \pi \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Re} (a_i a_j \bar{a}_{i+j}) r^{2(i+j)},$$

$$M_u = \pi \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Im} (a_i a_j \bar{a}_{i+j}) r^{2(i+j)}.$$

Thus, if we assume the convergence of the numerators, the formula for the centroid of the domain D becomes

$$\tag{1} \begin{aligned} \bar{u} &= \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Re} (a_i a_j \bar{a}_{i+j}) r^{2(i+j)} \right\} \Big/ \left\{ \sum_{n=1}^{\infty} n \mid a_n \mid^2 r^{2n} \right\}, \\ \bar{v} &= \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} j \operatorname{Im} (a_i a_j \bar{a}_{i+j}) r^{2(i+j)} \right\} \Big/ \left\{ \sum_{n=1}^{\infty} n \mid a_n \mid^2 r^{2n} \right\}. \end{aligned}$$

If the mapping function $f(z)$ is written in the form $f(z) = a_0 + \sum_1^\infty a_n z^n$, apply a translation and obtain*

$$\bar{u} = \operatorname{Re}(a_0) + \text{the fraction in formula (1) above,}$$

$$\bar{v} = \operatorname{Im}(a_0) + \text{the fraction in formula (1) above.}$$

Let us now consider a special case of the locus $C(r)$ of the centroid $(\bar{u}, \bar{v})$ as a function of r. Without loss of generality take $a_0 = 0$ and let the mapping function be $f(z) = \sum_1^\infty a_n z^n$. The locus $C(r)$ will then pass through the origin. When is $C(r)$ a straight line through the origin? Obviously, if all the coefficients a_n are real, then from equation (1) we have $\operatorname{Im}(a_i a_j \bar{a}_{i+j}) = 0$ for all values of i and j, and thus $\bar{v} = 0$ so that $C(r)$ coincides with a portion of the real axis. Thus a sufficient condition for $C(r)$ to be a straight line through the origin is that all the coefficients of $f(z)$ be real. This sufficiency condition, of course, is also obvious from the principle of reflection. A moment's thought will also show that the principle of reflection implies that the (essential) reality of the coefficients is also a necessary condition that $C(r)$ be a straight line through the origin. We now proceed to prove this result directly from equation (1). In fact we prove the following theorem:

THEOREM. *Let $f(z) = \sum_0^\infty a_n z^n = \sum_0^\infty |a_n| e^{i\theta_n} z^n$ be regular for $|z| \leqq R$. Let $C(r)$ be the locus of the centroid (given by (1)) of the domain D corresponding to $|z| = r < R$. Then $C(r)$ is a straight line if, and only if, all the coefficients a_n are (essentially) real.*

Proof. There results no loss of generality if we assume (a) $a_0 = 0$, by applying a translation, (b) $a_1 \geqq 0$ (*i.e.* $\theta_1 = 0$), by applying the rotation $f(ze^{-i\theta_1})$, and (c) $\alpha = 0$, by applying the rotation $e^{-i\alpha} f(z)$. The necessary and sufficient condition that $C(r)$ be a straight line through the origin with inclination $\alpha = 0$ is that $\bar{v} = 0$ for all $0 \leqq r < R$. Thus from (1), with a slight change in notation, we obtain

* Although the formulas for the area of the domain D and for the length of arc of the curve bounding the domain D are found in most textbooks on the subject, equations (1) are not to be found in readily available textbooks.

$$\text{Im} \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} j(a_j a_{n-j}) \bar{a}_n r^{2n} = 0; \qquad 0 \leqq r < R, \tag{2}$$

$$\text{Im} \left[a_1^2 \bar{a}_2 r^4 + (3a_1 a_2) \bar{a}_3 r^6 + (4a_1 a_3 + 2a_2^2) \bar{a}_4 r^8 + \cdots \right] = 0.$$

Equation (2) is satisfied if, and only if,

$$\phi = \arg \sum_{j=1}^{n-1} j a_j a_{n-j} \bar{a}_n \equiv \delta\pi; \qquad \delta = 0, 1; n = 2, 3, \cdots. \tag{3}$$

For $n=2$ equation (3) yields arg $(a_1^2 \bar{a}_2) \equiv \delta\pi$, or $2\theta_1 - \theta_2 \equiv \delta\pi$, or $\theta_2 \equiv 2\theta_1 + \delta\pi \equiv \delta\pi$. Similarly, for $n=3$ we obtain arg $(3a_1 a_2 \bar{a}_3) = \theta_1 + \theta_2 - \theta_3 \equiv \delta\pi$, or $\theta_3 \equiv \delta\pi$. Thus the relation $\theta_k \equiv \delta\pi$ holds for $k=2, 3$. By induction, assume $\theta_k \equiv \delta\pi$, $k=1, \cdots, n-1$. Then, arg $(j a_j a_{n-j}) = \theta_j + \theta_{n-j} \equiv \delta\pi + \delta\pi \equiv \delta\pi$, that is, each term of the sum $\sum j a_j a_{n-j}$ has the same argument, namely $\delta\pi$. Substituting in (4), we obtain $\phi = \delta\pi - \theta_n \equiv \delta\pi$, or $\theta_n \equiv \delta\pi$. Hence it follows that $\theta_n \equiv \delta\pi$, $n=2, 3, \cdots$. Therefore, the coefficients of $f(z)$ are all real. This completes the proof of the theorem.

As a generalization of the above theorem, we state that if $f(z)$ has the form $f(z) = \sum_k^\infty |a_n| e^{i\theta_n} z^n$, and α is arbitrary, then $C(r)$ is a straight line through the origin with inclination α if, and only if, any one of the following four equivalent conditions is true:

(a) $e^{-i\alpha} f[z e^{i(\alpha - \theta_k)/k}] \equiv$ function with all real coefficients,

(b) $\theta_n = [n\theta_k - \alpha(n-k)]/k; \qquad n = k, k+1, \cdots,$

(c) $\theta_n = (n-k)\theta_{k+1} - (n-k-1)\theta_k; \qquad n = k, k+1, \cdots,$

(d) $\text{Im} \left[a_n a_k^{(n-k-1)} \bar{a}_{k+1}^{(n-k)} \right] = 0; \qquad n = k, k+1, \cdots.$

References

1. S. D. Bernardi, A survey of the development of the theory of schlicht functions, Duke Math. J., vol. 19, 1952, pp. 263–287.

2. P. Dienes, The Taylor Series, Clarendon Press, Oxford, 1931.

CHARACTERISTIC ROOTS OF SEMI-MAGIC SQUARE MATRICES

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In this note we consider the matrices formed by the magic and semi-magic squares. A square matrix $A = (a_{rs})$ of order n is called a magic square matrix if

$$\sum_{r=1}^{n} a_{rs} = \sum_{r=1}^{n} a_{sr} = S(A), \text{ for } s = 1, 2, \cdots, n; \text{ and} \tag{1}$$

$$\sum_{r=1}^{n} a_{rr} = \sum_{r=1}^{n} a_{r,n-r+1} = S(A). \tag{2}$$

However, if only (1) holds, A is said to be a semi-magic square matrix, and,

following Weiner, will be called an S-matrix of order n. In [4] Weiner has considered the matrix algebra R_n of S-matrices of order n and has also determined the structure of the algebra R_n. The main purpose of this note is to determine the bounds for the characteristic roots of S-matrices. Throughout this note the entries of the S-matrices lie in an ordered field of characteristic zero.

We first prove the following:

THEOREM 1. *If A is in G', the set of all nonsingular S-matrices of order n, then A^{-1} is also in G', and $S(A^{-1})=(S(A))^{-1}$.*

Proof. To each element A of G', there exists a matrix $B\equiv(b_{ij})=A^{-1}$, such that $AB=BA=I$, the identity matrix. Considering $AB=I$, we have $\sum_{r=1}^{n} a_{ir}b_{rj}=\delta_{ij}$, where δ_{ij} is the Kronecker symbol, which equals 1 when $i=j$ and equals 0 when $i\neq j$. *i.e.*, $\sum_{i=1}^{n}\sum_{r=1}^{n} a_{ir}b_{rj}=\sum_{i=1}^{n}\delta_{ij}$, or, $\sum_{r=1}^{n} b_{rj}\ \sum_{i=1}^{n} a_{ir}=1$, or, $S(A)\sum_{r=1}^{n} b_{rj}=1$. Therefore, $\sum_{r=1}^{n} b_{rj}=1/S(A)$, a fixed quantity, for $j=1, 2, \cdots, n$.

Similarly, proceeding with $BA=I$, it can be seen that $\sum_{j=1}^{n} b_{rj}=1/S(A)$, for $r=1, 2, \cdots, n$.

Thus $B=A^{-1}$ belongs to G' and is such that $S(A^{-1})=\{S(A)\}^{-1}$. This completes the proof.

It may be observed here that the set G' of all nonsingular S-matrices of order n is a non-abelian multiplicative group. Further, it can be observed that every element A, distinct from the zero element (null matrix), belonging to G, the set of all S-matrices of the same order, generates a cyclic group of infinite order with addition as the rule of combination.

We shall now prove a theorem which gives an upper bound for the absolute value of any characteristic root of an S-matrix, A, with positive numbers as its elements.

THEOREM 2. *If λ is any characteristic root of A, then*

$$|\lambda| \leqq S(A), \tag{3}$$

***i.e.**, the absolute value of λ is not greater than the sum of the elements of A along any row or column.*

Proof. By a well known theorem of I. Schur, [3], there exists an orthogonal unitary matrix $U=(u_{ij})$ which transforms A into a triangular matrix T. The principal diagonal of T consists of the characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, not necessarily all distinct. Then from

$$T = UAU^*, \quad \text{and} \quad UU^* = U^*U = I,$$

it follows that $t_{ij}=\sum_{r,s=1}^{n} u_{ir}a_{rs}\bar{u}_{js}$, where t_{ij} is the element of T in the (i, j)th place.

That is, the elements of T are of the form $\sum_{r,s=1}^{n} a_{rs}x_r\bar{x}_s$, [1, p. 150], where

the set of (complex)numbers $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ is such that $\sum_{i=1}^{n} x_i\bar{x}_i = 1$.

Let the absolute values of x_i be denoted by ξ_i $(i=1, 2, \cdots, n)$. Then the set $(\xi_1, \xi_2, \cdots, \xi_n)$ is the set of real numbers such that $\sum \xi_i^2 = 1$. Also, since ξ_i, ξ_j are all real, $\xi_i\xi_j \leqq \frac{1}{2}(\xi_i^2 + \xi_j^2)$. Hence, $|t_{ii}| = |\sum_{r,s=1}^{n} a_{rs}x_r\bar{x}_s| \leqq \sum_{r,s} |a_{rs}| \cdot |x_r| \cdot |\bar{x}_s| = \sum_{r,s} a_{rs}\xi_r\xi_s \leqq \frac{1}{2}\sum_{r,s} a_{rs}(\xi_r^2 + \xi_s^2) = \frac{1}{2}[\sum_r \xi_r^2 \sum_s a_{rs} + \sum_s \xi_s^2 \sum_r a_{rs}] = \frac{1}{2}[S(A)\sum_r \xi_r^2 + S(A)\sum_s \xi_s^2] = S(A)$.

Therefore, $|t_{ii}| \leqq S(A)$. Hence, $|\lambda_i| \leqq S(A)$, $i=1, 2, \cdots, n$. This establishes the theorem.

We now generalize the above theorem in the following form:

THEOREM 3. *Let $r(x) = f_1(x)/f_2(x)$ be a rational function of the scalar indeterminate x and A be an S-matrix, with positive numbers as its elements, such that $f_2(A)$ is nonsingular. If λ is a characteristic root of A, then for any characteristic root $r(\lambda)$ of $r(A)$,*

$$|r(\lambda)| \leqq r\{S(A)\}. \tag{4}$$

Proof. Since the S-matrices of the same order form an algebra (Theorem 1, [4]), $f_1(A)$ and $f_2(A)$ are S-matrices, the latter being nonsingular. It may further be verified that $S\{f_1(A)\} = f_1\{S(A)\}$, and $S\{f_2(A)\} = f_2\{S(A)\}$, whence $S\{r(A)\} = S\{f_1(A)(f_2(A))^{-1}\} = S\{f_1(A)\} \cdot S\{(f_2(A))^{-1}\} = S\{f_1(A)\} \cdot \{S(f_2(A))\}^{-1} = f_1\{S(A)\}/f_2\{S(A)\} = r\{S(A)\}$.

Thus $r(A) = f_1(A)/f_2(A)$ is an S-matrix. Also, by a well known theorem of Frobenius [2, pp. 22, 23], if λ is a characteristic root of A, $r(\lambda)$ is a characteristic root of $r(A)$. In order to see that $r(\lambda)$ is defined, we observe that since the matrix $f_2(A)$ is nonsingular, no characteristic root of $f_2(A)$ is zero. That is, $f_2(\lambda) \neq 0$, for it is a characteristic root of $f_2(A)$.

Now, applying (3), we have $|r(\lambda)| \leqq r\{S(A)\}$, and the theorem is proved.

Finally, I am indebted to Professor S. M. Shah for his help in the preparation of this paper and to the referee for helpful criticism.

References

1. W. L. Ferrar, Finite Matrices, Oxford, 1951.

2. C. C. MacDuffee, The Theory of Matrices, Chelsea, New York, 1946.

3. I. Schur, Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. Math. Ann., vol. 66, 1909, pp. 488–510.

4. L. M. Weiner, The algebra of semi-magic squares, this MONTHLY, vol. 62, 1955, pp. 237–239.

CLASSROOM NOTES

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AN EARLY INTRODUCTION TO THE MAPPING CONCEPT

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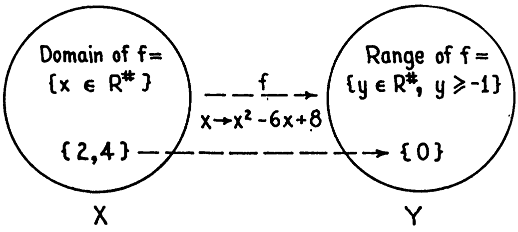
It is evident from the development in Part I of *Universal Mathematics** that the concept of mapping should be introduced at an early stage in mathematical training. Variations of the following method have been used by a number of teachers early in the first semester of their freshman courses.

The graph of a familiar polynomial such as $y=x^2-6x+8$ is made by the usual method which the student has learned in high school. A function is then defined as a triple $(X, Y; f)$ consisting of two sets X and Y and a relation f in the Cartesian product $X\times Y$ such that no $x\in X$ is the first element of two pairs (x, y) and (x, y'). The quadratic polynomial x^2-6x+8 is now considered in the light of this definition. In this case we have two sets of real numbers X and Y where X is mapped onto Y as follows:



It is pointed out that by the law f: $x\to x^2-6x+8$ we have performed a mapping process by which the open set of all real numbers is mapped onto the half-open set of real numbers equal to or greater than -1. With a slight apology to a future mathematics major, the subset $\{2, 4\}$ of X, which is mapped into $\{0\}$ by f, may be called the kernel of the map. The student is happy to tie this notion in with his idea of the roots of an equation.

Other simple examples of algebraic functions may be done in the same way, and this method is helpful in the introduction to the transcendental functions. For example, by using the map $x\to\log x$, the student sees that we have a homomorphic mapping of the multiplicative set of positive real numbers onto the additive set of real numbers.

* Universal Mathematics, Part I, Functions and Limits, Student Union Book Store, Univ. of Kansas, 1954.

SYNTHETIC APPROACH TO THE THEORY OF THE ENVELOPE

A. R. AMIR-MOÉZ, University of Idaho

DEFINITION. *Let* (C) *be a curve varying according to a certain rule. If there is a curve* (γ) *tangent to* (C) *in all its positions, then* (γ) *is called the envelope of* (C).

The basic approach to the problem of finding envelopes is similar to that used in differential geometry. In this paper, instead of the analytic treatment, we give the synthetic solutions of a few problems. In some cases, where the equation of the envelope is quite complicated, it is easier to describe the envelope synthetically.

Consider (C_1) and (C_2), two positions of (C), so that $(C_1)\cap(C_2)=K_1$. Let (C_2) approach (C_1). Then K_1 approaches K which is called the characteristic point. The locus of K is the envelope of (C) if it exists.

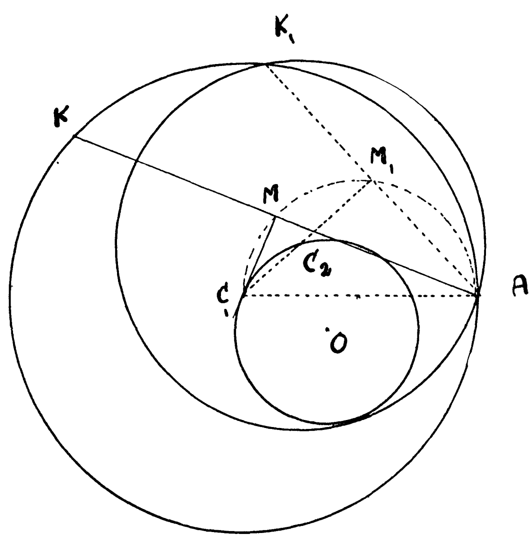


FIG. 1

Problem I. Let (O) be a fixed circle with center O, and A a fixed point in the plane of (O). Circle (C), in the plane of (O), passes through A and its center C moves on (O). Find the envelope of (C).

Solution. Let (C_1) and (C_2) be two positions of (C) (Fig. 1). Clearly C_1C_2 is the perpendicular bisector of AK_1, the common chord of (C_1) and (C_2). It is observed that M_1, the midpoint of AK_1, is on the circle with diameter AC_1. As C_2 tends to C_1, C_1C_2 tends to the tangent C_1M to (O) at C_1, and M_1 approaches M, the point of intersection of C_1M and the circle with diameter AC_1. But AM intersects (C_1) at K which is the limit of K_1. Therefore the locus of K is the envelope.

This locus is easily described as follows:

$AK=2AM$, and M is the vertex of a right angle with one side passing through A and the other side tangent to (O).

Draw a perpendicular through O to AM, (Fig. 2). The foot of the perpendicular, N, is on the circle with diameter OA, and NM is equal to the radius of (O). This describes a limaçon L. Therefore the locus of K is a limaçon, homothetic of L with ratio 2.

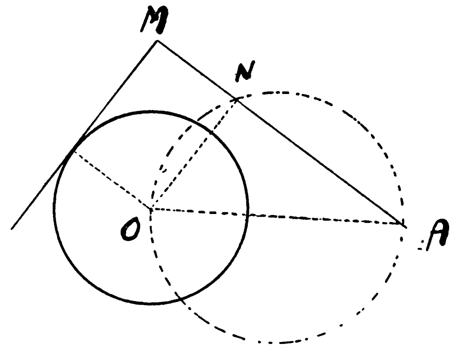


FIG. 2

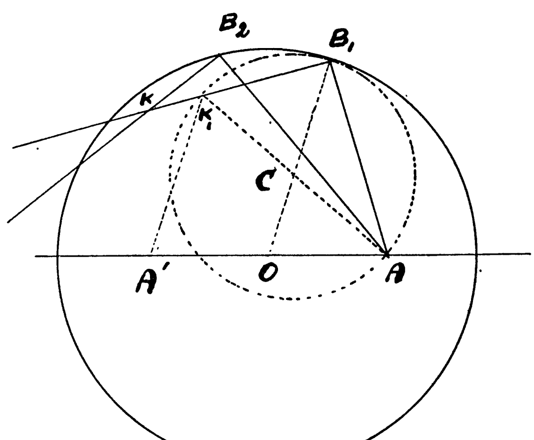


FIG. 3

Problem II. The circle (O) with center O, and the point A, on the plane of (O), inside (O), are fixed. The envelope of the second side of a right angle is desired such that its first side passes through A and its vertex moves on (O), (Fig. 3).

*Solution.** Let AB_1K and AB_2K be two positions of the right angle where K

* This solution is due to Dr. M. Hachtroudi, Professeur à l'Université Teheran.

is the point of intersection of the second sides of these angles. It is obvious that A, B_1, B_2, K are on a circle. As B_2 approaches B_1 the circle through A, B_1, B_2, K tends to the circle (C) through A, B_1 and tangent to (O) at B_1. This circle intersects B_1K at K_1, *i.e.* K_1 is the characteristic point.

It is clear that $K_1A = 2CB_1$. If we draw a line through K_1 parallel to CO, this line intersects OA at A_1, and $K_1A_1 = 2OC$. Therefore $K_1A + K_1A_1 = 2OB_1$, *i.e.* the envelope is an ellipse.

Problem III. Find the envelope of circle (C), with the center C on a given hyperbola with foci F_1 and F_2, and tangent to the fixed circle (F_1) with center F_1.

Solution. Let (C_1) and (C_2), two positions of (C), intersect at K_1 and H_1, (Fig. 4). Then K_1H_1 is perpendicular to C_1C_2. Let C_2 tend to C_1. Then C_1C_2 becomes tangent C_1T to the hyperbola at C_1, and H_1 tends to H, the point of

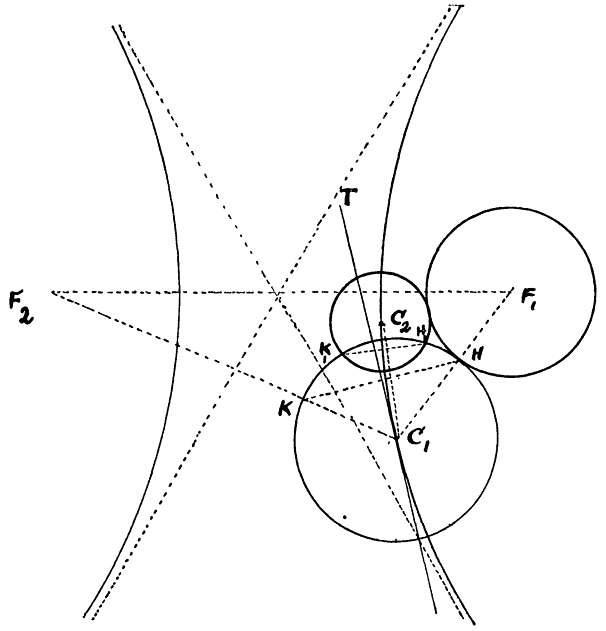


FIG. 4

tangency of (C_1) and (F_1). Therefore K_1 approaches K, the point of intersection of the line through H perpendicular to C_1T. So K is the characteristic point.

Now it is observed that $F_2K = C_1F_2 - C_1K = C_1F_2 - (C_1F_1 - HF_1) = (C_1F_2 - C_1F_1) - HF_1 =$ constant. Therefore the envelope is the circle with center F_2 and radius F_2K.

Examples:

(1) Find the envelope of the second side of the right angle such that its first side passes through a fixed point and its vertex moves on a fixed straight line.

(2) Solve (1) if the angle is equal to a fixed angle α, instead of the right angle.
(3) Find the envelope of the free side of a right angle whose vertex is on a fixed circle and the other side passes through a fixed point outside of the fixed circle.
(4) The same as (3) except use an angle equal to α.
(5) Solve Problem II for angle α instead of right angle.
(6) Solve Problem III where an ellipse or a parabola is used instead of a hyperbola.
(7) Find the envelope of circles with the center on a fixed circle and tangent to another fixed circle.
(8) Find the envelope of circles with center on a fixed circle and tangent to a fixed line.
(9) Find the envelope of circles with center on a fixed line and tangent to a fixed circle.
(10) Find the envelope of circles with center on a fixed line and tangent to a given curve (C).

A PASCAL TRIANGLE FOR THE COEFFICIENTS OF A POLYNOMIAL

RICHARD A. MILLER, Convair, Fort Worth

It is well known that the coefficients, c_i, of a polynomial, $f(x)=x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n$, may be determined by the elementary symmetric functions of its zeros, r_i, *i.e.*,

$$\sum_i^n r_i = -c_1, \sum r_1r_2 = c_2, \cdots, \sum r_1r_2\cdots r_n = (-1)^n c_n.$$

The purpose of this note is to develop an algorithm for the formation of the coefficients from a given set of numbers $\{r_i\}$, $i=1, \cdots, n$.

We denote the sum of the products, taken j at a time, of the first k elements of the set by $S(r_i, j, k)$. We observe that a simple recursion formula can be established by induction,

$$(1)\qquad S(r_i, j, k) = S(r_i, j, k-1) + r_kS(r_i, j-1, k-1) \qquad \text{for } j > 1,\ k > 1.$$

We define $S(r_i, 0, k)=1$ for $k>0$, and $S(r_i, j, k)=0$, for $k<j$.

Verification for $k=2$ is immediate, for we have $S(r_i, 1, 2)=S(r_i, 1, 1)+r_2S(r_i, 0, 1)=r_1+r_2$, and $S(r_i, 2, 2)=S(r_i, 2, 1)+r_2S(r_i, 1, 1)=0+r_2r_1$.

We assume validity for k and deduce it for $k+1$. We note that

$$S(r_i, 1, k+1) = S(r_i, 1, k) + r_{k+1}S(r_i, 0, k) = (r_1 + \cdots + r_k) + r_{k+1},$$

$$r_{k+1}S(r_i, k, k) = S(r_i, k+1, k+1).$$

INTERPRETATIONS OF THE PEANO POSTULATES

M. D. DARKOW, Hunter College

When students first deal with an abstract postulate system, their understanding profits from exercises involving models that satisfy the system. This paper contains suggestions for such exercises in connection with the Peano postulates for the natural numbers.

The Peano postulates are given in terms of an undefined concept, called a natural:

1) To every natural n, there corresponds a natural n', called the consequent of n.

2) There exists a natural, denoted by 1, which is not a consequent.

3) Naturals having equal consequents are equal.

4) If a set of naturals contains 1, and contains the consequent of every natural in the set, it is the set of all naturals.

The interpretations or models of these postulates utilize the students' prior knowledge.

Model A: The numbers $a, a+d, a+2d, \cdots$ of an arithmetic progression (in which a, d are real numbers, $d \neq 0$) satisfy the Peano postulates if a is the Peano 1, and n' is defined as $n+d$.

Model B: The numbers $a, ar, ar^2, \cdots$ of a geometric progression (in which a, r are arbitrary real numbers, $0 \neq a$, $0 < r \neq 1$) satisfy the Peano postulates if a is the Peano 1, and $n' = nr$. (A simple adjustment takes care of the case $0 > r \neq -1$.)

THEOREM. *There is one and only one operation* $\circ$ *combining two naturals* b *and* c *into a natural, such that* $b \circ 1 = b'$ *and* $b \circ c' = (b \circ c)'$. *This operation is called Peano addition and is denoted by* $(+)$. *For it*

$$b(+)1 = b' \text{ and } b(+)c' = [b(+)c]'. \tag{1}$$

The students are then asked to find the proper definitions for Peano addition in models A and B. Since it is no secret to them that the natural numbers are defined by the Peano postulates, they might proceed as follows:

A. If $x = a + (h-1)d$, $y = a + (k-1)d$, then $z = x(+)y$ should equal $a + (h+k-1)d$. Elimination of h and k produces

$$x\,(+)\,y = x + y - a + d. \tag{2}$$

The students then prove that this definition actually has properties (1).

B. If $x = ar^{h-1}$, $y = ar^{k-1}$, then $z = ar^{h+k-1}$ should define $x(+)y$. Hence $xyr/a = ar^{h+k-1} = z$ and

$$x\,(+)\,y = rxy/a. \tag{3}$$

This definition must be proved to satisfy (1).

THEOREM. *There is one and only one operation* $*$ *combining two naturals* b *and* c *into a natural, such that* $b * 1 = b$ *and* $b * c' = b * c(+)b$. *This operation is*

called Peano multiplication and is denoted by $(\cdot)$. *Hence*

$$b\,(\cdot)\,1 = b \quad \text{and} \quad b\,(\cdot)\,c' = b\,(\cdot)\,c\,(+)\,b. \tag{4}$$

The students then find the proper definitions of Peano multiplication for models A and B.

$A: x(\cdot)y = [a+(h-1)d](\cdot)[a+(k-1)d]$ should equal $z=[a+(hk-1)d]$. Elimination of h and k yields

$$z = x\,(\cdot)\,y = a - d + (x - a + d)(y - a + d)/d. \tag{5}$$

This definition must be shown to satisfy (4).

B: $x(\cdot)y = ar^{h-1}(\cdot)ar^{k-1}$ should equal $z=ar^{hk-1}$. Elimination of h and k yields $\log_r rz/a = \log_r rx/a \cdot \log_r ry/a$. Hence

$$z = x\,(\cdot)\,y = ar^E, \text{ where } E = (1 + \log_r x/a)(1 + \log_r y/a) - 1. \tag{6}$$

The students then show that this definition has property (4).

Suitable definitions of the Peano order relation may also be asked in A and B.

When the categorical nature of the postulates has been established, the students are asked to demonstrate the isomorphism of models A and B.

Any sequence of distinct complex numbers may be similarly treated.

METHOD OF UNDETERMINED COEFFICIENTS

M. S. WEBSTER, Purdue University

Consider the special linear differential equation of order n, $(D-a)^n y = x^k e^{ax}$ where n is a positive integer, k is a nonnegative integer, a is any real constant, and D is the operator d/dx. The usual rule* given for finding a particular solution by the method of undetermined coefficients is to multiply $e^{ax}(A_0x^k + A_1x^{k-1} + \cdots + A_{k-1}x + A_k)$ by an appropriate power of x(x^n in this case) and then to determine the constants $A_0, A_1, \cdots, A_k$ so that $x^n e^{ax}(A_0x^k + \cdots + A_k)$ is a solution. This procedure will do, but, because of the special nature of the left member of this differential equation, it may be greatly simplified in this case.

Actually the equation has a solution of the form $Ax^{n+k}e^{ax}$ where A is a constant. This may be shown by means of the shifting rule $e^{-ax}P(D)y = P(D+a)\cdot(e^{-ax}y)$ where $P(D)$ is a polynomial in D. The differental equation becomes $D^n(e^{-ax}y) = x^k$, from which the result follows.

* Apparently first given by A. Cohen in *An Elementary Treatise on Differential Equations*, 1906.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1261. *Proposed by C. S. Ogilvy, Hamilton College*

If the sectorial area bounded by a curve and the radii vectores to any two points on the curve equals the area bounded by the curve, the x-axis, and the ordinates of the two points, find the curve.

E 1262. *Proposed by A. J. Goldman, National Bureau of Standards*

For which real values of A do all roots of $z^3-z^2+A=0$ obey $|z|\leqq 1$?

E 1263. *Proposed by W. B. Carver, Cornell University*

Let T_0 be any triangle none of whose angles is a multiple of 45°. The tangents to the circumcircle of T_0 at its vertices form a new triangle T_1; and repetition of this process gives an infinite sequence of triangles $\{T_n\}$. If the angles of T_0 in degrees are integers, show that for $n\geqq 2$, T_{n+12} is similar to T_n. Is there a similar theorem for the case when the angles are rational in degrees?

E 1264. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If an interior point P of a tetrahedron $ABCD$ is projected orthogonally into A', B', C', D' on the planes of the faces BCD, CDA, DAB, ABC, and if the areas of these faces are denoted by A, B, C, D, show that

$$A(PA)+B(PB)+C(PC)+P(PD)\geqq 3[A(PA')+B(PB')+C(PC')+D(PD')].$$

E 1265. *Proposed by F. L. Wolf, Carleton College*

Is there a nonnegative, nontrivial, continuous function $f(x)$ such that

$$\int_0^x f(t)dt \geqq f(x)$$

for all x on $0\leqq x\leqq 1$?

SOLUTIONS

A Surface of Revolution

E 1231 [1956, 577]. *Proposed by A. W. Walker, University of Toronto*

Show that $x^3+y^3+z^3-3xyz=a^3$ is a surface of revolution.

I. *Solution by E. W. Marchand, Eastman Kodak Company.* The equation may be written $(x+y+z)[3(x^2+y^2+z^2)-(x+y+z)^2]=2a^3$. Hence the intersection of this surface with any plane of the form $x+y+z=c$ lies on a sphere $x^2+y^2+z^2=\text{constant}$ and so must be a circle having the normal to this plane from the origin as axis.

II. *Solution by Leonard Carlitz, Duke University.* One easily verifies the identity

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)[(x - y/2 - z/2)^2 + 3(y - z)^2/4].$$

The transformation

$$x' = (x + y + z)/\sqrt{3}, \quad y' = \sqrt{2/3}(x - y/2 - z/2), \quad z' = (y - z)/\sqrt{2},$$

is orthogonal with determinant $+1$. Thus the equation of the surface in the new coordinate system becomes $3\sqrt{3}x'(y'^2+z'^2)=2a^3$. The surface is obtained by revolving the curve $3\sqrt{3}x'y'^2=2a^3$ about OX'.

Also solved by Norman Anning, A. P. Boblétt, T. Y. Chow, Hazel E. Evans, Edward Fleisher, L. R. Ford, Michael Goldberg, A. S. Hendler, Frank Herlihy, A. R. Hyde, I. M. Isaacs, J. B. Johnston, M. S. Klamkin, D. C. B. Marsh, John Rausen, L. A. Ringenberg, Robin Robinson, Azriel Rosenfeld, David Zeitlin, and the proposer. Late solution by Sidney Glusman.

Editorial Note. The surface has the shape of an infinite horn; it is asymptotic to the plane $x+y+z=0$ and to the axis of revolution $x=y=z$.

A Line Through the Orthocenter of a Triangle

E 1232 [1956, 577]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Arbitrary, parallel lines drawn through the vertices A, B, C of a triangle intersect the circumcircle in A', B', C'. Show that A'', B'', C'', the symmetrics of these points with respect to the midpoints of BC, CA, AB, respectively, lie on a line perpendicular to the parallel lines and passing through a fixed point of the triangle.

I. *Solution by John Rausen, New York University.* Let O be the circumcenter and H the orthocenter of triangle ABC, and let M be the midpoint of BC. Let AH intersect the circumcircle at H'. By a well known theorem, H' is the symmetric of H with respect to BC. Let H'' be the other end of the diameter through A. Then $H'H''$ is perpendicular to AH', hence parallel to BC, and chords BC, $H'H''$ have the same perpendicular bisector, OM. Thus H'' is the symmetric of H' with respect to OM. By composition of two symmetries about perpendicular lines, H'' is the symmetric of H with respect to M. Therefore lines $A'H''$ and $A''H$ are symmetric with respect to M, and hence are parallel. But $A'H''$ is perpendicular to AA' (since AH'' is a diameter). Therefore so is $A''H$.

Similarly $B''H$ is perpendicular to BB' and $C''H$ is perpendicular to CC'. This proves all at once that (1) A'', B'', C'' lie on a line m, (2) m is perpendicular to AA', BB', CC', and (3) m passes through H, a fixed point in triangle ABC.

II. *Solution by A. R. Hyde, West Hartford, Conn.* If the origin is taken at the center of the circumscribed circle, and the x-axis is taken parallel to the parallel lines through the vertices, then the vertices are

$$A:(r \cos \alpha, r \sin \alpha), \quad B:(r \cos \beta, r \sin \beta), \quad C:(r \cos \gamma, r \sin \gamma),$$

where α, β, γ are the angles between the positive x-axis and the radii to A, B, C respectively. A', B', C' are the mirrors of A, B, C across the y-axis. Hence A'', B'', C'' all have x-coordinate equal to $r(\cos \alpha+\cos \beta+\cos \gamma)$, and $A''B''C''$ is a straight line perpendicular to the parallel (horizontal) lines through the three vertices. The altitudes of triangle ABC intersect on this line, as can be shown analytically.

Also solved by J. W. Clawson, J. C. W. De la Bere, J. T. Humphrey, Josef Langr, D. C. B. Marsh, Beckham Martin, Robert Sibson, Sister M. Stephanie, and the proposer.

Clawson, De la Bere, and Sister Stephanie gave proofs in the complex plane. Clawson stated the following generalization: If $AB \cdots N$ is an n-gon inscribed in a circle (O), and if a set of parallel lines are drawn through the vertices $A, B, \cdots, N$ to intersect (O) at $A', B', \cdots, N'$, and if $A_1, B_1, \cdots, N_1$ are the mean centers of the polygons obtained by omitting in turn $A, B, \cdots, N$, then if $A'A_1, B'B_1, \cdots, N'N_1$ are extended $1/(n-2)$ of these lengths to $A'', B'', \cdots, N''$, these latter points lie on a straight line perpendicular to the parallel lines and passing through the point which lies on the line joining O to M, the mean center of the polygon, extended $2/(n-2)$ of the length of OM.

On p. 6 of his book *Parmi les belles figures de la géomêtrie dans l'espace*, the proposer has extended the problem to a tetrahedron and its circumsphere; here A'', B'', C'', D'', are the mirrors of A', B', C', D' in the centroids of the faces BCD, CDA, DAB, ABC, and A'', B'', C'', D'' are coplanar on a plane perpendicular to the given parallel lines and passing through the Monge point of the tetrahedron.

A Sequence of Triangles

E 1233 [1956, 578]. *Proposed by Joseph Andrushkiw, Seton Hall University*

Denote the sides and inradius of a triangle by a_0, b_0, c_0, and r_0. The points of contact form a new triangle whose sides and inradius are a_1, b_1, c_1, and r_1. Repeating the process one obtains the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{r_n\}$. Show that as $n \to \infty$

$$\lim r_n/a_n = \lim r_n/b_n = \lim r_n/c_n = \sqrt{3}/6.$$

I. *Solution by N. J. Fine, University of Pennsylvania.* Let A_n, B_n, C_n be the angles opposite a_n, b_n, c_n respectively, for $n \geqq 0$. It is easy to see that $a_n/r_n = \cot (B_n/2)+\cot (C_n/2)$, and similarly for the other two ratios. Now $A_{n+1} = (\pi - A_n)/2$. An easy induction shows that $A_n = \pi/3+(-1/2)^n(A_0-\pi/3) \to \pi/3$, and similarly $B_n \to \pi/3$, $C_n \to \pi/3$. Hence a_n/r_n, b_n/r_n, c_n/r_n all converge to $2 \cot (\pi/3) = 2\sqrt{3}$, from which the required result follows.

II. *Solution by Leon Bankoff, Los Angeles, Calif.* Let A_i denote the vertices opposite a_i $(i=0, 1, \cdots, n)$. It is easily shown that the acute angle between two chords of contact of an incircle is equal to the arithmetic mean of the angles at the two adjacent vertices of the parent triangle. It follows that

$$2A_n = B_{n-1} + C_{n-1}, \quad 2B_n = A_{n-1} + C_{n-1}, \text{ and } |A_{n-1} - B_{n-1}| = 2|B_n - A_n|.$$

Then, as $n \to \infty$, $|B_n - A_n| \to 0$. Similarly $|A_n - C_n| \to 0$ and $|C_n - A_n| \to 0$, with the result that A_n, B_n, C_n each tend toward 60° as $n \to \infty$. This leads at once to the desired conclusion.

Also solved by W. B. Carver, J. C. W. De la Bere, A. R. Hyde, I. M. Isaacs, J. B. Johnston, P. G. Kirmser, M. S. Klamkin, Sadia Makky, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, D. S. Passman, John Rausen, Azriel Rosenfeld, R. E. Shafer, Michael Skalskyj, A. V. Sylwester, Chih-yi Wang, and the proposer. Late solution by J. W. Harter.

The convergence of the angles A_n, B_n, C_n to $\pi/3$ is quite rapid. Carver remarked that if we start with *any triangle whatever*, all the angles of triangle $A_7B_7C_7$ will differ from 60° by less than 1°.

Two Unique Sets of Numbers

E 1234 [1956, 578]. *Proposed by Leo Moser and J. R. Pounder, University of Alberta*

(a) Find 3 distinct positive integers, relatively prime in pairs, such that the sum of any two is divisible by the third. Prove that these integers are unique.

(b) Find 3 distinct positive integers such that the product of any two leaves a remainder of 1 on division by the third. Prove that the numbers are unique.

Solution by E. P. Starke, Rutgers University. First we consider three positive integers $x < y < z$ and note that if z is a divisor of $x+y$, then $z = x+y$ because $x+y<2z$. If also y divides $x+z(=2x+y)$, then y divides $2x$ which, with $2y>2x$, implies $y=2x$, whence $z=3x$.

(a) By the above, if the integers x, $2x$, $3x$ are relatively prime by pairs, they must be 1, 2, 3. (Note that we have not used the hypothesis that $y+z$ is a multiple of x.)

(b) Let the integers be $a<b<c$, and let $ab-1=cx$. Then x is an integer less than a. Now $x(ac-1)=a^2b-a-x$ divisible by b implies $a+x$ divisible by b; and similarly $b+x$ divisible by a. By the initial result above, we have $a=2x$, $b=3x$. Finally $ab-1=cx$ gives $6x^2-1$ divisible by x, whence $x=1$ and the required unique integers must be $a=2$, $b=3$, $c=5$.

Also solved by W. J. Blundon, Anne W. and Jean M. Calloway (jointly), Leonard Carlitz, Monte Dernham, Underwood Dudley, Hazel E. Evans, N. J. Fine, Virginia S. Hanly and Ernest Kanning (jointly), Roger Hou and Lincoln Teng (jointly), J. B. Johnston, Joe Lipman, D. C. B. Marsh, P. J. McCarthy, Azriel Rosenfeld, E. D. Schell, R. E. Shafer, and the proposers. Late solutions by D. A. Breault and J. W. Harter.

A Uniformly Convergent Series

E 1235 [1956, 578]. *Proposed by D. S. Greenstein, University of Pennsylvania*

Let $f_0(x)$ be bounded and integrable over $a \leqq x \leqq b$, and let

$$f_n(x) = \int_a^x f_{n-1}(t)dt, \qquad n = 1, 2, \cdots, \qquad a \leqq x \leqq b.$$

Evaluate $\sum_{n=1}^{\infty} f_n(x)$.

I. *Solution by T. S. Chihara, Seattle University.* $f_n(x)$ being an n-fold iterated integral of $f_0(x)$, we have

$$f_n(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f_0(t)dt, \qquad |f_n(x)| \leqq \frac{(x-a)^{n-1}}{(n-1)!} \max |f_0(x)|,$$

for $a \leqq x \leqq b$. The interchange of summation and integration thus being justified, we have

$$\sum_{n=1}^{\infty} f_n(x) = \int_a^x e^{x-t} f_0(t)d \; = e^x \int_a^x e^{-t} f_0(t)dt.$$

II. *Solution by P. G. Kirmser, Kansas State College.* It may be shown (as in Solution I) that the series converges uniformly, to $y(x)$, say. We then find that $y'(x) = y(x) + f_0(x)$. Solving this differential equation gives

$$y(x) = e^x \int_a^x e^{-t} f_0(t)dt.$$

Also solved by Thomas Erber, A. S. Hendler, A. R. Hyde, J. B. Johnston, M. S. Klamkin, A. E. Livingston, Marshall Luban, C. M. Luján, Y. L. Luke, D. C. B. Marsh, R. M. Meisel, C. S. Ogilvy, F. D. Parker, John Rausen, Azriel Rosenfeld, R. E. Shafer, Michael Skalskyj, H. S. Valk, Albert Wilansky, L. K. Williams, David Zeitlin, and the proposer. Late solution by Sidney Glusman.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

4733. *Proposed by Albert Wilansky, Lehigh University*

Give an example of a compact Hausdorff space which is separable (has a dense sequence) but not completely separable (does not satisfy the second axiom of countability).

4734. *Proposed by George Brauer, University of Minnesota*

Construct an ordinary Dirichlet series $F(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \lim_{\sigma \to 0^+} |F'(\sigma + i\tau)| = \infty$$

for almost all points on the τ-axis ($s=\sigma+i\tau$). (A Taylor series with analogous properties is given in Rudin's paper, *On a problem of Bloch and Nevanlinna*, Proc. Amer. Math. Soc. vol. 6, 1955, pp. 202–204.)

4735. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let $A_1A_2A_3A_4A_5$ be a simple 5-point plane figure, and let d be any line in the plane of the figure. Let the common point of the line d and the side a_i opposite to A_i be denoted by B_i, and the common point of the lines A_iB_{i+1}, B_iA_{i+1} by C_{i+3}. Then the five lines A_iC_i have a point D in common.

4736. *Proposed by M. v. Ments, Jerusalem, Israel*

An encyclopaedia consists of N volumes, and in a certain town p persons originally possessed complete sets. But by some event, a lot of volumes of the various sets were destroyed at random, so that only n_i volumes remain in the possession of the ith person. If all the townspeople now pool their books, what is the probability that M volumes (out of the total N) are available somewhere in town?

4737. *Proposed by E. P. Starke, Rutgers University*

For any modulus m, let $V(m)$ be the number of values of $r(0\leq r\leq m-1)$ for which the congruence $x^2\equiv r \pmod{m}$ has at least one solution. Determine the form of $V(m)$.

SOLUTIONS

Disjoint Permutations

4647 [1955, 447]. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

Editorial Notes. I. No solution has been received, but K. Eisemann points out that the problem is easy to solve for $n=p$, a prime, and that this solution is the basis for an article by H. P. Lawther, Jr., *An application of number theory to the splicing of telephone cables*, this MONTHLY, vol. 42, 1935, pp. 81–91 (also noted in Ore, *Number Theory and Its History*, pp. 302–310.)

In fact the least positive residues (mod p) of the numbers $1+rk$, ($r=0, \cdots, p-1$; $k=1, \cdots, (p-1)/2$) give a complete set. Furthermore, the set just described can be made into a satisfactory set for $n=p+1$ by inserting the number $(p+1)$ into each chain according to the rule: in the kth chain, for k odd, put $(p+1)$ between $(k+1)/2$ and $(2p-k+1)/2$; for k even, put $(p+1)$ between $(p+k+1)/2$ and $(p-k+1)/2$.

II. Experimentation with small values of n makes plausible an affirmative

answer to the proposer's question. Starting with a set for $n=n_1$, we form a set for $n=n_1+1$ by inserting the new number(n_1+1) into each chain of the old set. For n_1 odd, every number is adjacent once and only once to every other number and, provided n_1 is not large, it is fairly easy to choose from each chain a pair to be separated by the new number (n_1+1) in such a way that all the numbers are different. In fact, this may be done in many ways as n_1 increases. For n_1 even, each number touches all other numbers but one—which already provides pairs which can be used to form the additional chain needed for n_1+1; the remaining pairs come when (n_1+1) is inserted in a chain, thus separating two numbers which can be brought together again in the new chain to be formed.

No simple procedure has developed by which this can be systematized so as to provide a proof of our conjecture by induction. The question is still open and a proof (or counter-example) is desired.

The following set of n-chains for $n=17$ was developed step by step starting with $n=3$. (Letters are used instead of numerals for ease in transcribing.) The intermediate steps may be recaptured by dropping any number of the last letters and retaining only the required number of chains at the beginning. It will be noted that at no step do these sets follow the scheme outlined in I.

abcdefghijklmnopq	*bnloigjmqehcadfkp*	*cmgnkqdlhjboeafip*
dpmiekocjlqfnhagb	*hkiajpndogelcfmbq*	*hmkalpgqcnejofbid*
homanqilbeckgdjfp	*jqoapemdkbhflgcin*	

A Conjecture Concerning Primes

4683 [1956, 258]. *Proposed by Edgar Karst, Endicott, N. Y.*

Where fails the "interesting conjecture" (so called by D. H. Lehmer in a letter to the proposer, March 1952): If 2^p-1 is prime, then also $2^p-1+100$ is prime?

Solution by J. L. Selfridge, University of California, Los Angeles. For $p<20$, the conjecture is true, as can be checked by examining a table of primes. For each p in the accompanying table, we list the least factor f of 2^p+99. This table includes all values of $p>19$ for which 2^p-1 is known to be prime (R. M. Robinson, *Mersenne and Fermat numbers*, Proc. Amer. Math. Soc. vol. 5, 1954, p. 842.)

p	31	61	89	107	127	521	607	1279	2203	2281
f	1933	149	139	1609	83	23	47	373	191	23

The thing most interesting about this and similar conjectures is whether they can be disproved without resorting to extended numerical computations. The factors here listed were found using the SWAC, but, once discovered, they are easily verified by elementary computation.

What of the conjecture that, when $p>19$ and 2^p-1 is prime, then 2^p+99 is always composite?

Also solved by H. F. Bennett, Sidney Kravitz, D. C. B. Marsh, R. E. Shafer, Alan Wayne, and the proposer.

A Limit of a Summation

4684 [1956, 258]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Prove that $a_n \to 0$ as $n \to \infty$, where

$$a_n = 1 - \frac{n-1}{1!} + \frac{(n-2)^2}{2!} - \frac{(n-3)^3}{3!} + \cdots + \frac{1}{(n-1)!}.$$

Solution by Daniel Shanks, Naval Ordnance Laboratory, Silver Spring, Md. If

$$f(z) = (e^z - z)^{-1} = e^{-z}(1 - ze^{-z})^{-1} = e^{-z} + ze^{-2z} + z^2e^{-3z} + \cdots$$

is expanded into the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum a_n$ converges and $a_n \to 0$ as desired, provided every singularity of $f(z)$ satisfies $|z| > 1$.

Now if $z = e^z$ with $|z| \leqq 1$ we will have

$$x + iy = e^{x+iy} = e^x(\cos y + i \sin y), \qquad x^2 + y^2 \leqq 1,$$

so that $y^2 < (\pi/2)^2$ and $x = y/\tan y > 0$. Thus $|z| = e^x|e^{iy}| = e^x > 1$. This contradiction completes the proof.

Also solved by Leonard Carlitz, Peter Henrici, W. J. Pervin, H. S. Shapiro, and the proposer.

The n-dimensional Polytope

4685 [1956, 258]. *Proposed by L. J. Mordell, University of Toronto*

Prove that the n-dimensional polytope

$$(1) \qquad |x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqq 2,$$

has $2^{n+1} - 2$ faces of $n-1$ dimensions and contains an n-dimensional volume $(2n)!/(n!)^3$.

Solution by Jože Ulčar, Institute of Mathematics, Skoplje, Yugoslavia. The equation (1) defines a convex polytope, since every line $x_i = k_i t$ $(i = 1, 2, \cdots, n)$ intersects its boundary in two points which are symmetric to the origin and have coordinates $x_i = \pm 2k_i/(\sum|k_i| + |\sum k_i|)$.

1. The coordinate hyperplanes separate the Euclidean space R_n into 2^n parts. In each of the parts where all coordinates do not have the same sign, there lie two $(n-1)$-dimensional faces of the polytope (1). For example, in that part where $x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_i}$ are positive and $x_{\nu_{i+1}}, x_{\nu_{i+2}}, \cdots, x_{\nu_n}$ are negative, they are the faces

$$(2) \qquad x_{\nu_1} + x_{\nu_2} + \cdots x_{\nu_i} = 1, \qquad x_{\nu_{i+1}} + x_{\nu_{i+2}} + \cdots + x_{\nu_n} = -1.$$

In the part in which all x_i are positive, the polytope (1) has only one $(n-1)$-dimensional face, namely

$$(3) \qquad x_1 + x_2 + \cdots + x_n = 1,$$

and in the part in which all x_i are negative, only the face

$$x_1 + x_2 + \cdots + x_n = -1. \tag{4}$$

Altogether, then, the number of $(n-1)$-dimensional faces of the polytope (1) is $(2^n-2)\cdot 2+2=2^{n+1}-2$.

2. The polytopes which are bounded by the coordinate hyperplanes and by either of the hyperplanes (3) and (4), are simplexes. Each has the volume $1/n!$.

The polytope which is bounded by the coordinate hyperplanes and the hyperplanes (2) is a simplotope* whose n-dimensional volume is equal to the product of the i-dimensional volume and the $(n-i)$-dimensional volume determined respectively by the hyperplanes (2) and the coordinate hyperplanes; hence the volume is $(1/i!)(1/(n-1)!)$. Since there are $\binom{n}{i}$ parts of R_n for which i coordinates are positive, the volume of the entire polytope (1) equals

$$V = \sum_{i=1}^{n-1} \frac{1}{i!(n-i)!}\binom{n}{i} + \frac{2}{n!} = \sum_{i=0}^{n} \frac{1}{i!(n-i)!}\binom{n}{i} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}^2.$$

This last form for V reduces to the proposed form upon observing that

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

Two Summations

4686 [1956, 258]. *Proposed by Leonard Carlitz, Duke University*

Show that

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n}(2k-1)\tanh\frac{(2k-1)\pi}{2} - n^2\right\} = -1/12. \tag{1}$$

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\coth k\pi - \frac{n(n+1)}{2}\right\} = 1/12 - 1/4\pi. \tag{2}$$

Solution by Peter Henrici, University of California, Los Angeles. In view of

$$\tanh x = 1 - 2/(e^{2x}+1), \qquad \coth x = 1 + 2/(e^{2x}-1),$$

the stated relations are equivalent to

$$2\sum_{k=1}^{\infty}\frac{2k-1}{e^{(2k-1)\pi}+1} = \frac{1}{12}, \qquad 2\sum_{k=1}^{\infty}\frac{k}{e^{2k\pi}-1} = \frac{1}{12} - \frac{1}{4\pi}.$$

But these are known identities; see, e.g., H. F. Sandham, *Some infinite series*, Proc. Amer. Math. Soc. vol. 5, 1954, pp. 430–436, equations (4.22) and (6.221). See also Problem 4453 [1952, 706].

Also solved by J. R. Hatcher, R. G. Stoneham, Chih-yi Wang, and the proposer.

* See H. S. M. Coxeter, Regular Polytopes, New York and Chicago, 1948, footnote p. 124.

A Double Summation

4693 [1956, 426]. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!n!}{(m+n+2)!}=\frac{\pi^2}{6}.$$

Solution by J. S. Frame, Michigan State University. Multiplying each term by $(m+n+2)-(n+1)$ and dividing by $(m+1)$, the summation over n can be written as a telescoping series, and the double sum becomes

$$\sum_{m=0}^{\infty}\frac{m!}{m+1}\sum_{n=0}^{\infty}\left(\frac{n!}{(m+n+1)!}-\frac{(n+1)!}{(m+n+2)!}\right)$$
$$=\sum_{m=0}^{\infty}\frac{m!}{m+1}\frac{0!}{(m+1)!}=\sum_{m=0}^{\infty}\frac{1}{(m+1)^2}=\frac{\pi^2}{6}.$$

Also solved by R. P. Agnew, Fred Brafman, K. A. Bush, Leonard Carlitz, P. L. Chessin, Leopold Flatto, M. L. Frimer and A. L. Tritter, R. K. Guy, J. R. Hatcher, Peter Henrici, S. S. Holland, Jr., W. C. Janes, J. B. Johnston, M. A. Kirchberg, P. G. Kirmser, M. S. Knebelman, W. R. Knight, A. E. Livingston, Marshall Luban, Y. L. Luke, D. C. B. Marsh, Kovina Milosevich, D. R. Morrison, Ingram Olkin, N. J. Phillips, E. J. F. Primrose, Chih-yi Wang, R. E. Wild, Henri Yerly, David Zeitlin, Judith Zoffmann, and the proposer. Late solutions by Žiradin Pantić and J. R. Trollope.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Engineering Mathematics. By Kenneth S. Miller. Rinehart & Co., New York, 1956. xii+417 pp. $6.50.

This text is intended for the first year of graduate work for the engineering student with "a little college algebra, analytics, calculus, and a smattering of differential equations." It assumes a more substantial maturity, however, for it covers a wide selection of topics, with a thoroughness which should lighten the burden of teaching. In perusing this exposition one finds many pet teaching devices incorporated into the text. Although the author seems anxious to argue that each topic introduced is necessary for a coherent presentation, it seems that the various sections are largely independent in treatment, so that, by proper

selection, the book could be adapted to a variety of courses, including the post-calculus course for undergraduates which has become popular in recent years. There is ample material for two four-semester-hour undergraduate courses, or six hours on the graduate level.

The first chapter proves the basic theorems on determinants, but there is only a brief glimpse of matrices. Another preliminary chapter is devoted to functions defined by integrals such as gamma, beta, sine-integral, and elliptic functions, and such topics as Jacobians, Stirling's formula and Euler's constant, are concisely treated. Breezing through a review of linear differential equations with constant coefficients, the author then gives a careful exposition of the method of variation of parameters leading to the Green's function. The method of Frobenius is given a heuristic treatment, showing how the coefficients may be successively computed, and postulating the convergence of the series by quoting an appropriate existence theorem. The Legendre and Bessel equations are solved, but their derivation from Laplace's differential equation in spherical, and cylindrical coordinates is implied, but not explicitly shown. Fourier series and integrals are derived and used. The Laplace transform chapter includes the convolution theorem. The unit on network theory includes many engineering applications. The section on statistics seems more advanced than the others, and it is questionable whether an engineering student could benefit by this study of stochastic and ergodic processes without additional pre-requisites. (The algebra and calculus of vectors is not included.)

The author asserts that "the level of rigor is at the presently accepted standard for engineering schools." This apparently means that limiting processes are interchanged with the validity of the answer accepted as sufficient justification. This reviewer would take exception to such statements as: "Given the function $\int_0^x e^{-x^2}dx$, one cannot find a function $\psi(x)$ such that $\psi'(x)=e^{-x^2}$." But though one may quibble over wording, or the motivation for introducing the successive topics, the expositions of the steps necessary to solve the type of problems selected are careful and explicit. The exercises are relatively few but no student could work all of them. The exposition is aided by a number of well-chosen figures, and each chapter terminates with a bibliography. Three appendices, answers, and an index are included in this volume.

Any teacher of engineering mathematics will enjoy reading this compact book.

ARTHUR BERNHART
University of Oklahoma

A Portrait of 2. By L. A. Ringenberg. Washington, National Council of Teachers of Mathematics, 1956. 42 pp. $.75.

This is not, as the title might imply, a discussion of the special properties of the integer 2. Rather it is a development of the number system of algebra from the concept of ordinal and cardinal numbers through the complex numbers, with

2 used as an example to illustrate the evolving generalizations. Thus we first see 2 as an ordinal number and then as a cardinal number. Next $+2$ is seen as the class of all ordered pairs of natural numbers (a, b) for which $a-b=2$ and it is pointed out that, contrary to the average beginning algebra student's ideas (and even to conventional mathematical notation), 2 (the natural number) and $+2$ (the integer) are not identical but are related through the concept of isomorphism.

A similar type of discussion lets us see a view of 2 as a rational number, as a real number (both as a Dedekind cut and as a Cantor class of sequences), and as a complex number.

While similar developments of the number systems of elementary algebra may be found in many places, few are as detailed and as motivated as this one and none, to my knowledge, bring the discussion to bear so closely on the high school teacher's problems (as, for example, in the author's discussion of the dual role of the $+$ sign). Although the treatment is kept on as elementary a level as possible necessary technicalities are not left out. Thus, for example, it is clearly pointed out that $\sqrt{2}$ is not just a sequence of rational numbers but, rather, a *class* of sequences. Although the various proofs involved in the development are not generally given, it is made clear what proofs are needed and, in most cases, how they might be carried out. Suggestions for further readings are also given.

If every high school teacher were to read and master the contents of this little book the level of instruction in high school algebra should rise considerably.

ROY DUBISCH
Fresno State College

Numerical Analysis. (Proceedings of Symposia in Applied Mathematics, vol. 6). Edited by John H. Curtiss. McGraw-Hill, New York, 1956, for the American Mathematical Society. vi+303 pp. $9.75.

In 1943 a proposal to hold a symposium on numerical analysis would probably have met with consternation, to say the least, especially if the proposal were made to the American Mathematical Society. The expression itself would have been unfamiliar, for the Institute for Numerical Analysis had not yet come into existence. However, if one could imagine such a symposium being held, it might have been sparked by the table makers of the Bureau of Standards (formerly a WPA project), and by the ballisticians of Aberdeen, and perhaps Dahlgren. Doubtless the astronomers would have been invited along with the geodesists and map makers, and perhaps actuaries and statisticians. Topics would have included the calculation of trajectories and orbits, least square fitting and smoothing of data, interpolation and numerical quadrature, and perhaps the solution of nonlinear equations. Methods of relaxation for solving problems in elasticity and electrostatics might have come in for some discussion, with a bow to the engineers. There may or may not have been discussion

of characteristic roots and vectors, since who would have had anything novel to contribute? And almost certainly the trivial problem of solving linear algebraic systems would not have come up unless someone wanted to offer a new variant of the Doolittle method. If the scope were broadened slightly to bring in the psychometrists, they could have talked about principal axes, rotations, and simple structure, concepts related to positive definite symmetric matrices. But the objectives here would often have been poorly defined, hence the techniques fuzzy, and most of the audience might have felt more confused than enlightened. If a scattering of physicists and chemists had attended, they would have been there mainly to listen, but the mathematicians would have been most conspicuous by their absence.

In 1943 there was some use of punched card machines by the astronomers, but for the most part desk computers were the most elaborate computing aids then in use. The Mark I, the ENIAC, and the relay machines of the Bell Laboratories were far from complete. The literature on computing was scattered and concerned with special techniques, some of which were repeatedly rediscovered and described in varied settings, except in the areas of interpolation and finite differencing, which gave rise to and bordered on a well elaborated mathematical theory. Elsewhere theory was sparse, in spite of the occurrence of such names as Gauss, Bernoulli, and Newton.

In 1953, the Mark I, the ENIAC and the Bell relay machines were already obsolete; the SEAC was to begin its fourth year of operation, Univacs were appearing, and many other electronic digital machines were at varying stages of development. The Institute for Numerical Analysis, set up under the National Bureau of Standards, had brought together outstanding mathematicians and outstanding professional computers and encouraged the introduction of mathematical rigor into the evaluation of numerical techniques. Much of the support for this and for other related projects had come from the armed services, especially the Office of Naval Research.

In this atmosphere the sixth Symposium in Applied Mathematics, arranged by the American Mathematical Society, was held at Santa Monica and devoted to numerical analysis. Of the twenty-one papers delivered, nineteen are published in the volume here under review. Contributors are exclusively mathematicians and physicists. Nowhere in either the index or the table of contents is there mention of orbits, trajectories, finite differences, or smoothing, and only two references are made to interpolation. However, there are three papers dealing with the solution of linear algebraic systems, and a fourth dealing with those special systems which appear as partial difference equations.

Broadly, the topics fall into two classes, techniques and applications, although most papers give some consideration to both. The applications are interesting. One paper by Clutterham and Taub, deals with a problem in hydrodynamics, but there is at most passing reference to specific problems in electrostatics, elasticity, or other phases of classical applied mathematics. On the other hand, there are three papers, by Bruck, Emma Lehmer, and Taussky, on ap-

plications to number theory, a paper by Bellman on dynamic programming, one by Motzkin on the assignment problem, and one by Tompkins on problems whose variables are permutations. In 1943, dynamic programming had not been invented, the assignment problem had been stated as a mathematical problem but without recognized practical application, the problems discussed by Tompkins would probably have been regarded as without interest. As for the number theorists, they had for some time resorted to computing for suggesting and testing conjectures, but these efforts were not commonly reported. It is the electronic computer that makes the difference.

Generally speaking, the papers are all at an advanced level, and the methodological papers especially so. Bergman discusses boundary value problems of linear partial differential equations. The longest paper of all is by Rosenbloom on the method of steepest descent, in quite abstract terms. The method has a host of applications. Sard describes his method of obtaining remainder terms for functional approximations, Walsh discusses *Best-approximation Polynomials of Given Degree*, and the Hastings approximations, based upon the Chebyshev theory, are discussed by Hastings. Particular mention should be made of a paper by Wielandt entitled *Error Bounds for Eigenvalues of Symmetric Integral Equations*, since it deals with assessments in a situation where few useful results are available.

Until a very few years ago only two distinct methods of solving linear systems were in common use. One of these is direct, the other iterative, and both go by the name of Gauss. The so-called Doolittle method is a particular form of Gaussian elimination. A number of new methods are now known with varying degrees of distinctiveness, but perhaps the most significant is the method of conjugate gradients, developed independently by Stiefel and by Hestenes. In this volume, Hestenes discusses the conjugate gradient method, and Fischbach discusses "gradient methods," including the method of conjugate gradients. A third paper on linear systems, by David Young, discusses iterative methods, with special reference to linear difference equations. Frankel discusses linear partial difference equations, but confines himself to discursive remarks on the stability problem.

Only two papers remain to be mentioned. These are by Wasow on transforming distributions, and by Warschawski on conformal mapping.

Evaluation of such a volume as this should be in terms of what it contains. It should not be condemned for its omissions. But for purposes of orientation some of the more obvious ommissions might be mentioned. Ordinary differential equations and nonlinear systems of algebraic and transcendental equations are considered only indirectly, in the sense that the method of steepest descent can be applied to both. Certain aspects of partial differential equations come in for consideration by Bergman, Frankel and Young, but the treatment is very limited. Considerably more could be said on the solution of linear systems, and the characteristic value problem for matrices is not discussed at all except indirectly by Wielandt. Finally, there is only passing reference to the Monte Carlo

method.

Thus the book is not one for the novice who wishes a uniform development of numerical analysis in its present state. Neither is it a book for the practicing computer who wishes to find a collection of special techniques to be applied routinely. Its greatest appeal will be to the mathematician who would like to learn about some of the mathematical problems that arise in this important field. Such a person, whether or not he has had experience in computing, will find much to interest him.

A. S. HOUSEHOLDER
Oak Ridge National Laboratory and
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NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER PROGRAM OF CANADIAN MATHEMATICAL CONGRESS

A joint Summer Seminar of the Canadian Mathematical Congress and the Theoretical Physics Division, Canadian Association of Physicists, will be held at the University of Alberta, Edmonton, August 12–30, 1957.

Congress research lectures will be given by the following: A. D. Alexandrov, University of Leningrad; Philip Hall, Cambridge University; D. H. Lehmer, University of California, Berkeley; Jean Dixmier, Institut Henri Poincaré; E. P. Wigner, Princeton University.

Congress instructional lectures are as follows: Leo Moser, University of Alberta, introduction to the theory of numbers; Abraham Robinson, University of Toronto, an introduction to mathematical logic; A. W. Tucker, Princeton University, theory of games; H. J. Zassenhaus, McGill University, introduction to Lie algebras.

A joint Congress of the two organizations will be held at the School of Fine Arts, Banff, Alberta, September 1–7, 1957.

Inquiries about the Seminar and the Congress should be addressed to Canadian Mathematical Congress, Secretariat, Chemistry Building, McGill University, Montreal, Quebec.

Inquiries regarding accommodation at Edmonton should be sent to Professor E. S. Keeping, Department of Mathematics, University of Alberta, Edmonton, Canada; and at Banff, to Professor D. R. Crosby, Department of Mathematics, or to Professor D. D. Betts, Department of Physics, University of Alberta, Edmonton, Canada.

MEMBERSHIP PROGRAM OF INSTITUTE OF MATHEMATICAL SCIENCES

The Institute of Mathematical Sciences, New York University, offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields

include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dynamics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as hydromagnetics and reactor theory.

The temporary membership program is designed primarily as a means of alleviating the present critical shortage of scientists trained in mathematical physics, applied mathematics, and related fields of mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms.

Temporary members may participate freely in the research projects, the advanced graduate courses and the research seminars of the Institute, and they will have the opportunity of using the computational facilities.

The temporary members will receive a stipend commensurate with their status. Membership will be awarded for one year, but it may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, New York.

NATIONAL HIGH SCHOOL AND JUNIOR COLLEGE MATHEMATICS CLUB

A National High School and Junior College Mathematics Club has been formed. It is anticipated that high school mathematics clubs, if properly qualified, will join the new organization. The purpose of the club is to engender keener interest in mathematics, to develop sound scholarship in the subject and promote enjoyment of mathematics among high school and junior college students. For further information write to Secretary-Treasurer, Josephine P. Andree, Box 1127, University of Oklahoma, Norman.

Other officers are: President, H. L. Alder, University of California, Davis; Vice-President, E. L. Walters, William Penn Senior High, York, Pennsylvania; Governors-General, Nellie M. Kitchens, Hickman High School, Columbia, Missouri; Dr. J. R. Mayor, Director, Science Teaching Improvement Program, Washington, D. C.; Virginia L. Pratt, Central High School, Omaha, Nebraska.

CONFERENCE ON CAREERS IN THE MATHEMATICAL SCIENCES

A conference on Careers in the Mathematical Sciences was held by the Institute of Mathematical Sciences, New York University, on January 5, 1957. High school mathematics chairmen and vocational guidance counselors from New York, New Jersey, and Connecticut participated in the conference.

The program included a discussion of employment opportunities at various levels, a description of some of the work in which mathematicians are involved, and a consideration of the role of high-speed electronic computing machines in science and technology. The speakers were: Professor Richard Courant, Dr. L. W. Cohen, Dr. C. R. DeCarlo, Professor Harold Grad, Professor Morris Kline, President C. V. Newsom, Dr. Mina S. Rees, Dean H. W. Stoke, and Professor J. J. Stoker.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1957.

Columbia University, Teachers College, July 7 to August 16: Professor Rosskopf, department seminar in teaching mathematics; Dr. Sobel, professionalized subject matter in advanced secondary school mathematics, materials and models in mathematics education; teaching of geometry; mathematics in the junior high school. Under the National Science Foundation: Professor Levi, sets, algebra and analysis; Professor Lott, Jr., probability and statistics; Professor Rosskopf and Mr. Rourke, teaching modern concepts in the secondary school.

DePaul University, June 24 to August 3: Professor DeCicco, abstract metric geometry, abstract set theory; Mr. Merkes, elementary differential geometry; Dr. G. Weiss, topics in conformal mapping.

Kent State University, June 17 to July 20: Professor Iwanchuk, selected topics for elementary school teachers; Professor Bush, introduction to modern algebra; Professor Jenkins, introduction to mathematical statistics. July 22 to August 24: Professor Dressler, solid analytic geometry; Professor Johnson, history of mathematics; Professor Stapleford, advanced methods of teaching mathematics in high school.

Michigan State University, June 25 to August 2: Professor Blair, differential equations, functions of a complex variable; Professor Faith, college geometry; Professor Gaddum, advanced calculus III; Professor Nordhaus, potential theory; Professor Powell, concepts in mathematics (this course is part of a summer institute in mathematics, physics and chemistry sponsored by the National Science Foundation). June 25 to August 23: Professor Campbell, differential equations; Professor Herzog, functions of a complex variable III; Dr. Larcher, advanced mathematics for engineers II; Professor Oehmke, introduction to higher algebra III; Professor Parkus, advanced mathematics for engineers III, mathematical theory of elasticity, gas dynamics; Professor Peirce, vector analysis, computer coding I; Professor Reid, advanced calculus I, differential equations II; Professor Weeg, theory of matrices and groups, finite differences.

New York University, Institute of Mathematical Sciences, June 17 to July 26: Dr. Rubin, advanced calculus and applications I; Research Professor Magnus, special functions of mathematical physics; Professor Shapiro, special topics in linear programming (all half-courses). July 29 to September 6: Professor Shapiro, special topics in number theory; Staff, advanced calculus and applications II, Laplace transform and Heaviside calculus (half-courses).

Northwestern University, June 22 to August 17: Engineering mathematics I, III; numerical methods in mathematics; advanced calculus; geometry for teachers; complex variables for applications; topics in modern mathematics for teachers; modern algebra.

Stanford University, June 25 to August 18: differential equations; introduction to the functions of a complex variable; selected topics from analysis; graduate seminar; Professor Herriot, computer laboratory; Staff, advanced reading and research.

University of Buffalo, July 1 to August 9: Professor Bienert (Hobart College), finite differences, topics in geometry; Professor Gehman, history of mathematics; Professor Schneckenburger, theory of sets.

University of Chicago, June 24 to August 30: Program emphasizes algebraic topology; Professor MacLane, introduction to algebraic topology; Professor Spanier, homology of locally compact spaces; Professor Eilenberg (Columbia University), foundations of fiber bundles; Professor Thom (University of Strassbourg, France), homotopy of functional spaces. Other advanced courses: Professor Chern and Kuranishi, infinite continuous pseudo-groups; Professor Segal, analysis in topological groups; Staff, advanced calculus; algebra; projective geometry; set theory; complex variables.

University of Florida, June 14 to August 10: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professors Lang, Blake, and Rohde, higher mathematics for engineers and physicists; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Kokomoor, history of elementary mathematics; Professor Hadlock, advanced topics in calculus; Professor Moore, theory of groups of finite order; Professor Pirenian, vector analysis; Professor Hutcherson, synthetic projective geometry; Professor Butson, general topology.

University of Illinois, June 17 to August 10: Professor Munroe, functions of real variables; Professor Hamilton, elementary geometry from a modern viewpoint; Dr. Zaring, group theory; Professor Landin, the teaching of selected mathematical topics in secondary schools.

University of Michigan, June 24 to August 16: Professor Coburn vector analysis;

Professor Coxeter, college geometry, higher geometry; Professor Craig, probability, theory of statistics II; Professor Dushnik, advanced calculus; Professor Harary, foundations of mathematics; Professor Hewitt, functions of real variable, abstract harmonic analysis; Professor Higman, theory of matrices, Galois theory; Professor Jones, teaching of algebra, history of elementary collegiate mathematics; Professor Rainville, intermediate differential equations; Professor Rothe, operational mathematics, integral equations; Professor Shields, Fourier series, dimension theory; Professor Titus, advanced calculus for engineers; Professor Ullman, functions of complex variable with applications; Professor Wesler, calculus of finite differences, theory of statistics I.

University of Minnesota, College of Science, Literature and Arts, June 17 to July 20: Professor Gibbens, solid analytic geometry; Professor Shapiro, probability, topics in topology. July 22 to August 24: Professor Olmsted, introduction to the theory of functions, advanced algebraic theory; Professor Munro, theory of numbers, calculus of finite differences. An Institute for High School Teachers of Mathematics will be conducted from June 17 to August 10.

University of Nebraska, June 13 to August 2: Dr. Cree, differential equations; Dr. Schweppe, projective geometry; Professor Leavitt, theory of equations; Professor Basoco, Laplace transforms; Staff, reading course, thesis.

University of North Carolina, June 6 to July 13: Professor Mackie, theory of equations; Professor Linker, differential equations; Professor MacNerney, partial differential equations, topics in analysis. July 15 to August 21: Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Garner, introduction to higher geometry; Professor Hill, elementary mathematical statistics; Professor Lasley, analytic geometry from a higher standpoint; Professor Wall, some recent results in algebra.

University of Oklahoma, June 6 to August 3: Professor Bernhart, college geometry; Professor Springer, elementary differential equations, ordinary and partial differential equations; Professor LaFon, vector analysis; Professor Brixey, theory of groups.

University of Pennsylvania, June 24 to August 3: Professor Gottschalk, axiomatics of number systems; Professor Epstein, numerical methods and differential equations, theory of functions of a complex variable; Professor Yang, introduction to higher algebra.

University of Pittsburgh, June 10 to July 19, July 22 to August 30: Professor Barsotti, differential equations; Professors Blumberg and Christiano, advanced calculus; Professor Taylor, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Bompiani, algebraic geometry; Professor Laush, calculus of variations; Professor Levine, topology. June 17 to August 9 (evenings): Professors Edwards and Knipp, differential equations; Professors Bryson and Laird, mathematics of modern engineering; Professor Laush, integral equations; Professor Levine, functions of a complex variable (second half of two semester course). July 1 to August 9: Professor Elyash, differential equations for students of engineering; Professor Teats, history of mathematics; Professor Kehl, introduction to the mathematics of digital computation; Professor Myers, recreational mathematics for teachers, teaching of secondary school mathematics; Professor Kachun, navigation for teachers; Professor DeSua introduction to the foundations of mathematics.

University of South Carolina, June 10 to August 10: Professor Lee, theory of equations; Professor Novak, college geometry; Professor Williams, advanced calculus, theory of functions of complex variables; Professor Hedberg, introduction to modern algebra. A Summer Institute for Teachers of High School Mathematics will be held simultaneously with the Summer Session; Director, Professor W. L. Williams.

University of Texas, June 4 to July 16: Professor Craig, applications of tensor analysis; Professor Moore, theory of sets; Professor Ettlinger, research in integral equations. July 17 to August 27: Professor Lane, continued fractions; Professor Wall, analytic functions, infinite processes.

University of Virginia, June 24 to August 17: Professor Plunkett, foundations of

algebra (for 6 weeks beginning July 8), differential equations and applied mathematics; Professor Botts, introductory analysis; Professor Floyd, analytic topology.

University of Washington, June 24 to August 22: Professor Walter, linear algebra; Professors McFarlan, Kingston, and Leipnik, differential equations; Professor Avann, advanced calculus and vector analysis, applications of vector analysis; Professor Cramlet, topics in applied analysis; Professor Livingston, advanced euclidean geometry; non-euclidean geometry; Professor Pierce, foundations of mathematics.

University of Wyoming, June 10 to July 12: Professor Walsh, theory of determinants and matrices; Staff, ordinary differential equations; Professor Barr, seminar in analysis; Professor Neubauer, history of mathematics. July 15 to August 16: Professor Varineau, advanced calculus, fundamental concepts of mathematics; Professor S. R. Smith, partial differential equations, Fourier series and boundary value problems.

Virginia Polytechnic Institute, Southern Regional Graduate Summer Session in Statistics, June 12 to July 20: Dr. DeLury (Ontario Research Foundation, Toronto, Canada) and Dr. McHugh (Virginia Fisheries Laboratory), sampling of biological populations; Dr. Williams (Commonwealth Scientific and Research Organization, Melbourne, Australia), analysis of variance; Professor Ash, statistical inference; Professor Brenna, engineering statistics; Professor Bradley, rank order statistics; Professor Clunies-Ross, probability; Professor J. E. Freund, stochastic processes; Professor R. J. Freund, sampling; Professor Kramer, statistical methods; Professor Wine, theory of least squares. Director: Professor Boyd Harshbarger.

Wayne State University, Computation Laboratory, June 3 to 8: introduction to computers and their applications. June 10 to 15: data processing in business and industry. September 9 to 14: industrial and management computer applications. Experts from various parts of the country and the staff of the laboratory will participate. A Conference on Matrix Computations will be held September 3 to 6 (see May MONTHLY).

PERSONAL ITEMS

Professor J. M. Barbour, Michigan State University, has been elected President of the American Musicological Society for the term 1957–59.

Professor Emeritus George Polya, Stanford University, has been elected to honorary membership in the London Mathematical Society.

College of Arts and Sciences, Baghdad, Iraq: Professor H. Schatz has been appointed Professor; Dr. M. W. Al-Dhahir has been promoted to Assistant Professor; Mr. Waleed Al-Salam has been given a scholarship to study at Duke University; Mr. Y. Al-Duri is studying at the University of Oregon.

Michigan State University: Dr. Heinrich Larcher, Institute Montana, Zugerberg, Switzerland, has been appointed Instructor; Dr. W. H. Peirce, University of Wisconsin, has been appointed Assistant Professor; Professor Henry Parkus, Institute of Technology, Vienna, has been appointed Visiting Professor for the spring and summer terms of 1957; Dr. S. K. Berberian and Dr. J. H. McKay have been promoted to Assistant Professors.

University of Virginia: Dr. R. P. Goblirsch, Teaching Assistant, University of Wisconsin, has been appointed Instructor; Associate Professor E. E. Floyd has been promoted to Professor.

Mr. A. T. Bharucha-Reid has been appointed Instructor at the University of Oregon.

Associate Professor O. C. Collins, University of Nebraska, is on leave of absence and is a resident consultant with General Mills, Minneapolis, Minnesota.

Dr. Martin D. Davis, Ohio State University, has been appointed Assistant Professor at Hartford Graduate Center, Rensselaer Polytechnic Institute.

Dr. D. O. Ellis, Electronics Division, National Cash Register Company, Hawthorne, California, has a position as a research scientist and head of the analysis staff of Litton Industries, Beverly Hills, California.

Assistant Professor R. L. Gay, Wake Forest College, has been promoted to Associate Professor.

Dr. W. A. Howard, University of Chicago, has accepted a position as a mathematician with Bell Telephone Laboratories, Whippany, New Jersey.

Mr. O. C. Juelich is employed now as a computing engineer at North American Aviation, Columbus, Ohio.

Dr. J. Lawrence Katz, Polytechnic Institute of Brooklyn, has been appointed Assistant Professor of Physics at Rensselaer Polytechnic Institute.

Dr. Manfred Kochen, Columbia University and the Institute for Advanced Study, has accepted a position as an associate mathematician with the International Business Machines Corporation Research Center, Poughkeepsie, New York. Dr. Kochen was formerly a Ford Foundation fellow at Harvard University.

Associate Professor J. A. Larrivee, Worcester Polytechnic Institute, is employed as a mathematical analyst by Lockheed Aircraft Corporation, Burbank, California.

Dr. W. S. Mahavier, Defense Research Laboratory, University of Texas, has been appointed Instructor at Illinois Institute of Technology.

Mr. H. W. Martin, Clarendon Junior College, has been appointed Instructor at Amarillo College.

Miss Ethelyn L. McBee, University of Florida, has been appointed Associate Professor and Head of the Department of Mathematics at Wesleyan College.

Mr. R. P. Mitchell, White Sands Signal Corps Agency, has accepted a position as a mathematician with the Naval Ordnance Laboratory, Corona, California.

Professor C. N. Moore, University of Cincinnati, is a visiting lecturer at the University of South Carolina for the second semester of 1956–57.

Dr. W. L. Nicholson, Princeton University, has accepted a position as a research statistician with General Electric Company, Richland, Washington.

Mr. R. J. Pipino has been appointed Instructor at Western Reserve University.

Mr. G. F. Simmons, Yale University, has been appointed Assistant Professor at the University of Rhode Island.

Mr. W. W. Varner, Head, Computation Laboratory, University of Colorado, has a position as Head of the Flight Simulation Laboratory and Computers, Convair Astronautics, General Dynamics Corporation, San Diego, California.

Miss Marion I. Walter, Teaching Assistant, Cornell University, has been appointed Assistant Professor at Simmons College.

Professor Emeritus E. P. Adams of Princeton University died on December 31, 1956. He was a charter member of the Association.

Dr. Tobias Dantzig, who had retired from the position of Chairman of the Department of Mathematics, University of Maryland, died on August 9, 1956.

Professor F. W. Reed of Ohio University died on November 22, 1956. He was a member of the Association for thirty-seven years.

Professor Emeritus Carrie B. Taliaferro of State Teachers College, Farmville, Virginia, died on September 14, 1955. She was a member of the Association for thirty-four years.

Professor Emeritus H. S. Uhler of Yale University died on December 6, 1956. He was a charter member of the Association.

Professor Emeritus F. B. Wiley of Denison University died on December 14, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1956

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1956. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

There was a deficit of $4,111 in the Current Fund of the Association for the year 1956. This deficit was anticipated by the Finance Committee when the Budget for 1956 was adopted. It is believed that with increased receipts for dues and other items, the Current Fund will have no deficit during 1957.

Assets of the Association	January 1, 1956	December 31, 1956
M & T Trust Co., Buffalo	12,842.52	30,857.02
Buffalo Savings Bank	20,000.00	19,297.28
Securities	123,832.00	111,828.64
	156,674.52	161,982.94
Funds of the Association		
Current Fund	2,561.37	450.03
Carus Fund	22,140.94	23,763.26
Chace Fund	13,839.25	12,369.13
Houck Fund	11,496.05	12,082.88
Chauvenet Fund	1,446.41	1,420.06
Dunkel Fund	18,954.24	19,917.92
General Fund	41,509.40	40,518.72
	111,947.66	110,522.00
Visiting Lecturers Fund	23,162.53	42,942.44
Fund for Committee on Undergraduate Program	21,564.33	8,069.77
Fund for Committee on Films	—	448.73
	156,674.52	161,982.94

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 171 persons have been elected to membership by the Board of Governors on applications duly certified.

R. J. Adler, Student, Carnegie Institute of Technology

S. J. Adler, Ph.D. (Columbia) Chemist, Brookhaven National Laboratory, Upton, N. Y.

William Allan, B.A. (Amherst) Actuary, Home Life Insurance Co., New York, N. Y.

Rev. R. W. Allen, S.J., Ph.D. (St. Louis) Chm., Department of Mathematics, Xavier University.

Yale Altman, M.S. (Boston U.) Field Representative, Addison-Wesley Publishing Co., Reading, Mass.

Maurice Anderson, A.B. (Wayne S.T.C.) Grad. Asst., University of Nebraska.

Mansur Arbabi, B.S. in E.E. (California State Poly. C.) Instr., Seattle University.

R. J. Arguello, B.E.E. (Poly. Inst. of Brooklyn) Electrical Design Engr., Otis Elevator Co., Brooklyn, N. Y.

J. W. Armstrong, A.B. (San Jose S.C.) Grad. Student, San Jose State College.

Lawrence Arnold, Student, Occidental College.

F. W. Ashley, Jr., B.S. (Oklahoma) Lexington, Massachusetts.

M. H. Barnebey, M.S. (Ohio S.U.) Asso. Professor, Tougaloo Southern Christian College.

G. D. Bisbey, M.S. (Iowa S.C.) Instr., Wisconsin State College, River Falls.

J. T. Bliss, B.S. (Black Hills T.C.) Math., Directorate of Research and Development, Holloman Air Development Center, N. Mex.

G. C. Branche, M.S. (Rochester) Grad. Asst., University of Rochester.

Rev. E. W. Brande, S.J., M.S. (St. Louis) Grad. Student, St. Louis University.

R. D. Branstetter, Ph.D. (Iowa S.C.) Asst. Professor, San Diego State College.

G. A. Brown, Grad. Student, Rutgers University.

A. J. Burda, Jr., B.S. (U. S. Naval Acad.) Instr., University of Maryland.

Mrs. Marjorie V. Butcher, M.A. (Michigan) Instr., Trinity College, Connecticut.

Jean S. Campbell, Math., Office of Naval Research, Washington, D. C.

D. L. Carlstrom, Student, Eastern Montana College of Education.

S. D. Chatterji, M.S. (Lucknow U.) Grad. Asst., Syracuse University.

Evan Chenet, Jr., M.A. (Columbia) New York, New York.

Rev. A. A. Clarke, S.J., M.A. (Fordham) Instr., Fordham University.

E. H. Connell, B.A. (McMurry) Res. Scientist, Missile Systems Division, Lockheed Aircraft Corp., Palo Alto, Calif.

H. H. Corson, Ph.D. (Duke) Instr., Tulane University.

H. L. Dachslager, M.S. (Wisconsin) Grad. Student, University of Wisconsin.

J. O. Danley, M.A. (Oklahoma) Instr., East Central State College.

C. T. Daub, Jr., Student, Carleton College.

H. J. Davis, M.S. (Stanford) Instr., Pomona College.

J. D. DePree, B.A. (Hope C.) Instr., University of Colorado.

R. L. Duncan, M.A. (Pennsylvania S.U.) Instr., Pennsylvania State University.

Janet M. Dunning, M.A. (Columbia U., Teachers C.) Head, Department of Mathematics, Ramapo Regional High School, N. J.

Philip Dwinger, Ph.D. (Leiden) Asst. Professor, Purdue University.

R. L. Edwards, Jr., B.E.E. (Alabama P.I.) Grad. Student, Johns Hopkins University.

Herman Elson, M.A. (N.Y.U.) Teacher, Technical High School, Buffalo, N. Y.; Instr., University of Buffalo.

Carlos Fallon, Lieutenant (Natiónal Military Acad., Colombia) Engr., Vitro Corporation of America, Silver Spring, Md.

H. L. Farris, Jr., B.A. (Texas) Math., Douglas Aircraft Co., Tulsa, Okla.

Walter Feit, Ph.D. (Michigan) United States Army.

Thorleif Fostvedt, M.S. (Chicago) Asso. Professor, University of Alberta.

Charles Fox, D.Sc. (London, England) Professor, McGill University.

L. B. Fuller, M.A. (Michigan) Grad. Asst., University of Rochester.

S. E. Ganis, M.S. (Michigan) Asso. Professor, Ohio Wesleyan University.

Dot J. Gifford, M.A. (Oklahoma) Asst. Professor, Oklahoma College for Women.

J. H. Gissel, M.S. (North Dakota) Instr., General Motors Institute, Flint, Mich.

I. I. Glick, B.A. (Johns Hopkins) Res. Asst., University of Maryland.

Esther E. Guerin, B.S. (Douglass) Teacher, Roxbury High School, Succasunna, N. J.

Mary Hagen, B.A. (Coll. of St. Elizabeth) Palisade, New Jersey.

D. L. Hartford, B.A. (Kentucky) Instr., University of Kentucky.

E. L. A. Hayes, B.S. (Seton Hall) United States Army.

J. W. Haynes, Jr., M.A. (California, Berkeley) Grad. Student, University of California.
Leon Henkin, Ph.D. (Princeton) Asso. Professor, University of California, Berkeley.
Mrs. Dagmar R. Henney, M.S. (Miami) Junior Instr., University of Maryland.
J. N. Herrmann, Res. Stress Analyst, Bendix Products Division, South Bend, Ind.
Aaron Herschfeld, M.A. (Columbia) Asst. Professor, Canisius College.
J. C. Hickman, M.S. (S.U. of Iowa) Actuarial Asst., Bankers Life Co., Des Moines, Iowa.
E. T. Hodges, M.A. (North Carolina) Asso. Professor, College of William and Mary.
Frederick Hoffman, Student, Georgetown University.
H. E. Homesley, Jr., B.S. (Jacksonville S.C.) Grad. Student, University of Florida.
V. E. Howes, Doctorat d'Universite (Paris) Instr., Fresno State College.
L. C. Huffman, M.Ed. (Midwestern) Instr., Midwestern University.
Elaine L. Johnson, Student, Carleton College.
M. C. Johnson, Jr., M.S. (Illinois) Editor, College Department, McGraw-Hill Book Co., New York, N. Y.
D. L. Jones, Student, University of Colorado; Math. Aide, National Bureau of Standards, Boulder, Colo.
J. P. Jordan, B.S. (Fordham) Teaching Asst., Rutgers University.
F. J. Kammel, Student, St. John's University.
G. A. Kandall, Student, Princeton University.
Y. H. Kao, B.S. (National Taiwan U., China) Grad. Asst., Oklahoma Agricultural and Mechanical College.
Seymour Kass, B.A. (Brooklyn C.) Grad. Student, Stanford University.
Costas Kassimatis, M.A. (Toronto) Instr., Lehigh University.
N. D. Kazarinoff, Ph.D. (Wisconsin) Asst. Professor, University of Michigan.
Mildred Keiffer, M.A. (Columbia) Supervisor of Math., Cincinnati Board of Education, Ohio.
Mrs. May E. R. Kinsolving, M.S. (Michigan) Instr., Harpur College.
W. A. Kirby, M.A. (Wyoming) Teaching Asst., University of Texas.
R. B. Kirchner, Student, Carleton College.
R. M. Kozelka, Ph.D. (Harvard) Asst. Professor, University of Nebraska.
Capt. W. H. Lake, B.S. (U. S. Military Acad.) U. S. Naval Academy.
E. F. Lang, M.D. (Michigan) Radiologist, L. Reynolds and Assoc., Detroit, Mich.
R. L. Larson, Student, San Diego State College.
G. M. Lehnen, A.B. (Northern Michigan C. of Ed.) Equipment Engr., Western Electric Co., Chicago, Ill.
L. L. Lichti, M.A. (Nebraska) Registrar, Hesston College.
C. E. Linderholm, Student, University of Chicago.
P. W. Lindsey, Jr., B.S. (Alabama P.I.) Teaching Fellow, Alabama Polytechnic Institute.
Shirley T. Loeven, B.S. in Ed. (Central Missouri S.C.) Asst. Instr., University of Kansas.
Joyce M. Lutz, Student, University of Kentucky.
A. G. MacLean, M.A. (Oxford, England) Mgr., Statistical Services, California Test Bureau, Los Angeles, Calif.
J. C. Mairhuber, M.S. (Rochester) Instr., University of Rochester.
J. E. Martin, M.S. (Vanderbilt) Asst. Professor, Virginia Military Institute.
M. S. Matheson, B.S. (Maine) Instr., Pemetic High School, Southwest Harbor, Me.
J. R. McCarthy, M.A. (Boston C.) Instr., College of the Holy Cross.
R. W. McChesney, B.S. (Union) Teaching Asst., University of Rochester.
H. B. McClung, M.S. (West Virginia) Instr., West Virginia University.
R. E. McGill, B.A. (Wooster) Res. Asst., University of Maryland.
D. D. McKeachie, M.S. (Michigan) Instr., General Motors Institute, Flint, Mich.
L. D. McLean, Jr., M.S. (Arizona) Instr., University of Arizona.
Brockway McMillan, Ph.D. (M.I.T.) Asst. Director, Systems Engineering, Bell Telephone Labs., New York, N.Y.
J. L. Ménard, B.S. (Montreal) Student, University of Montreal.
R. J. Merikangas, B.A. (Tulane) Ensign, United States Navy.

Rosemary Milkovitch, M.A. (Montana) Professor, Eastern Montana College of Education.

R. P. Mitchell, M.A. (Denver) Math., U. S. Naval Ordnance Lab., Corona, Calif.

Franklin Mohr, M.A. (Columbia) Summit, New Jersey.

W. O. J. Moser, M.A. (Minnesota) Instr., University of Saskatchewan.

I. H. Mufti, M.S. (Karachi) Grad. Asst., University of British Columbia.

M. G. Mundt, B.A. (Luther C.) Grad. Asst., Iowa State College.

S. E. Natelson, Student, Carleton College.

L. F. Nattinger, Field Engr., Westinghouse Electric Corp., Calif.

W. L. Nicholson, Ph.D. (Illinois) Statistician, General Electric Co., Richland, Wash.

R. Z. Norman, Ph.D. (Michigan) Asst. Professor, Dartmouth College.

Torsten Norvig, B.A. (Copenhagen) Teaching Asst., Brown University.

Ellen Oliver, Student, Brooklyn College.

Mrs. Vivienne H. Olson, M.S. (Pittsburgh) Teacher, Chicago City Junior College, Wilson Branch.

W. E. Olson, Jr., Student, Carleton College.

C. D. Papakyriakopoulous, Doctor (Athens, Greece) Institute for Advanced Study.

K. C. Park, Professor, Seoul National University, Korea.

S. K. Patton, B.S. (Washington & Lee) Grad. Asst., Syracuse University.

C. Y. Pauc, Ph.D. (Paris) Visiting Professor, Purdue University.

R. J. Pegis, Student, University of Toronto.

Martha M. Pennell, Student, Cornell University.

C. A. Perkins, B.S. in B.A. (Boston U.) Asst. Section Chief, Connell, Price & Co., Boston, Mass.

O. W. Peterson, M.E. (Montana S.U.) Asso. Professor, Eastern Montana College of Education.

R. L. Pierce, Student, Carleton College.

R. J. Pipino, M.A. (Johns Hopkins) Instr., Western Reserve University.

Mrs. Judith M. S. Prewitt, Math. Analyst, Burroughs Corp., Philadelphia, Pa.; Student, Swarthmore College.

Mrs. Mabel G. Reavis, Ph.D. (Duke) Asst. Professor, Texas Christian University.

Heinz Renggli, D.Sc. (Swiss Federal I.T.) Res. Instr., Newcomb College, Tulane University.

R. A. Restrepo, Ph.D. (California I.T.) Lecturer, University of British Columbia.

C. R. Riehm, Student, University of Toronto.

R. H. Riffenburgh, M.S. (William and Mary, Richmond Div.) Asst. Professor, Virginia Polytechnic Institute.

E. W. Robinson, B.S. in Ed. (Northeastern S.C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

C. C. Robusto, Ph.D. (Fordham) Asso. Professor, St. John's University.

H. M. Roensch, Sr., LL.B. (Texas) Linneus, Missouri.

Milton Rosenthal, M.D. (California) Los Angeles, California.

T. S. Rowland, M.S. in Ed. (Kentucky) Machine Programming Math., Ames Aeronautical Lab., Moffett Field, Calif.

I. D. Ruggles, M.A. (Wyoming) Grad. Asst., Iowa State College.

Rev. J. P. Ruth, C.S.B., M.A. (Michigan) Professor, St. John Fisher College.

D. B. Saylors, Student, University of Tulsa.

A. H. Schlenker, A.M. (Indiana) Acting Instr., University of Tulsa.

Hans Schneider, Ph.D. (Edinburgh) Asst. Professor, State College of Washington.

R. A. Schwabe, B.A. (Denison) Grad. Asst., Syracuse University.

G. B. Seligman, Ph.D. (Yale) Instr., Yale University.

M. W. Shelly, II, Ph.D. (Wisconsin) Res. Asso., Ohio State University.

Alfonso Shimbel, Ph.D. (Chicago) Asst. Professor, Illinois Institute of Technology.

Marvin Shinbrot, M.A. (Syracuse) Aero. Res. Scientist, National Advisory Committee for Aeronautics, Moffett Field, Calif.

R. E. Simmons, B.A. (San Diego S.C.) Asst. Instr., San Diego State College.

F. W. Sinn, Jr., Student, Temple University.

Sister Mary Leonora Bagley, M.A. (St. Bonaventure) Villa Maria College, Erie, Pa.

Alliston Slade, M.A. (Mississippi) Instr., University of Mississippi.

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THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the College of William and Mary, Williamsburg, Virginia, on December 1, 1956. Professor R. C. Yates, Chairman of the Section, presided at the morning and afternoon sessions.

There were 70 persons in attendance, including 40 members of the Association.

By unanimous vote of members present, the Section approved the following resolution:

Whereas the Mathematical Association of America has voted to sponsor a nation-wide mathematics contest for high schools in 1958;

And, Whereas, the Md-DC-Va. Section has operated a successful local contest in these three states each year since 1954;

And, Whereas, certain advantages can be gained by coordination in contest activities, thus securing better support, prestige, and financial backing:

Therefore, be it resolved that the Md-DC-Va. Section does hereby—

(1) Express interest in participating in, and hereby offer its services on the National Standing Committee on High School Contests to be appointed by the President of the Association;

(2) Authorize its present contest committee to serve as the Sectional Committee on High School Contests, as mentioned in section IIIa (1) of the National Committee recommendations;

(3) Authorize this committee to continue to conduct our present contest program until transition to the national program is approved by our executive committee.

The following papers were presented:

1. *A unique formula for the ratio of the areas of two triangles, the one formed by transversals through the vertices of the other,* by Professor S. T. Gormsen, Virginia Polytechnic Institute.

A formula was developed which expresses the ratio of the area of a given triangle to the area of a second triangle, formed by passing three transversals through the vertices of the given triangle and dividing each of the opposite sides into two segments. As a consequence of this formula, proofs for at least four important geometric theorems about concurrency of lines, such as the converse of "Ceva's Theorem," follow immediately.

2. *The place of mathematics in Austrian education*, by Professor Herta T. Freitag, Hollins College.

Curricular offerings and teaching procedures on all levels of mathematics in the Austrian school system were sketched as experienced from 1914 to 1938, and observed in 1954.

3. *The geometry of the compass*, by Professor L. S. Shively, Bridgewater College.

This paper presented a brief exposition of the geometric constructions of Mascheroni (1797) in which the compass only is used. It was demonstrated that any construction which can be made with straight edge and compass can be made with the compass only. Also, the constructions with five fixed compasses, for the division of the circle into equal arcs, as made by Mascheroni, were given. One of these (division into 4 equal arcs) is the so-called problem of Napoleon.

4. *A theorem concerning the Bernstein polynomials*, by Mr. H. W. Gould, University of Virginia.

It was shown that if $f(x)$ is a polynomial of degree ν in x, then the Bernstein polynomials of $f(x)$ are given by $B_k^f(x) = f(x) + \sum_{\alpha=1}^{\nu} k^{-\alpha} Q_\alpha^\nu(x)$, where the $Q_\alpha^\nu(x)$ (independent of k) are polynomials of degree $\leqq \nu$ which can be given explicitly in terms of the Stirling numbers of the first and second kinds. The existence of $Q_\alpha^\nu(x)$ was proved by E. J. McShane, *Ann. of Math.*, Study 31, page 119.

5. *On the Gibb's phenomenon in the eigenfunction series associated with a non-self-adjoint differential equation*, by Professor L. I. Mishoe, Morgan State College.

This paper investigated the Gibb's phenomenon of the series expansion of an arbitrary function $f(x)$ in terms of the eigenfunctions of the system $(A+\lambda B)u=0$, $u(a)=u(b)=0$, where A is the differential operator $(d^2/dx^2)+q(x)$ and B is the operator $(d/dx)+p(x)$, and showed that such an expansion will exhibit the Gibb's phenomenon whenever the classical Fourier series does so.

6. *Contribution to the theory of the blunderbuss*, by Mr. Peter Treuenfels, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Special solutions of the equations of nearly one-dimensional time dependent compressible isentropic flow of a perfect gas are studied, subject to the assumptions that the Mach number is independent of time and that the gas velocity varies inversely with time. The system of differential equations considered in this paper is non-singular at sonic velocity. The solutions can be completely described in terms of quadratures of elementary functions. Numerical values are presented. It is believed that the material would form a suitable topic of discussion for an undergraduate mathematics club.

7. *The National High School Mathematics Contest*, Panel Discussion by Professor R. P. Bailey, U. S. Naval Academy (Former Chairman, National Contest Committee), Mr. W. H. Norris, Norfolk Public Schools (Chairman, MD-DC-Va. Contest Committee), and Professor D. B. Lloyd, D. C. Teachers College (Former Chairman, NCTM Contest Committee).

The speakers discussed the implications of the National Contest Recommendations recently approved by the Board of Governors and considered the general question of the participation of the Section in the project.

8. *A plea for changed attitudes toward science*, by Dr. B. C. Dees, Assistant Director of Scientific Personnel and Education, National Science Foundation (invited address).

We must create a more favorable climate for science education in the United States. As we succeed in presenting science and mathematics to students and to the public in more meaningful, more accurate ways, we will create the kind of respect for science and scientists which will improve attitudes toward science. Clearly, we must train our high school teachers more adequately, and we must see to it that they are given more status and more pay. We must make sure that the right kinds of science curricula are made available to students at all educational levels—and that students are properly counseled so that they do not rule out for themselves careers in science by choosing unwisely the courses they take in high school.

R. P. BAILEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS, University of Kansas, Lawrence, April 13, 1957.

KENTUCKY, Berea College, Berea, April 27, 1957.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.

METROPOLITAN NEW YORK, Hunter College, New York, April 27, 1957.

MICHIGAN

MINNESOTA, Carleton College, Northfield, May 11, 1957.

MISSOURI, Southeast Missouri State College, Cape Girardeau, April 27, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 26, 1957.

NEW JERSEY, Fall, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 28, 1957.

NORTHERN CALIFORNIA, January, 1958.

OHIO, University of Cincinnati, April 20, 1957.

OKLAHOMA, University of Arkansas, Fayetteville, April 12–13, 1957.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.

PHILADELPHIA, November 28, 1957.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.

SOUTHEASTERN

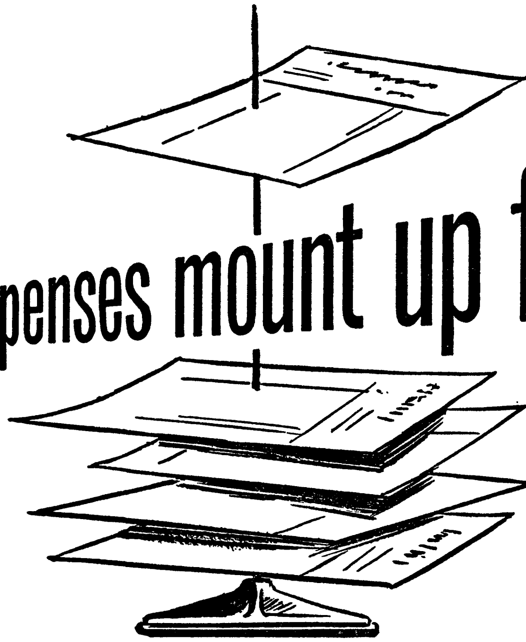
SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April 26–27, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.



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NONASSOCIATIVE NUMBER THEORY

TREVOR EVANS, Emory University

Introduction. In several papers I. M. H. Etherington has studied the algebra of exponents of the general element in a nonassociative linear algebra and he has called these systems *logarithmetics*. If the linear algebra does not satisfy any identities the corresponding logarithmetic bears a close resemblance to the natural numbers. In fact, in [3] Etherington has shown that the elements of this particular logarithmetic can be defined in terms of partitioned classes in complete analogy to the Frege-Russell definition of the natural numbers as classes of classes. It is not too surprising then that we can also characterize this logarithmetic by a set of postulates analogous to the Peano postulates for the natural numbers. We do this in Section 1 and develop the basic properties of the logarithmetic in a manner paralleling the usual development of the natural numbers.*

We proceed to study the number theory of this logarithmetic. Several of the theorems in Section 1 and 2 including the "fundamental theorem of arithmetic" have been obtained by Etherington, although our derivations are in general quite different. Some of the standard theorems and conjectures of ordinary number theory have trivial analogues in this new number theory but a little more effort is needed to prove Fermat's Last Theorem.

By analogy with the extension of the natural numbers to the ring of positive and negative integers, we extend the logarithmetic to a system in which subtraction is always possible. The system so obtained is the left neoring of Bruck's recent paper [2]. The fundamental theorem of arithmetic has to be proved anew, since there are many more primes than just the original primes and their associates. In this new system we are also able to introduce the analogues of finite arithmetics and congruence by using some of the results of [6].

We conclude by mentioning a few problems and possible directions for further work.

1. Peano-like postulates for the nonassociative natural numbers. Peano's postulates characterize the natural numbers as a set closed under a unary operation and satisfying certain other conditions. If we replace the unary operation by a binary operation and make the corresponding changes in the postulates we obtain the following system.

Undefined terms: The set of nonassociative natural numbers, the elements of which we will just call numbers;† the binary operation of addition.

* See, for example, the first few pages in [8].

† We will always use the words "positive integer" in referring to the natural numbers of ordinary arithmetic.

Postulates:

(1)
- (i) 1 is a number,
- (ii) to every pair of numbers a, b there corresponds a third called the sum of a and b and written $a+b$,
- (iii) there are no numbers a, b such that $a+b=1$,
- (iv) if the numbers a, b and c, d are such that $a+b=c+d$, then $a=c$ and $b=d$,
- (v) if a set of numbers contains 1, and if whenever it contains numbers a, b then it contains $a+b$, then the set contains all numbers. (The principle of nonassociative induction.)

Thus our numbers are $1, 1+1, 1+(1+1), (1+1)+1, 1+(1+(1+1)), \cdots$. By postulate (v), every number except 1 can be expressed as the sum of two other numbers and postulate (iv) implies that this can be done in only one way. Also, by postulate (iv), addition is in general noncommutative since $a+b=b+a$ implies $a=b$. Following Etherington, we will denote $1+1$ by 2, $1+(1+1)$ by 3, $1+(1+(1+1))$ by 4, $\cdots$.

As an example of nonassociative induction we prove:

THEOREM 1. *For all numbers a, b, $a \neq a+b$.*

Proof. Let S be the set of all values of a such that $a\neq a+b$ for any b. S contains 1 by postulate (iii). Let $m, n \in S$. If there exists a number b such that $m+n=(m+n)+b$, then $m=m+n$ by postulate (iv), in contradiction to the assumption that $m\in S$. Thus $m+n\in S$ and so by the principle of nonassociative induction, S contains all numbers.

An immediate consequence of this theorem is that there are no numbers a, b, c such that $a+(b+c)=(a+b)+c$. That is, addition is completely nonassociative. Because of the lack of commutativity and associativity in addition, introducing an order or partial order into the system does not seem to be very fruitful. One fairly reasonable definition is as follows. We first define "well-formed part" of a number by (i) the only well-formed part of 1 is 1 itself, (ii) if $a=b+c$, the well-formed parts of a are a itself and the well-formed parts of b and c. Now we define $x\leq y$ if x occurs as a well-formed part of y, and $x<y$ if $x\leq y$ but $x\neq y$. With these definitions we get a partial ordering* between numbers but unfortunately $x<y$ does not imply $x+z<y+z$ or $z+x<z+y$. However, with the definition of multiplication given below, $x<y$ does imply $z\cdot x<z\cdot y$.

We introduce multiplication $a\cdot b$ (or ab) into our number system by

* The partial order for N can be extended to a complete ordering as was pointed out to me recently in conversation by R. H. Bruck and D. R. Hughes. Define $a<b$ in N if (i) $|a|<|b|$; (ii) $|a|=|b|$ but $a_1<b_1$, where $a=a_1+a_2$, $b=b_1+b_2$; (iii) $|a|=|b|$ and $a_1=b_1$, but $a_2<b_2$. Unlike the partial ordering given above, this ordering has all the usual properties. Another complete ordering of N is obtained by interchanging a_1 and a_2, b_1 and b_2 in (ii), (iii).

$$\text{(2)} \qquad \text{(i) } a\cdot 1 = a, \qquad \text{(ii) } a\cdot(b+c) = a\cdot b + a\cdot c.$$

Clearly this defines a product between every pair of numbers. We leave to the reader the proof of the next two theorems. Nonassociative induction is used on a in the first and c in the second.

THEOREM 2. *$1\cdot a = a$ for all numbers a.*

THEOREM 3. *$(ab)c = a(bc)$ for all numbers a, b, c.*

Some examples of calculation in our system are

$$2\cdot 2 = 2 + 2, \qquad 3\cdot 2 = 3 + 3 = (1 + (1 + 1)) + (1 + (1 + 1)),$$
$$2\cdot 3 = 2 + (2 + 2) = (1 + 1) + ((1 + 1) + (1 + 1)),$$
$$((3 + 2) + (3 + 2)) + (3 + 2) = ((1 + 2) + 2)\cdot(2 + 1).$$

As an immediate consequence of the definition of multiplication the left-distributive law is satisfied. The cancellation properties of multiplication are given in the following theorems.

THEOREM 4. *If $xa = ya$, then $x = y$.*

Proof. This is true for $a = 1$. Assume that it is true for $a = m$ and $a = n$. Now, if $x(m+n) = y(m+n)$, expanding each side we get $xm+xn = ym+yn$. By postulate (iv), $xm = ym$, and so by our inductive hypothesis, $x = y$. Thus the theorem is true for $a = m+n$, and so for all values of a by nonassociative induction.

In order to prove the other cancellation law it is useful to introduce the concept of length of a number n. We mean by this the positive integer obtained from n by regarding $+$ in the expression for n as the addition of ordinary arithmetic. We will denote the length of n by $|n|$. The following relations hold.

$$\text{(3)} \qquad |m + n| = |m| + |n|, \qquad |m\cdot n| = |m|\cdot|n|,$$

i.e., $m \to |m|$ is a homomorphism onto the positive integers.

THEOREM 5. *If $ax = ay$, then $x = y$.*

Proof. We use induction on x. When x is 1, consideration of the lengths of the two sides of the equation $a = ay$ shows that $y = 1$. Consider the equation $a(m+n) = ay$. By the preceding sentence y cannot be 1 and so $y = s+t$ for some numbers s, t. Then $a(m+n) = a(s+t)$ or $am+an = as+at$. By postulate (iv), $am = as$ and $an = at$.

Hence, if $am = as$ implies $m = s$, and $an = at$ implies $n = t$, then $a(m+n) = ay$ implies $m+n = y$. The theorem follows by nonassociative induction.

We now have a fairly complete picture of our nonassociative number system. Every number in it can be obtained from 1 by a finite number of nonassociative additions. Multiplication, $u(1)\cdot v(1)$, of two of these numbers satisfies $u(1)\cdot v(1) = v(u(1))$, in complete analogy with multiplication in ordinary

arithmetic. Addition satisfies the uniqueness law, $a+b=c+d$ implies $a=c$ and $b=d$. Multiplication is associative, has 1 as an identity, is connected with addition by the left-distributive law, and satisfies the usual cancellation laws. In the language of modern algebra, this system can be described as additively the free groupoid generated by 1 with a multiplication introduced by $a\cdot b=b\phi_a$ where ϕ_a is the endomorphism of the groupoid determined by mapping 1 into a.

From now on, we will denote this system by N and call it *nonassociative arithmetic.*

2. Number theory. We can now proceed with the development of the number theory of N. In view of the noncommutativity of multiplication we need the concepts of *left-factor* and *right-factor*. If $a=b\cdot c$, then b is called a left-factor of a and c is called a right factor of a. If b, c are not equal to 1 or a, we call them *proper* left- or right-factors. A number, other than 1, having no proper left-factors is called a *prime number*. Clearly, a prime number has no proper right-factors either. A striking property of factors in nonassociative arithmetic is given in the next theorem.

THEOREM 6. *If p is a proper left-factor of a, and $a=b+c$, then p is a left-factor of b and a left-factor of c.*

Proof. Since $pq=a$ for some q and $p\neq a$, then $q\neq 1$. Thus $q=m+n$ for some m, n. Then $p(m+n)=a$ and so $pm+pn=b+c$. By postulate (iv), $pm=b$, $pn=c$. That is, p is a left-factor of both b and c. We note that this theorem is not true if we consider right-factors instead of left-factors.

In ordinary arithmetic we have the theorem that if a prime is a factor of a product it is a factor of one of the numbers. The following theorem is similar.

THEOREM 7. *If the prime p is a left-factor of the product $a\cdot b$, where a is not* 1, *then it is a left-factor of a.*

Proof. We use nonassociative induction on b. For $b=1$ the theorem is certainly true. Now if it is true for m, n and if $b=m+n$, then p is a left-factor of $a(m+n)=am+an$. But $p\neq a(m+n)$ since p is prime and so by Theorem 6, p is a left-factor of am. Hence p is a left-factor of a by our inductive hypothesis.

The corresponding result for right factors is also true, but it is most easily obtained as a consequence of the following theorem.

THEOREM 8. (*The fundamental theorem of nonassociative arithmetic.*) *There is only one way in which a number can be written as a product of primes.*

Proof. Let $p_{|1|}p_{|2|}\cdots p_{|s|}$, $q_{|1|}q_{|2|}\cdots q_{|t|}$ be two products of primes such that $p_{|1|}p_{|2|}\cdots p_{|s|}=q_{|1|}q_{|2|}\cdots q_{|t|}$. By Theorem 7, $p_{|1|}$ is a left-factor of $q_{|1|}$ and since $q_{|1|}$ is prime, $p_{|1|}=q_{|1|}$. Then, by Theorem 5, $p_{|2|}\cdots p_{|s|}=q_{|2|}\cdots q_{|t|}$. Continuing this, we get $p_{|2|}=q_{|2|}$, $p_{|3|}=q_{|3|}$, $\cdots$. There must be the same number of factors in each product since otherwise we would eventually have 1 expressed as a product.

COROLLARY. *If the prime p is a right factor of the product $a \cdot b$, where b is not* 1, *then it is a right-factor of b.*

For $a \cdot b$ when written as a product of primes must end with p by the theorem, and since this product of primes can be obtained by writing a and b separately as products of primes, p must be the last factor in the expression of b as a product of primes.

The concept of factor can be extended by defining m to be a *factor* of a if $a = smt$. The concept of mutually prime in ordinary arithmetic has several analogues in nonassociative arithmetic. Two numbers, a, b are *mutually left-prime* if they have no common proper left-factor, *mutually right-prime* if they have no common proper right-factor, *mutually prime* if one is not a factor of the other, and no proper right-factor of one is a left-factor of the other. It is easy to verify the following generalizations of Theorems 6, 7:

(i) If m is a proper factor of a, not a right-factor, and $a = b + c$, then m is a factor of b and of c, (ii) let m be a factor of $a \cdot b$; if m, b are mutually prime, then m is a factor of a, or if a, m are mutually prime, then m is a factor of b.

If a, b are two mutually left-prime numbers then a, $a+b$, $(a+b)+b$, $((a+b)+b)+b$, $\cdots$ have no nontrivial left-factors by Theorem 6, and so are all prime. Hence there are an infinite number of primes. An example of such an infinite sequence of primes is 2, 3, 4, $\cdots$. The twin primes conjecture of ordinary arithmetic has a trivial generalization for, if k is any number, there exists an infinite number of pairs of primes of the form n, $n+k$. The analogue of Goldbach's conjecture fails to hold by virtue of postulate (iv). However, another famous conjecture of ordinary arithmetic is provable in nonassociative arithmetic. In fact, an even stronger result than the original is true.

THEOREM 9. (*Fermat's Last Theorem*). *There are no numbers x, y, z such that $x^{|n|} + y^{|n|} = z^{|n|}$ for any positive integral $|n|$ greater than $|1|$.*

Proof. We obtain a proof by contradiction. Let x, y, z be numbers such that $x^{|n|} + y^{|n|} = z^{|n|}$, where $|n|$ is a positive integer greater than $|1|$. We note that (i) neither x nor y can be 1 since this would imply that $x^{|n|} + y^{|n|}$ is a prime, (ii) $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$.

Since $|n|$ is greater than $|1|$, $z^{|n|}$ has z as a left-factor and so by Theorem 6, both $x^{|n|}$ and $y^{|n|}$ have z as a left-factor. Let x and z be expressed as a product of primes in the form $x = p_{|1|}p_{|2|} \cdots p_{|s|}$, $z = q_{|1|}q_{|2|} \cdots q_{|t|}$. Then, since $zu = x^{|n|}$ for some number u, we have $q_{|1|}q_{|2|} \cdots q_{|t|}u = p_{|1|}p_{|2|} \cdots p_{|s|}v$, where $v = x^{|n-1|}$.

By Theorem 7, $q_{|1|} = p_{|1|}$, $q_{|2|} = p_{|2|}$, $\cdots$. Now $|s| < |t|$, for otherwise $|z| \leq |x|$, in contradiction to $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$. Hence, $z = xa$ for some a and, similarly, $z = yb$ for some b.

We have then $|z| = |x| \cdot |a|$, $|z| = |y| \cdot |b|$. Substituting in $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$ for $|x|$ and $|y|$, we get $|1|/|a|^{|n|} + |1|/|b|^{|n|} = 1$. This is a contradiction since for $|n| > |1|$, no positive integers $|a|$, $|b|$ satisfy such a condition.

3. Introduction of "negative integers." In ordinary arithmetic, zero and the negative integers are introduced in order that subtraction always be possible. The same problem arises naturally in nonassociative arithmetic also. In N, subtraction can be defined between some pairs of numbers as follows. If $m=p+n$ we can introduce the operation of right-subtraction $m-n$ between m and n and write $m-n=p$. If $m=n+q$, we can introduce the operation of left-subtraction* $-n+m$ between m and n and write $-n+m=q$. With these definitions we get the following properties

$$(4) \qquad \begin{aligned} (m+n)-n &= m, & n+(-n+m) &= m, \\ (m-n)+n &= m, & -n+(n+m) &= m. \end{aligned}$$

Clearly these are properties we would like subtraction to have, and in a general nonassociative system they are the most for which we can hope. The problem now is to find a system containing N and such that left- and right-subtraction is possible between every pair of elements. More specifically, we want a system with two operations $+$, $\cdot$, and such that (i) the equations $a+x=b$, $y+a=b$ have unique solutions, (ii) multiplication is associative, (iii) the multiplicative identity 1 generates the system, (iv) the cancellation laws hold, (v) the left-distributive law holds, (vi) with respect to the operation $+$, 1 generates a subsystem isomorphic to N. Such a system is of the type discussed by Bruck in a recent paper [2] and called by him a left neoring. However, there are many left neorings satisfying the above conditions. We will choose the one which seems to be the most natural extension of N.

Let L be the free monogenic loop† generated by 1 with the operation written as addition. This is the nonassociative analogue of the additive group of integers. The mapping $1\to a$ where a is any element of L determines an endomorphism ϕ_a of L and we can introduce a multiplication into L by defining $a\cdot b=b\phi_a$. We will denote the resulting system by I and call it the left neoring of nonassociative integers. In this section "number" will refer to an element of I.

An immediate consequence of this definition of multiplication is that $a(b+c)=ab+ac$ for all a, b, c. In addition, as is shown in [2], multiplication is associative and the two cancellation laws of multiplication are satisfied. Since, additively, I is a loop, we do have the required subtraction properties, and the subsystem of I consisting of 1, $1+1$, $1+(1+1)$, $\cdots$, *etc.* is isomorphic to N. We refer the reader to [2], [6], for a discussion of the algebraic structure of I. We wish to introduce here some analogues of ordinary number theory in I. For this reason we will use another approach to the system which has the advantage of an explicit representation of its elements.

Consider all expressions which can be generated by 0 and 1 with the three

* Note that the $-$ and $+$ here do not exist independently, but are each part of the notation for the binary operation of left-subtraction.

† For a discussion of free loops see [1], [4], [5].

binary operations of addition $a+b$, left subtraction $-a+b$, and right subtraction $a-b$. We call such expressions *numerical expressions*. An example is $(4+(0-1))-((1+1)+(1+(-2+1)))$, where 2, 4 have the usual meaning as abbreviations.

Two numerical expressions are equal if and only if their equality follows from the following

$$
(5)\qquad
\begin{array}{ll}
\text{(i)}\ a + 0 = 0 + a = a, & \\
\text{(ii)}\ a - a = - a + a = 0, & \\
\text{(iii)}\ a - 0 = - 0 + a = a, & \\
\text{(iv)}\quad (a + b) - b = a, & -b + (b + a) = a, \\
\text{(v)}\quad (a - b) + b = a, & b + (-b + a) = a, \\
\text{(vi)}\ a - (-b + a) = b, & -(a - b) + a = b,
\end{array}
$$

where a, b are numerical expressions.

Clearly (i), (ii), (iii) are properties we wish 0 to have, (iv) and (v) are the properties of subtraction we already have in N. Equations (vi) are actually consequences of the preceding equations and we list them merely for their usefulness in computation. We remark that $(-a+b)$ is the unique solution of $a+x=b$ and $b-a$ is the unique solution of $y+a=b$.

Our nonassociative integers are now defined as the classes of equal numerical expressions. A multiplication is introduced into the system by $u(1)\cdot v(1)=v(u(1))$ where u, v are numerical expressions.

That this system is I is a consequence of the results of [4], [5]. Another result from [4, Theorem 2.2], shows that in each class of equal numerical expressions there is a unique expression of shortest length (here "length" refers to the number of 0's and 1's in the expression). Such a shortest numerical expression is characterized by the property that there is no application of equations (5) to the expression which will shorten it. We will call this the normal form of the class of equal numerical expressions and refer the reader to [4], [5] for a full discussion of these ideas.

The following examples illustrate the rules of computation in I and some specific computations.

$$
(6)\qquad
\begin{array}{l}
\left.\begin{array}{l}
\text{(i)}\ a\cdot 1 = 1\cdot a = a, \\
\text{(ii)}\ a\cdot(m + n) = a\cdot m + a\cdot n, \\
\text{(iii)}\ a\cdot(m - n) = a\cdot m - a\cdot n, \\
\text{(iv)}\ a\cdot(-m + n) = - a\cdot m + a\cdot n,
\end{array}\right\}\ \text{by the definition of multiplication,} \\
\text{(v)}\ a\cdot 0 = a(1 - 1) = a - a = 0, \\
\text{(vi)}\ 0\cdot a = 0, \text{ by induction on the length of } a, \\
\text{(vii)}\ a\cdot(0 - 1) = 0 - a, \qquad a\cdot(-1 + 0) = - a + 0,
\end{array}
$$

$$\text{(viii)}\ (0-1)\cdot(-1+0) = -(0-1)+(0-1)\cdot 0 = -(0-1)+0 = 1,$$

$$\begin{aligned}\text{(ix)}\ (1-2)\cdot((0-1)+2) &= (1-2)\cdot(0-1)+(1-2)\cdot(1+1)\\ &= (0-(1-2))+((1-2)+(1-2)).\end{aligned}$$

The discussion of the number theory of I is complicated by the existence of units. As usual we define a unit to be an element possessing a multiplicative inverse. In the ring of integers of ordinary arithmetic there are only two units but the left neoring I contains an infinite number.

We will call the elements $0-1$, $0-(0-1)$, $0-(0-(0-1))$, $\cdots$ the first, second, third, $\cdots$ right negatives of 1 and similarly, $-1+0$, $-(-1+0)+0$, $-(-(-1+0)+0$, $\cdots$ the first, second, third, $\cdots$ left negatives of 1. It is easily verified from equations (5) and (6) that the product of the nth left negative and nth right negative is 1.

It is not quite so easy to show that these are the only units in I. We recall that the product of two elements $u(1)$, $v(1)$ of I is defined by $u(1)\cdot v(1)=v(u(1))$. Hence we have to show that the left- and right-negatives of 1 are the only elements of I which satisfy $v(u(1))=1$. This is an immediate consequence of Lemma 2 in [5].

Since $(0-1)^2=0-(0-1)$, $(0-1)^3=0-(0-(0-1))$, $\cdots$ and $(-1+0)^2=-(-1+0)+0$, $(-1+0)^3=-(-(-1+0)+0)+0$, $\cdots$, the units of I are exactly the powers of $0-1$.

We collect these results as a theorem.

THEOREM 10. *The multiplicative group of units of I is the infinite cyclic group generated by* $0-1$.

As before b will be called a *left-factor* of a if there exists an element c of I such that $b\cdot c=a$. If neither b nor c is a unit and $a\neq 0$, we say that b is a *proper left-factor* of a. In the same way we define *right-factor* and *proper right-factor*. We note that any number is a factor of 0. Two numbers a, b in I will be called *associates* if $xay=b$ where x, y are units.

LEMMA 1. *If a, b are left- (right-) factors of each other, then they are associates.*

Proof. If $ax=b$, $by=a$, then $axy=a$ or $xy=1$. Hence x, y are units. The proof for right-factors is similar.

If a is a left- (right-) factor of b and b is a right- (left-) factor of a, then $a=b$ unless both a and b are units. A proof of this leans heavily on the results of [4], [5], and so we omit it.

In ordinary arithmetic, the primes in the ring of integers are simply the original primes in the set of natural numbers multiplied by the units. This situation does not carry over to nonassociative arithmetic. In fact, a rather complicated situation exists in I. We define, in the usual way, a *prime number* of I to be a number without proper factors. Then all the primes of N are primes in I.

We also have primes such as $0-(1+1)$ consisting of the product of the prime $(1+1)$ in N and the unit $0-1$. But other primes such as $1-(1+1)$ exist in I, not the product of a unit and a prime of N. In addition, there is a special subclass of the primes of I with the property that no prime in this subclass can be written as a product of two numbers of shorter length. For want of a better name, we will call these *special primes*. The number $(0-(1+1))$ is a prime but it is not a special prime since $0-(1+1)=(1+1)\cdot(0-1)$. However, $(1+1)$ is a special prime and, more generally, all the primes of N are special primes in I. Examples of other special primes are $(1-2)$, $(1-3)$, $(1-4)$, $\cdots$.

We now state some theorems, giving only brief outlines of the proofs, which are basic in the further development of the number theory of I.

THEOREM 11. *Let a be an element of I represented by a numerical expression in normal form so that a has one of the forms $m+n$, $m-n$, $-m+n$ where m, n are numerical expressions. Then any proper left-factor of a is a left-factor of m and n.*

Proof. This corresponds to Theorem 6 for N. The proof proceeds by ordinary induction on the length of a, coupled with the fact that the representation of a number as a numerical expression in normal form is unique.

THEOREM 12. *If the prime p is a left-factor of the product ab, where $a\neq 1$, $b\neq 0$, then p is a left-factor of a.*

Proof. By induction on the length of b, and by the previous theorem.

THEOREM 13. *If a number a can be written as a product of primes in two ways, say, $a=p_{|1|}\cdots p_{|s|}$ and $a=q_{|1|}\cdots q_{|t|}$, then $|s|=|t|$ and $p_{|i|}$, $q_{|i|}$ are associates $(|i|=|1|,\cdots,|s|)$.*

Proof. By Lemma 1, Theorem 12, and the left-cancellation law for I.

We conclude our discussion of I by introducing the concept of congruence in it. In ordinary arithmetic, a homomorphic image of the ring of integers is obtained by adding the relation $|m|=|0|$ to the ring. Then two integers are congruent mod $|m|$ if they map onto the same element under this homomorphism. It is shown in [6] that if $m(=u(1))$ is an element of I, then adding the relation $u(1)=0$ to I determines a left neoring which is a homomorphic image of I. We define two numbers in I to be congruent mod m if they map onto the same element under this homomorphism. Alternatively, we can define two numbers in I to be congruent mod m if their difference lies in the fully invariant normal subloop, generated by m, of the additive loop of I. The relation between these two points of view is discussed briefly in [6] and can be studied in detail using the techniques of [4], [5]. The homomorphic images of I described above are the nonassociative analogues of finite arithmetics.

With the above definition of congruence in I, some of the elementary properties of congruence in ordinary arithmetic carry over without difficulty (see, *e.g.*, Chapter 1 in [8]). The author does not know whether the same is true of

some of the deeper theorems involving congruence.

4. Further developments. The ideas introduced in this paper can be developed in several directions. There are many problems for the arithmetic N, *e.g.*, obtaining an analogue of the prime number theorem. This seems quite feasible since estimates of the number of nonassociative natural numbers of given length are available.

In some of our proofs of properties of N we used properties of ordinary arithmetic including induction. Can this be avoided completely and all properties of N obtained from the postulates for N given in Section 3? One way to do this is to develop ordinary arithmetic within N. Define nonassociative powers of numbers in N by $a^1=a$, $a^{m+n}=a^m\cdot a^n$ where $m, n\in N$. An equivalence relation, $\equiv$, between numbers in N can be defined by $m\equiv n$ if $a^m=a^n$ for all $a\in N$. We now show that the set of these equivalence classes satisfies Peano's postulates for the ordinary natural numbers. This is a consequence of the Peano-like postulates which N satisfies. The length of a number n in N is defined as the equivalence class containing n. In this way all of ordinary arithmetic and in particular those parts which we have used in discussing N can be developed inside N. It follows that if we set up N as a formal system, there will be a Gödel incompleteness theorem for the system. We leave to the interested reader the detailed carrying out of the above ideas. A related topic which may be interesting is the theory of recursive functions of nonassociative natural numbers.

There are many concepts involving congruence in ordinary arithmetic which should have interesting analogues in the arithmetic I. In particular we can ask such questions as the following. For what congruences does the quotient arithmetic (i) satisfy the cancellation law, (ii) allow division, (iii) allow unique division, (iv) satisfy the commutative laws of addition and multiplication? Other problems are (i) what is the structure of the multiplicative semigroup of I, (ii) can I be embedded in a system with division?

If we add to N the identical relation $(a+b)+(c+d)=(a+c)+(b+d)$ we get an arithmetic S with many interesting properties. In this system, which is the free symmetric groupoid generated by 1 (see [7]), we define multiplication as usual by $u(1)\cdot v(1)=v(u(1))$. Then multiplication is commutative and so both distributive laws hold. This arithmetic is extremely close to ordinary arithmetic, differing only in the replacing of the associative and commutative laws by the single law $(a+b)+(c+d)=(a+c)+(b+d)$. A study of the number theory of S should lead to some interesting problems. Another problem which presents itself is the obtaining of a set of Peano-like postulates which characterize S.

References

1. Grace Bates, Free loops and nets and their generalizations, Amer. J. Math., vol. 69, 1947, pp. 499–550.

2. R. H. Bruck, Analogues of the ring of rational integers, Proc. Amer. Math. Soc., vol. 6, 1955, pp. 50–57.

3. I. M. H. Etherington, Non-associative arithmetics, Proc. Roy. Soc. Edinburgh, vol. 62, 1949, pp. 442–453.

4. Trevor Evans, On multiplicative systems defined by generators and relations, I. Normal form theorems, Proc. Cambridge Philos. Soc., vol. 47, 1951, pp. 637–649.

5. ———, On multiplicative systems defined by generators and relations, II. Monogenic loops, Proc. Cambridge Philos. Soc., vol. 49, 1953, pp. 579–589.

6. ———, Some remarks on a paper by R. H. Bruck, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 211–220.

7. Orrin Frink, Symmetric and self-distributive systems, this MONTHLY, vol. 62, 1955, pp. 697–707.

8. C. C. MacDuffee, An Introduction to Abstract Algebra, New York, 1940.

MAINTAINING COMMUNICATION*

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Some twenty years ago a professor of philosophy spoke to the mathematics club at the University of Virginia. The speech was followed by a warm discussion of the paradoxes of Zeno. Some regarded the mathematical explanation of the paradoxes as completely adequate, others disagreed, and needless to say, each disputant emerged triumphantly bearing the opinion he had carried in. But one remark of a professor of physics has stayed with me ever since. He said that he could easily conceive that someone could arrive in his own mind at a perfect solution of the paradoxes and still be unable to convey the explanation to anyone else.

Let us at least temporarily suspend disbelief in this philosopher with the incommunicable thoughts, and not boggle over reasons for acceding to him a belief that we deny to Fermat and his celebrated proof that was too long for the book's margin. There remains the fact that to the body of philosophy he remains exactly as useless as though he had never existed. Perhaps he has derived intense personal satisfaction from his brilliant reasoning, but the rest of the world may as well ignore him.

In recent years I have been troubled by a suspicion that this image of the uncommunicative philosopher may be a parable of an approaching state of mathematics. Fortunately, rather than a parable it is an overdrawn caricature, but as in all caricatures some features are recognizable. For in mathematics, as in the sciences, the communication of ideas becomes steadily more difficult. This is a matter that concerns all of us, and each of us should try to help in keeping the lines of communication open.

To begin with, there are the mechanical and financial difficulties involved

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in the publication of a flow of mathematical research whose volume grows steadily. I do not intend to say much about this. Competent people are trying hard to find a solution, and it is not entirely clear that a solution exists. As someone said to me earlier this month, we have caught a geometric progression by the tail. Nevertheless, as matters now stand the author of a reasonably good paper can expect to have it published with reasonable speed. I propose to talk about the author and the paper, rather than about its publication.

I wonder if any one here has already silently objected that I seem to be speaking of research, and not of teaching, which is a more proper activity of the Association. If so, his objection is itself an example of a hindrance to communication. The normal activities of a mathematician should include a mixture of three ingredients. One is teaching; another is research; and the third, which as Professor Duren says is all too often left unmentioned, is scholarship. Any one of these three may be omitted, but in my opinion not without loss. No one can confine himself to any single one of the three without injury to himself, and perhaps to others too. When this Association was formed in 1915 it was for the satisfaction of a need. To put it bluntly, the American Mathematical Society had refused to consider anything but mathematical research as being in its sphere of activity. Nevertheless, I consider it regrettable that the need existed, and that the activities of teaching and research are thus separated. One of the visible results of the separation is seen in the content of mathematics courses in colleges, and even more in high school courses. The tremendous changes in matter and method of research have produced hardly a ripple at the freshman level, and less in the high schools.

Let me enlarge a bit on this topic. During our lives mathematicians have transferred their chief interest to new topics, and they study these topics with changed thought-patterns. Abstraction and generalization have increased, not (at least not always) merely for the sake of generality itself, but because the abstract approach exposes the essential ideas and clears away what is merely fortuitous, and the generality yields new fields of application of the ideas. Set-theoretic concepts have pervaded mathematics, both pure and applied. The basic ideas of topology are the common language of mathematicians in all fields. Linear processes and matrix algebra are now matters of great interest to applied mathematicians, and so on, through a list which is quite long if we include more specific items. Yet there are modern books on "mathematics for engineers" which explain, as did my grandfather's calculus text, that there are two kinds of zero, the "nothing-at-all" kind obtained by subtracting a number from itself, and another mysterious kind, smaller than any positive number but yet not reconciled to being nothing at all, that we meet when we "evaluate the indeterminate expression 0/0." Most mysteriously, this nonsense is written in a presumably sincere effort to help the reader understand the subject. Fortunately, there now exists a respectable minority of calculus texts in which definitions and proofs are carefully and correctly stated. At a lower level, the advent first of the electric computing machine and then of the high-speed electronic computer has

diminished the importance of some older computational devices. Horner's method can profitably be forgotten; yet in some good high schools several weeks are allotted to it. Logarithms are not nearly as important as they once were, and the various "time-saving" devices for solving triangles by formulas adapted to computation with logarithms are hardly worth the time they cost. Yet, if I hear a chorus of mathematicians' voices (and I hear my own voice in the chorus) asking "Why do these people spend irreplaceable time on the less valuable subjects and ignore the better ones?," I also hear voices (including the voice of my conscience) asking, "Did you bother to tell them?."

Here, at least, I can add a more cheerful note. The Association's Committee on the Undergraduate Program is composed of mathematicians who are active participants in research and are also interested in undergraduate teaching, and its aim is to provide the means of changing the present undergraduate teaching by making some of the more recent gains in mathematics available to college students. Also, there are other committees and organizations working on related problems, so we may reasonably hope for improvement.

Communication between the teacher and the scholar of mathematics is so important that I can imagine only one satisfactory arrangement: the two must wear the same skin. The man who is neither researcher nor scholar, and nevertheless is listed on some payroll as teacher of mathematics, is probably falsely listed. At best he is a transmitter of information given him in the more or less remote past; at worst he is drill-master for the problems in some third-rate textbook. In either case, he hasn't bothered to come to this meeting. The man who is a research specialist and teacher, without being a scholar, is easy enough to find. Sometimes he is a teacher only by necessity, giving all his enthusiasm to his research. Sometimes he is an enthusiastic teacher of advanced students, spreading forth the pleasures of specialization in his own field. In this case he may produce students with great enthusiasm but narrow views, whose broader education must be left to other more scholarly teachers or the hands of the gods. In either case, he gives weight to the definition in *Webster's Collegiate Dictionary*:

> **doctor** [O.F. *doctour*, fr. L. *doctor* teacher, fr. *docere* to teach] **1.** *Archaic.* A teacher; a learned man.

Quite another matter is the communication of thought between the research worker on the one hand and the teacher-scholar on the other. Now we are in a zone where the individual mathematicians can begin to feel their own share of guilt in a situation that is far from perfect, although a prevailing climate of thought is also largely to blame. Here is the domain of usefulness of the expository article. The advanced treatise or monograph may be serviceable as a means of showing all that has been accomplished in some field, but for the man who is principally a teacher or principally a researcher in some other field, the sheer bulk of the monograph may be discouraging. Besides, these are properly the fruits of long labor, and their writing is not lightly to be undertaken. But for expository writing on a smaller scale there is considerably more demand than

supply. In order to have at least one number to quote, to substantiate my claim that the supply is scant, I thumbed through the first hundred pages of the latest number of *Mathematical Reviews*, searching for papers dismissed with the comment "Expository paper." I found seven of them, that is roughly one expository article to a hundred research articles. But more careful investigation shows that two of the seven were reports of conferences, hence hardly to be considered expository in the present sense. Of the remaining five, four were in Russian. Are we then to dismiss the scholar, and the research worker in some other field, and the physicist or chemist or interested layman, with the advice "Learn Russian and subscribe to the *Uspehi*"?

Certainly this trouble has been recognized before, and various efforts made to cure it. For example, the Carus Monographs and the Slaught Papers are sponsored by this Association, and some of the "What is . . . ?" series in the MONTHLY were excellent. But it has always been difficult to keep up the supply.

A chief reason for this scantiness is easy to find. Every normal human being wants recognition for his work. Even the hypothetical philosopher with his incommunicable resolution of Zeno's paradoxes would probably like to hear a word of praise from some believing soul. Since a research paper, even on the narrowest and most special of topics, is ordinarily looked on with more reverence than even an excellent expository paper, it is natural and human that a mathematician should be inclined to spend all his available working time on research. This is particularly true of the younger mathematicians. With them, recognition and promotion may well depend more on published research than on scholarship or teaching. Besides this, the writing of a good expository article calls for breadth of view and historical perspective, which are hardly the correlates of youth.

But these reasons apply much less cogently to the mature mathematician, who has reached a secure position and a level of esteem not likely to be greatly raised or lowered. By "mature" I do not mean superannuated. I am not recommending the writing of expository papers as a sort of pastime for gentlemen (young, old or middle-aged) who have determined by careful self-examination that they haven't a research paper left in their systems. A man of thirty may have attained position and recognition and broad knowledge; a man past seventy may be active in research, as the current volume of the *Proceedings of the National Academy of Sciences* will show.

The fund of scholarship that supports the writing of a good expository paper comes from having read, marked, learned, and inwardly digested many articles in some field. Usually this reading was prompted by an interest in some research problem, but this is not a necessary condition. However, the mere intensive reading of many papers is not a sufficient condition either. One can be so imbued with interest in one special problem that everything is mapped on a sort of polar coordinate system, with the one special problem at the origin and interest inversely proportional to r^2. Alternatively, one can read something new to find what it is in itself and how it relates to previous knowledge. Such reading, with

thoughtful rumination, is a natural source of scholarship, of good teaching, and in particular of good expository writing.

Leonard Eugene Dickson used to say (I have forgotten his exact words) that every mathematician owed a debt to mathematics that he should repay by one hard job of scholarly writing. His was the huge *History of the Theory of Numbers*. Not many of us could consider such a vast undertaking. But each of us owes the debt, and should not repudiate it if he is mathematically solvent.

The purpose of an expository mathematical paper is, of course, to convey information about some domain of mathematics to some audience, which might consist of specialists in a slightly different field who wish to add to their research capacities, or of mathematicians wishing to broaden their knowledge, or of teachers at any level from high school to graduate school. Obviously, the best written exposition fails if no one reads it. If any man wishes to consider himself a teacher and scholar of mathematics, it is his clear duty, and it should be his pleasant duty, to add continually to his knowledge, and in this he should be greatly helped by expository articles of the type appropriate for him. The teacher who neglects this is doing himself a grave wrong. What is worse is that he is doing an even graver wrong to his students and to his subject. I have heard of a professor (not a mathematician) who remarked that he had been lecturing twenty years from the same notes, and that the only change that he expected after another twenty years was that the pages would be yellower. I cannot conceive of any field in which this attitude would be harmless. But certainly any teacher of mathematics who would thus decide that mathematics belongs in the Valley of Dry Bones must inevitably convey the same impression to his students. The best of books is inferior to a human being as a means of conveying enthusiasm for and pleasure in a field of study, and if the teacher fails to show that the subject is alive and moving and fascinating, he fails in just that respect in which the responsibility is most peculiarly his own.

One form of communication which up to the Second World War had been in rather bad shape, but fortunately is now improving, is the contact between mathematicians and other scientists. Applied mathematics was not long ago regarded as the tedious solution of specific problems by known devices, and often without adequate logical justification. Even today the mathematical reasoning in the quantum theory of fields or in nuclear physics is apt to shock a mathematician trained in rigor. There are two ways of reacting to such a situation. One is to look on the physical theory as ludicrous and refuse to sully one's hands with it. The other is to observe that the illogical theory has yielded useful results, and is thus probably a sort of parody of a rigorous and coherent theory. This implies a challenge to find that rigorous theory. It is in such situations as this that the applications have benefited mathematics, by calling for new devices and new combinations of old devices to handle a problem which has presented itself not artificially but irrepressibly and clothed with its own importance.

Mathematics has retained its place among the chief subjects of education for

well over two thousand years, but this does not mean that it is surely immortal. It has stood firmly because it has stood on two legs. First, it is supported by its innate beauty and austere elegance. Second, it is supported by its usefulness to scientists and technicians of all kinds. If we try to make it stand exclusively on its usefulness, it becomes a mere tool for the use of non-mathematicians, and degenerates into dullness and eventually into uselessness. If we try to make it stand exclusively on its esthetic virtues, we not only make it useless to other sciences but reject the stimulus that it can receive from them. So it is desirable that among the research workers in mathematics there should always be some who are interested in its applications. Likewise, the scholars and the teachers should not ignore the many uses of mathematics. Right now this places quite a demand on the scholars and teachers, for mathematics has entered new fields, sometimes in rather unexpected ways. As examples, I cite genetics and the theory of games.

So far, all the aspects of communication that I have discussed have borne the typical earmarks of a Mathematical Association activity—they have all called for much individual enthusiasm and activity and very little cash. I now wish to say something about another activity which now involves millions of dollars, but at its outset had the typical Association earmarks. I am referring to the Institutes for teachers. A while ago I made the obvious remark that enthusiasm for a subject is easiest conveyed by personal contact with an enthusiastic teacher. It is reasonable to assume that a teacher of mathematics brings some store of enthusiasm to his teaching. But we ask a great deal if we expect that teacher to keep up his enthusiasm and increase his scholarship by reading and study if he himself has no contact with some other enthusiast. Clearly it is desirable to rekindle the enthusiasm and increase the learning of teachers by giving them the chance to study occasionally with leading mathematicians. This is the motive of the Institutes for teachers. These have sprung up in various forms; in time-scale, they varied from short conferences up to the one developed at Notre Dame, which is devised especially for teachers and has a program based on five summers' attendance and leading to the degree of M.S. The one about which I now want to speak took place in the summer of 1953, at the University of Colorado. The enthusiasm and personal dedication were certainly there; Burton W. Jones did two men's work. Other mathematicians were vigorous in their cooperation, and the University of Colorado supported the project well. The supply of funds was none too ample, but the personal contributions made the Institute a notable success. Its chief disadvantage, of course, was that it alone could not reach a vast number of teachers.

Only three years after the Colorado Institute, Congress, alarmed by our shortage of scientists and technicians, made an appropriation of millions of dollars for improvement in teaching of mathematics and science. The National Science Foundation had to find wise ways of using this money for the given purpose. It must have been quite a help to them (I speak from no expert knowledge)

that they did not have either to spend the time needed for a trial run or to take the risks of putting the money into new and untried devices. Summer Institutes had already been tried out and found successful, and the experience was there for guidance. Now from small beginnings we have progressed to a large enterprise. The scale is utterly beyond the Association's financial resources, but the personal contribution is still a vital need. Some of us may be needed as teachers in the institutes; others may wish to attend them for refreshing. I hope that each of us will do whatever he can to help with the project. It would be a serious error to think that the large scale of operation has diminished the importance of the individual contributor. Quite the opposite is true. The success of an Institute depends above all on the presence of workers who have both learning and enthusiasm, and the increase in the number of institutes calls for the cooperation of every one who can make a contribution.

We have been hearing a great deal recently about the proliferation of mathematical research. This certainly is no news to anyone who has noticed the steady growth in the size of *Mathematical Reviews*. About sixteen years ago I bought a house which had been unoccupied for five years. Summer after summer we fought a losing battle against honeysuckle; it grew faster than we could dig it out. Then came the discovery of the weed-killing properties of 2-4-D, and the battle was won. As I understand it, a broad-leaved plant sprayed with 2-4-D does not die at once; it begins to proliferate rapidly, growing quickly and without organization. As a result, it dies. I cannot look on the proliferation of mathematics as being in all circumstances an unqualified good. Each mathematical discipline needs to draw on the others; yet it is impossible for even the best of us to keep abreast of the research in more than a small part of the field. We are separating off into small groups of specialists with little intercommunication.

What the solution is, or whether there is a solution, I do not pretend to know. But I can propose a palliative. We can pay more attention to the quality of our writing. Within a generation there has been a regrettable decline in the style and clarity of mathematical composition, a decline visible in all countries. Even the French have slipped sadly from their previous excellence. This decline has made it harder for us to read articles in any field in which we are not specialists. A published paper is not properly an open letter to a half-dozen fellow-specialists who understand the motivation, possess the antecedent information and can readily fill in omitted definitions and proofs. An addition to mathematical knowledge has significance as a part of the body of all mathematical knowledge, and its place in that body should be clearly indicated. Unless the reader is one of the tiny group of those who have read everything in that specific subject, he may be quite capable of following the details of proofs and yet unable to realize why the author was led to study the problem, or how it complements past knowledge or points toward future advances. If the author has not bothered to establish the setting of his paper and convey some of the motive that impelled him, he has driven off potential readers and made everything harder for

those who stay with him. I am convinced that it is the duty of editors to demand, not to forbid, the writing of introductory paragraphs to provide motivation and background. This need not be long; even a little can be disproportionately helpful.

Within the body of the paper, it is often distressing that so little care is expended on presentation. Often the author has clearly worked hard on the mathematical content, removing all superfluous hypothesis, obtaining as much conclusion as possible without irrelevant steps. But after he has burnished the mathematics to a high polish, he has written it down hurriedly, with little thought for style. Long sentences drag their gangling dependent clauses across the page. Pronouns look back helplessly into a welter of nouns in the hope of finding an antecedent. Even worse for the uninitiated reader, there are expressions that proclaim the hopelessness of the search. Many papers bristle with "The expression . . . is defined in the obvious way," and "Clearly, . . . ," and "By the usual proof, . . . ," and "Mapping this onto A in the natural way," and "By a well-known theorem, ..." These are frequently not even space-savers; the uninformative words can be replaced by a precise statement, and the "well-known" theorem precisely located, with little if any cost in length and with great benefit to the intelligent but imperfectly informed reader.

Moving still further in this direction, we meet a mental attitude that regards communication as vulgar. There are mathematicians (fortunately few) who consider that a speaker has somehow "lost face" if he has spoken so as to be intelligible to any but the select few. One hears of "folk theorems," established (presumably) by some expert, communicated verbally or more likely mentioned in an off-hand way during some conversation with another expert or two, and thereafter unpublishable forevermore because no one would want to publish a "known theorem." This is not mere uncommunicativeness; it is active opposition to communication. I recently saw it nicely summarized in a sentence in one of the reviews in a recent number of *Mathematical Reviews*; I am quoting from memory because I have forgotten the name of the reviewer and prefer to forget just who said this: "This theorem seems not to have been published previously, but is probably known to some of the workers in this field."

I would much prefer to see such mathematicians converted to fellow-workers. However, if they wish to form a Society for Mutual Admiration, or a sort of Egyptian priesthood guarding their secret knowledge from profanation by the vulgar, we can only let them go their way. But it would be surely wrong to abet them in their self-satisfaction. If they choose to suppress their knowledge, they cannot ask for recognition because they possess it. No man deserves serious credit for having established a theorem until he has put the proof in plain sight, for the open criticism of other mathematicians. I know of at least two instances in which a report was widely circulated that Mr. X had a proof of this or that, and later it turned out that what he had was seriously defective. This is quite excusable; men do make mistakes. But it would have been inexcusable to ask for the acclaim of mathematicians on the strength of a "proof" with an

error in it; and it is likewise wrong for us to give acclaim on the strength of a proof that has not been openly presented for criticism and found sound. I feel that no mathematician should hesitate to publish a result because someone tells him that he thinks that Z did something like that two years ago but didn't write it up; nor should he falsify history by giving credit to Z for priority when Z has not established his right to the credit by letting us see his proof.

Everyone of us is touched in some way or other by the problems of mathematical communication. Every one of us can make some contribution, great or small, within his own proper sphere of activity. And every contribution is needed if mathematics is to grow healthily and usefully and beautifully.

A CURIOUS SEQUENCE OF SIGNS

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In this paper we give an algebraic identity which gives rise to a sequence of number theoretic functions, $u_b(n)$ with b a positive integer ≥ 2. For each such b, $u_b(n)$ appears in a rather interesting polynomial identity. Considering two special cases of this polynomial identity we find a new binomial identity and another identity which has application to the Tarry-Escott problem in number theory.

The function $u_2(n)$ also gives rise to a set of functions on $[0, 1]$ which constitute a lacunary subsequence of the Walsh functions and which have some properties similar to the Rademacher functions.

1. The functions $u_b(n)$. Let $\bar{G}$ be a commutative ring with unity such that there exists a mapping f of J^+, the collection of nonnegative integers, into a subset G of $\bar{G}$ satisfying $f(a+b)=f(a)f(b)$. Then when $b\geq 2$ we find that

$$\prod_{n=1}^{k} (1 - f(b^{n-1}))^{b-1} = \sum_{n=1}^{b^k} u_b(n)f(n-1), \tag{1}$$

where $u_b(n)$ is ± 1 times a product of binomial coefficients. In order to give an explicit expression for $u_b(n)$ we define two other functions of n.

$b_j(n)$ is the coefficient of b^j in the expansion of n to the base b,

$$v_b(n) = \sum_{j=0}^{\infty} b_j(n). \tag{2}$$

Using (2) we have

$$u_b(n) = (-1)^{v_b(n-1)} \prod_{j=0}^{\infty} \binom{b-1}{b_j(n-1)}. \tag{3}$$

Although $u_b(n)$ appears in (1) only for $1 \leq n \leq b^k$, the relation (3) defines $u_b(n)$ for all $n \geq 1$. Taking $1 \leq n \leq b^k$, $m \geq 0$ we find that $v_b(n-1+mb^k) = v_b(n-1)+v_b(m)$ and

$$\prod_{j=0}^{\infty}\binom{b-1}{b_j(n-1+mb^k)} = \prod_{j=0}^{\infty}\binom{b-1}{b_j(n-1)}\prod_{j=0}^{\infty}\binom{b-1}{b_j(m)}.$$

Hence from (3) we conclude that

$$u_b(n+mb^k) = u_b(n)u_b(m+1) \text{ for } 1 \leq n \leq b^k, \qquad m \geq 0. \tag{4}$$

Formula (4) enables us to calculate $u_b(n)$ inductively from $u_b(n)$ for $1 \leq n \leq b$ and these are readily determined from (3). We discuss the calculation of the $u_b(n)$ again in Section 3.

Taking $b=2$ we find $u_2(n) = (-1)^{v_2(n-1)}$ and therefore $u_2(n)$ is 1 or -1 according to whether the digit 1 appears an even or an odd number of times in the dyadic expansion of $n-1$. Writing $+$ for 1 and $-$ for -1 the sequence of $u_2(n)$ starts as follows:

(5) $+--+-++--++-+--+-++-+--++--+-++-$.

Note that the first 2^k digits are the negatives of the next following 2^k digits for all $k \geq 0$.

2. Translation operators and a polynomial identity. Let $D(a_1, \cdots, a_k; b_1, \cdots, b_k)$, a_i and b_i integers, be an operator which maps the polynomial $P(x)$ into the polynomial $a_1P(x+b_1)+ \cdots +a_kP(x+b_k)$. The totality of such operators is a commutative ring with unity. Letting this ring be $\overline{G}$ and taking G to be the subset of "translation operators" $E(b) = D(1; b)$, b a nonnegative integer, we find that the mapping f: $b \leftrightarrow E(b)$ from J^+ to G has the property $f(a+b) = f(a)f(b)$. Hence (1) of Section 1 becomes

$$\prod_{n=1}^{k}(1 - E(b^{n-1}))^{b-1} = \sum_{n=1}^{b^k} u_b(n)E(n-1). \tag{6}$$

Noting that $1-E(a)$ maps constants onto zero and reduces the degree of all other polynomials by one we see that the operator in (6) will map all polynomials of degree less than $k(b-1)$ onto zero; *i.e.* $\sum_{n=1}^{b^k} u_b(n)E(n-1)P(x) = 0$ for $P(x)$ a polynomial of degree less than $k(b-1)$. This then yields the identity

$$\sum_{n=1}^{b^k} u_b(n)P(x+n-1) = 0 \text{ for } P(x) \text{ of degree less than } k(b-1). \tag{7}$$

Formula (7), with $x = q/m$, yields the binomial identity

$$\sum_{n=1}^{b^k} u_b(n)\binom{j+q+m(n-1)}{q} = 0, \qquad 0 \leq q < k(b-1), \qquad j \geq 0, \tag{8}$$

when $P(x)$ is taken to be the polynomial $(j+mx)\cdots(j+mx-q+1)/q!$ of degree q. This identity (8) is especially nice when $b=2$.

In terms of the Pascal triangle, identity (8), with $b=2$, says that if we start choosing numbers spaced m rows apart ($m\geqq 0$) from the $q+1$st column, beginning anywhere ($j\geqq 0$), and take 2^k of them, where $k>q$, then if we append the first 2^k signs of (5) to the numbers obtained the resulting numbers sum to zero. For example, if $j=q=2$, $k=3$, $m=1$, we have $3-6-10+15-21+28+36-45=0$.

Taking $P(x)=(x+1)^m$, $0\leq m<k(b-1)$, in (7) gives

$$\sum_{n=1}^{bk} u_b(n)(x+n)^m = 0. \tag{9}$$

From (9) we obtain the following two identities by letting x be 0 and q/p, respectively.

$$\sum_{n=1}^{bk} u_b(n)n^m = 0 \text{ for } 0 \leq m < k(b-1). \tag{10}$$

$$\sum_{n=1}^{bk} u_b(n)(q+pn)^m = 0 \text{ for } 0 \leq m < k(b-1),\ p \text{ and } q \text{ arbitrary real numbers.} \tag{11}$$

3. A modified Pascal triangle. In this section we give a method for the rapid computation of the $u_b(n)$ and at the same time show in a more striking way the connection between the $u_b(n)$ and the binomial coefficients.

When $1\leq n\leq b$ equation (3) becomes

$$u_b(n) = (-1)^{n-1}\binom{b-1}{n-1}.$$

Therefore we can obtain the values of $u_b(n)$, $1\leq n\leq b$, by reading from left to right in the bth row of the following modified Pascal triangle.

1						
1	−1					
1	−2	1				
1	−3	3	−1			
1	−4	6	−4	1		
1	−5	10	−10	5	−1	
1	−6	15	−20	15	−6	1
.	.	.	.	.	.	.

Now if we take $k=1$ in (4) we obtain $u_b(n+mb)=u_b(n)u_b(m+1)$ for $1\leq n\leq b$, $m\geq 0$. Successive applications of this formula enable us to extend to the right the rows of the above modified Pascal triangle thereby obtaining the $u_b(n)$ for $n>b$. Thus at the first step we multiply each of the first b elements of the bth row ($b\geq 2$) by the 2nd element in that row and put these new numbers at the

end of the row one after the other. The completed first step yields

$$
\begin{array}{rrrrrrrrrrrr}
1 \\
1 & -1 & -1 & 1 \\
1 & -2 & 1 & -2 & 4 & -2 \\
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 \\
1 & -4 & 6 & -4 & 1 & -4 & 16 & -24 & 16 & -4 \\
1 & -5 & 10 & -10 & 5 & -1 & -5 & 25 & -50 & 50 & -25 & 5 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

At the second step each of the first b elements of the bth row is multiplied by the 3rd element in that row and the resulting numbers placed one by one at the end of the row. At the next step we multiply by the 4th element in each row, *etc.* The final resulting array is as follows.

$$
\begin{array}{rrrrrrrrrrrrrrrrr}
1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \cdots \\
1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1 & -2 & 4 & -2 & 4 & -8 & 4 & -2 & 4 \cdots \\
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 & 3 & -9 & 9 & -3 & -1 & 3 & -3 & 1 & -3 \cdots \\
1 & -4 & 6 & -4 & 1 & -4 & 16 & -24 & 16 & -4 & 6 & -24 & 36 & -24 & 6 & -4 & 16 \cdots \\
1 & -5 & 10 & -10 & 5 & -1 & -5 & 25 & -50 & 50 & -25 & 5 & 10 & -50 & 100 & -100 & 50 \cdots \\
1 & -6 & 15 & -20 & 15 & -6 & 1 & -6 & 36 & -90 & 120 & -90 & 36 & -6 & 15 & -90 & 225 \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot
\end{array}
$$

Thus this array of numbers contains all $u_b(n)$ for $b \geq 2$, $n \geq 1$.

4. Sums of powers of integers. The extension of the modified Pascal triangle discussed in the preceding section can be used to illuminate some of the results in Section 2. We consider here formula (10).

Since the $u_b(n)$ are just the numbers in the bth row of the final array of numbers given in Section 3 we have the following. Write below the numbers in the bth row of this array the successive mth powers of the integers. In the case $b=4$ we have

$$
\begin{array}{rrrrrrrrrrrrrrrrr}
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 & 3 & -9 & 9 & -3 & -1 & 3 & -3 & 1 & -3 \cdots \\
1^m & 2^m & 3^m & 4^m & 5^m & 6^m & 7^m & 8^m & 9^m & 10^m & 11^m & 12^m & 13^m & 14^m & 15^m & 16^m & 17^m \cdots.
\end{array}
$$

If we now multiply each number in the first row by the power directly below it in the second row and regard these products as summands then the sum of the first b products is 0 when $0 \leq m < b-1$, the sum of the first b^2 products is 0 when $0 \leq m < 2(b-1)$, $\cdots$, the sum of the first b^k products is 0 when $0 \leq m < k(b-1)$. In the example above with $b=4$ we have

$$
1^m - 3 \cdot 2^m + 3 \cdot 3^m - 4^m = 0, \qquad 0 \leq m < 3,
$$

$$
1^m - 3 \cdot 2^m + 3 \cdot 3^m - 4^m - 3 \cdot 5^m + 9 \cdot 6^m - 9 \cdot 7^m + 3 \cdot 8^m + 3 \cdot 9^m - 9 \cdot 10^m
$$
$$
+ 9 \cdot 11^m - 3 \cdot 12^m - 13^m + 3 \cdot 14^m - 3 \cdot 15^m + 16^m = 0, \quad 0 \leq m < 6,
$$
$$
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
$$

In the next section we consider sums of powers of the above kind which arise from (10) in the case $b=2$.

5. Application to the Tarry-Escott problem. The Tarry-Escott problem is concerned with finding sets of integers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that

$$\sum_{i=1}^{n} a_i^t = \sum_{i=1}^{n} b_i^t, \qquad 0 \leq t \leq k. \tag{12}$$

If (12) holds we shall write $a_1, \cdots, a_n \stackrel{k}{=} b_1, \cdots, b_n$.

From (10) in Section 2 we can obtain in the case $b=2$ the result

$$\sum_{n=1}^{2^{k+1}} u_2(n) n^m = 0 \text{ for } 0 \leq m \leq k. \tag{13}$$

Similarly from (11) we obtain

$$\sum_{n=1}^{2^{k+1}} u_2(n)(q + pn)^m = 0 \text{ for } 0 \leq m \leq k. \tag{14}$$

Using (13) we can give a very simple proof of the known proposition:

For all $k \geq 0$ there exist sets of integers a_i and b_i such that $a_1, \cdots, a_n \stackrel{k}{=} b_1, \cdots, b_n$.

We need only take $a_1, \cdots, a_n$ to be those integers between 1 and 2^{k+1} inclusive with $u_2(a_i) = 1$ and $b_1, \cdots, b_n$ to be those with $u_2(b_i) = -1$. Then $a_1, \cdots, a_n \stackrel{k}{=} b_1, \cdots, b_n$ by (13).

The usual way of proving this result is to use the following proposition due to Tarry.

If $a_1, \cdots, a_n \stackrel{k}{=} b_1, \cdots, b_n$ then, for all h,

$$a_1, \cdots, a_n, b_1 + h, \cdots, b_n + h \stackrel{k+1}{=} b_1, \cdots, b_n, a_1 + h, \cdots, a_n + h.$$

The advantage of our proof is that it manages to get at the a_i and b_i directly without having to use a stepwise procedure.

Our solution for k is, however, one possible result obtained by applying Tarry's theorem repeatedly and starting with $1 \stackrel{0}{=} 2$. Using Tarry's theorem with $h = 2^k$ we deduce from the equation $\sum_{n=1}^{2^k} u_2(n) n^m = 0$, valid for $0 \leq m < k$, the equation

$$\sum_{n=1}^{2^k} \left\{ u_2(n) n^m + u_2(n + 2^k)(n + 2^k)^m \right\} = 0,$$

valid for $0 \leq m \leq k$. But the left side of this last equation is

$$\sum_{n=1}^{2^k} u_2(n) n^m + \sum_{n=2^k+1}^{2^{k+1}} u_2(n) n^m = \sum_{n=1}^{2^{k+1}} u_2(n) n^m,$$

and this is just the left side of our solution for k. Hence, starting with our solution for $k=0$ and applying Tarry's theorem k times with $h = 2, 2^2, 2^3, \cdots$, we arrive at $\sum_{n=1}^{2^{k+1}} u_2(n) n^m = 0$.

Another known theorem is the following:

Any set of 2^{k+1} *integers in arithmetic progression can be split into equinumerous classes* $a_1, \cdots, a_{2^k}$ *and* $b_1, \cdots, b_{2^k}$ *such that* $a_1, \cdots, a_{2^k} \overset{k}{=} b_1, \cdots, b_{2^k}$.

We can generalize this theorem and at the same time make the splitting explicit by using (14). We get

Any set of 2^{k+1} *numbers in arithmetic progression can be split into equinumerous classes* $a_1, \cdots, a_{2^k}$ *and* $b_1, \cdots, b_{2^k}$ *such that* $a_1, \cdots, a_{2^k} \overset{k}{=} b_1, \cdots, b_{2^k}$ *and this splitting can be effected by* (14), *taking the* a_i *to be the positive terms and the* b_i *to be the negative terms.*

6. An orthonormal set of functions. Define the functions $u^{(n)}(x)$, $n \geq 0$, on $0 \leq x \leq 1$ by

$$u^{(n)}(x) = \begin{cases} 0 & \text{if } 2^{n+1}x \text{ is an integer,} \\ u_2(1 + [2^{n+1}x]) & \text{otherwise,} \end{cases} \tag{15}$$

where $[y]$ denotes the greatest integer $\leq y$. Defining the Rademacher functions $r_n(x)$, $n \geq 0$, on $0 \leq x \leq 1$ by

$$r_n(x) = \operatorname{sgn} \sin (2^{n+1}\pi x), \tag{16}$$

we can show that

$$u^{(n)}(x) = \prod_{i=0}^{n} r_i(x). \tag{17}$$

Noting that the Walsh functions $\psi_n(x)$, $n \geq 0$, on $0 \leq x \leq 1$ are defined by

$$\begin{aligned} \psi_0(x) &= 1, \\ \psi_N(x) &= \prod_{i=1}^{k} r_{n_i}(x), \qquad N = \sum_{i=1}^{k} 2^{n_i}, \end{aligned} \tag{18}$$

we see that, in virtue of (17), the functions $u^{(n)}(x)$ form a lacunary subsequence of the Walsh functions.

The theorem which states that $\sum_{i=1}^{\infty} a_i r_i(x)$ converges almost everywhere in $0 \leq x \leq 1$ when $\sum_{i=1}^{\infty} a_i^2 < \infty$ remains true when the $r_i(x)$ are replaced by $u^{(i)}(x)$ and with the same proof (6, pp. 126–7).

References

1. L. E. Dickson, History of the Theory of Numbers, vol. 2, ch. 24, New York.
2. ———, Introduction to the Theory of Numbers, Chicago, 1931, pp. 49–58.
3. H. L. Dorwart and O. E. Brown, The Tarry-Escott problem, this MONTHLY, vol. 44, 1937, pp. 613–626.
4. G. H. Hardy and E. M. Wright, Introduction to the Theory of Numbers, 3rd ed., Oxford, pp. 328–332.
5. A. Gloden, Mehrgradige Gleichungen, Groningen, 1944.
6. S. Kaczmarz and H. Steinhaus, Theorie der Orthogonalreihen, New York, 1951.

ON MORERA'S THEOREM

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The Theorem of Morera states that if f is a single-valued, continuous function from a region R of the z-plane to the complex numbers and if

$$\int_C f(z)dz = 0 \tag{1}$$

for every closed, rectifiable curve C in R, then f is holomorphic (that is, f is analytic and regular) in R. It was early recognized that one need not assume (1) for *every* closed rectifiable curve in C but only for some more restrictive class of curves. Osgood [5] gave a proof of Morera's theorem in which he assumed that (1) held on all "small" rectangles with horizontal and vertical edges lying within R. Rademacher [6] showed that the condition of continuity of f in Osgood's theorem may be replaced by Lebesgue integrability over R and linear integrability over horizontal and vertical segments lying in R. He then obtained that f is almost everywhere equal to a holomorphic function in R.

Looman [4, 9] showed that, for continuous f, (1) may be replaced by the condition that at each point $z_0 \in R$,

$$\limsup_{m(Q)\to 0} \frac{1}{m(Q)} \left| \int_Q f(z)dz \right| = \sigma(z_0) < \infty, \tag{2}$$

and $\sigma(z_0)=0$ for almost all $z_0 \in R$, where Q represents a square with center z_0 and horizontal and vertical edges, and $m(Q)$, denotes the area enclosed by Q. Wolff [10] and Ridder [8] relaxed the hypotheses of Looman's theorem somewhat.

In all of these works, the squares or rectangles with horizontal and vertical edges played an essential role. It is well known that harmonic functions are characterized by the mean value property on small circles [1] and we may ask whether holomorphic functions are likewise characterized by the Morera theorem for small circles. The problem is different from that for rectangles since circles do not lend themselves to building up paths between two points. The method we use to prove the Morera theorem for "small" circles is that of smoothing operators or areal means [2, 7]. This application of the method affords an easy way to become acquainted with this important tool that is so often used now in existence proofs.

We first prove the type of theorem we seek under stronger hypotheses than necessary and later weaken these hypotheses.

THEOREM 1. *Let f be a single-valued function from a region R in the z-plane to the complex numbers and let $f\in C^1(R)$ (i.e., f has continuous first partial derivatives, $\partial f/\partial x$ and $\partial f/\partial y$, in R). Assume that*

$$\lim_{r\to 0}\frac{1}{r}\int_0^{2\pi} f(z_0+re^{i\theta})e^{i\theta}d\theta = \lim_{r\to 0}\frac{1}{r^2}\int_{C_r(z_0)} f(z)dz = 0 \tag{3}$$

for each $z_0\in R$, where $C_r(z_0)$ denotes the circle $|z-z_0|=r$. Then f is holomorphic in R.

We observe that if $f=u+iv$, u, v real, then for any circle C in R,

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx).$$

According to Green's theorem, we have

$$\int_C f(z)dz = -\iint_D\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dxdy + i\iint_D\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy, \tag{4}$$

where D represents the region enclosed by C. If F is any continuous function in a neighborhood of z_0, then

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^{\rho}\int_0^{2\pi} F(z_0+re^{i\theta})rd\theta dr = \pi F(z_0).$$

Treating $\partial u/\partial y+\partial v/\partial x$ or $\partial u/\partial x-\partial v/\partial y$ as F, the relations (3) and (4) tell us that the Cauchy-Riemann equations hold for each $z_0\in R$, so that f is holomorphic in R.

COROLLARY 1.1. *If f is a single-valued function in R with $f\in C^1(R)$, and to each $z_0\in R$ corresponds an r_0 such that*

$$\int_{C_r(z_0)} f(z)dz = 0 \tag{5}$$

for $r<r_0$, then f is holomorphic in R.

Indeed, it is clear that (5) implies (3).

COROLLARY 1.2. *If the single-valued function $f\in C^1(R)$ satisfies at each point $z_0\in R$,*

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^{\rho}\int_0^{2\pi} f(z_0+re^{i\theta})e^{i\theta}d\theta dr = 0, \tag{6}$$

then f is holomorphic in R.

To prove Corollary 1.2, we set $G(r)=\int_0^{2\pi}f(z_0+re^{i\theta})e^{i\theta}d\theta$ and proceed to show that

$$\lim_{r\to 0} \frac{1}{r} G(r) = 0, \tag{7}$$

when

$$\lim_{\rho\to 0} \frac{1}{\rho^2} \int_0^\rho G(r)dr = 0. \tag{8}$$

We observe first that $G(0)=0$ and that since $f\in C^1(R)$, $G(r)$ has a continuous derivative G' for $0\leq r\leq r_0$ for some $r_0>0$. In fact, we have

$$G'(r) = \int_0^{2\pi} \frac{\partial f(z_0 + re^{i\theta})}{\partial r} e^{i\theta}d\theta.$$

Then $\lim_{r\to 0} G(r)/r = G'(0)$ while

$$\frac{1}{\rho^2}\int_0^\rho G(r)dr - \frac{1}{2}G'(0) = \frac{1}{\rho^2}\int_0^\rho \int_0^r [G'(s) - G'(0)]dsdr.$$

Since G' is continuous at $r=0$, for a given $\epsilon>0$, there exists an $r_0>0$ such that $|G'(s)-G'(0)|<\epsilon$ when $0\leq s\leq r_0$. Thus if $\rho<r_0$, we have

$$\left|\frac{1}{\rho^2}\int_0^\rho G(r)dr - \frac{1}{2}G'(0)\right| < \frac{1}{2}\epsilon,$$

proving that

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^\rho G(r)dr = \frac{1}{2}G'(0),$$

and (7) follows from (8). This implies that Corollary 1.2 follows from Theorem 1.

We now remove the differentiability hypothesis of Theorem 1 by introducing smoothing operators or areal means of a function. Let us first define

$$s_\rho(x+iy) = \begin{cases} k(\rho^2 - x^2 - y^2)^2 & \text{for } x^2+y^2<\rho^2, \\ 0 & \text{for } x^2+y^2\geq\rho^2, \end{cases} \tag{9}$$

where k is chosen so that

$$\int_0^{2\pi}\int_0^\rho s_\rho(re^{i\theta})rdrd\theta = 2\pi k\int_0^\rho(\rho^2-r^2)^2rdr = 1.$$

We say that a function f is Lebesgue integrable *in* a region R if it is Lebesgue integrable over every compact subset of R. Let R_ρ be the subset of R consisting of those z_0 such that the entire disk $|z-z_0|\leq\rho$ lies in R. Then if f is Lebesgue integrable in R, we define the areal mean of f to be

(10) $$M(f, \rho; z_0) = \int_0^{2\pi} \int_0^{\rho} f(z_0 + re^{i\theta}) s_\rho(re^{i\theta}) r dr d\theta$$

for $z_0 \in R_\rho$. Since $s_\rho(z) = 0$ for $|z| \geq \rho$, we may define $f(z) = 0$ for z in the complement of R and note that (10) may also be written as

(11) $$M(f, \rho; z_0) = \iint_{|\zeta|<\infty} f(\zeta) s_\rho(\zeta - z_0) d\xi d\eta, \qquad \zeta = \xi + i\eta.$$

We next state and prove several properties of the operator M defined in (10).

PROPERTY 1. *If f is Lebesgue integrable in R, then $M(f, \rho; z)$ is in class C^1 in R_ρ (i.e., $M(f, \rho; z)$ has continuous first partial derivatives in R_ρ).*

If we let $z = x + iy$, we may compute the difference quotient of $M(f, \rho; z)$ at the points z and $z + \Delta x$. This is

$$\frac{M(f, \rho; z + \Delta x) - M(f, \rho; z)}{\Delta x} = \iint_{|\zeta|<\infty} f(\zeta) \left[\frac{s_\rho(\zeta - z - \Delta x) - s_\rho(\zeta - z)}{\Delta x} \right] d\xi d\eta.$$

According to the mean value theorem applied to the function $s_\rho(\zeta - z)$, which has continuous partial derivatives in the whole plane, this may be rewritten as

$$\frac{M(f, \rho; z + \Delta x) - M(f, \rho; z)}{\Delta x} = \iint_{|\zeta|<\infty} f(\zeta) \frac{\partial}{\partial x} \{ s_\rho(\zeta - z - \theta \Delta x) \} d\xi d\eta,$$

where $0 < \theta < 1$. The function $\partial s_\rho(z)/\partial x$ vanishes for $|z| > \rho$, and being continuous, it is uniformly bounded in the whole plane. Thus, we may apply the Lebesgue convergence theorem as $\Delta x \to 0$ and obtain

$$\frac{\partial M(f, \rho, z)}{\partial x} = \iint_{|\zeta|<\infty} f(\zeta) \frac{\partial s_\rho(\zeta - z)}{\partial x} d\xi d\eta.$$

A similar expression holds for $\partial M(f, \rho, z)/\partial y$, which proves Property 1.

PROPERTY 2. *If f is holomorphic in R and $z_0 \in R_\rho$, then,*

(12) $$M(f, \rho; z_0) = f(z_0).$$

For any holomorphic function f and any z in $|z - z_0| \leq \rho$, we have $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} e^{in\theta} d\theta = 2\pi f(z_0),$$

for all $r \leq \rho$. From this we conclude that

$$M(f, \rho; z_0) = \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta}) s_\rho(re^{i\theta}) r d\theta dr$$

$$= k\int_0^\rho r(\rho^2 - r^2)^2 dr \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

$$= 2\pi f(z_0)k\int_0^\rho r(\rho^2 - r^2)^2 dr = f(z_0).$$

PROPERTY 3. *If f is continuous in R, then* $\lim_{\rho\to 0} M(f, \rho; z) = f(z)$ *uniformly on any compact subset of R.*

For $n = 1, 2, \cdots$, let $\tilde{R}_n$ denote those points of $|z| \leq n$ which are interior to R and at a distance greater than or equal to $1/n$ from the boundary of R. Then $\tilde{R}_n$ is compact, $\tilde{R}_n \subset \tilde{R}_{n+1}$, and $R = \bigcup_{n=1}^{\infty} \tilde{R}_n$. It suffices to prove Property 3 for the sets $\tilde{R}_n$. For any $z_0 \in \tilde{R}_n$, the disk $|z - z_0| \leq \rho$ lies entirely within $\tilde{R}_{2n}$ whenever $\rho < 1/(2n)$. Then

$$f(z_0) - M(f, \rho; z_0) = \int_0^{2\pi}\int_0^\rho [f(z_0) - f(z_0 + re^{i\theta})]s_\rho(re^{i\theta})rdrd\theta.$$

Since $\tilde{R}_{2n}$ is compact, f is uniformly continuous on $\tilde{R}_{2n}$ and given $\epsilon > 0$, there exists a $\delta > 0$ such that when $|z_1 - z_2| < \delta$, $z_1, z_2 \in \tilde{R}_{2n}$, we have $|f(z_1) - f(z_2)| < \epsilon$. Thus when $\rho < \delta$,

$$|f(z_0) - M(f, \rho; z_0)| \leq \epsilon \int_0^{2\pi}\int_0^\rho s_\rho(re^{i\theta})rdrd\theta = \epsilon,$$

which proves Property 3.

PROPERTY 4. *If f is Lebesgue integrable in R, then*

$$\lim_{\rho\to 0}\iint_D |f(z) - M(f, \rho; z)|\,dxdy = 0$$

for any compact subset D of R.

It suffices to prove this theorem for the $\tilde{R}_n$ defined in the proof of Property 3. The function f may be approximated "in the mean" by continuous functions on $\tilde{R}_n$; *i.e.*, given $\epsilon > 0$, there is a continuous function g on $\tilde{R}_n$ such that

$$\iint_{\tilde{R}_n} |f(z) - g(z)|\,dxdy < \epsilon.$$

Then we have for $\tilde{R}_n$

$$\iint_{\tilde{R}_n} |f(z) - M(f, \rho; z)|\,dxdy \leq \iint_{\tilde{R}_n} |f(z) - g(z)|\,dxdy$$

$$+ \iint_{\tilde{R}_n} |g(z) - M(g, \rho; z)|\,dxdy + \iint_{\tilde{R}_n} |M(g - f, \rho; z)|\,dxdy.$$

The first integral on the right is less than ϵ by the choice of g. The second integral is made less than ϵ by making ρ small enough, say $\rho<\rho_0$ (Property 3). It remains to prove that the third integral is also small.

By Fubini's theorem, we may interchange the orders of integration over $\tilde{R}_n$ and over the disk of radius ρ. Thus

$$\begin{aligned}\iint_{\tilde{R}_n} &| M(g-f), \rho; z) | \, dxdy \\ &\leq \iint_{\tilde{R}_n} \left[\iint_{|\zeta|<\rho} | g(\zeta+z) - f(\zeta+z) | \, s_\rho(\zeta) d\xi d\eta \right] dxdy \\ &= \iint_{|\zeta|<\rho} s_\rho(\zeta) d\xi d\eta \iint_{\tilde{R}_n} | g(z+\zeta) - f(z+\zeta) | \, dxdy \\ &\leq \epsilon \iint_{|\zeta|<\rho} s_\rho(\zeta) d\xi d\eta = \epsilon.\end{aligned}$$

This proves Property 4.

THEOREM 2. *Let f be a single-valued function in a region R such that f is Lebesgue integrable over any compact subset of R. Assume that for all $z \in R$,*

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \int_0^\rho \int_0^{2\pi} f(z + re^{i\theta}) e^{i\theta} d\theta dr = 0. \tag{13}$$

Moreover, assume that for each compact subset $K \subset R$, there exist positive numbers ρ_0 and M such that for any $z \in K$ and $\rho < \rho_0$,

$$\frac{1}{\rho^2} \left| \int_0^\rho \int_0^{2\pi} f(z + re^{i\theta}) e^{i\theta} d\theta dr \right| < M. \tag{14}$$

Then f is almost everywhere equal to a holomorphic function in R.

Let us introduce the notation

$$A(f, \rho; z) = \int_0^\rho \int_0^{2\pi} e^{i\theta} f(z + re^{i\theta}) d\theta dr. \tag{15}$$

We first show that the operators A and M commute; *i.e.*, given an integer n, then for $0<\sigma<1/(2n)$ and $0<\tau<1/(2n)$, we have

$$M(A(f, \sigma), \tau; z_0) = A(M(f, \tau), \sigma; z_0) \tag{16}$$

for all $z_0 \in \tilde{R}_n$. (Here $A(f, \sigma)$ denotes the function whose value at z is $A(f, \sigma; z)$ and similarly for $M(f, \tau)$). This is an immediate consequence of Fubini's theorem, for

$$M(A(f, \sigma), \tau; z_0) = \iint_{|\zeta|<\tau} s_\tau(\zeta) d\xi d\eta \int_0^\sigma \int_0^{2\pi} f(z_0 + \zeta + re^{i\theta}) e^{i\theta} d\theta dr$$

$$= \int_0^\sigma \int_0^{2\pi} e^{i\theta} d\theta dr \iint_{|\zeta|<\tau} f(z_0 + \zeta + re^{i\theta}) s_\tau(\zeta) d\xi d\eta = A(M(f, \tau), \sigma; z_0).$$

The function $\rho^{-2}A(f, \rho)$ satisfies (according to (13), (14)) $\lim_{\rho\to 0} \rho^{-2}A(f, \rho; z) = 0$ and corresponding to the compact set $\tilde{R}_{2n}$, there exist an M and $\rho_0 > 0$ such that $|\rho^{-2}A(f, \rho; z)| < M$ when $\rho < \rho_0$, $z \in \tilde{R}_{2n}$. Since for $\tau < 1/(2n)$, $\rho^{-2} A(M(f, \tau), \rho; z_0) = M(\rho^{-2} A(f, \rho), \tau; z_0)$ for $z_0 \in \tilde{R}_n$, we may apply the Lebesgue bounded convergence theorem to conclude that

$$\lim_{\rho\to 0} \frac{1}{\rho^2} (A(M(f, \tau), \rho; z_0) = 0$$

for all $z_0 \in \tilde{R}_n$.

We know, however, that $M(f, \tau) \in C^1$ in $\tilde{R}_n$ when $\tau < 1/(2n)$. According to Corollary 1.2, $M(f, \tau)$ is holomorphic in $\tilde{R}_n$. This leaves us with the task of showing that $M(f, \tau)$ being holomorphic implies that f is almost everywhere equal to a holomorphic function in $\tilde{R}_n$.

We begin by showing that $M(f, \tau; z)$ is independent of τ; *i.e.*, if $\tau, \sigma < 1/(2n)$, then

$$M(f, \tau; z) = M(f, \sigma; z) \tag{17}$$

for $z \in \tilde{R}_{n/2}$. From Property 2 of M, we deduce that

$$M(M(f, \tau), \sigma; z) = M(f, \tau; z) \tag{18}$$

and

$$M(M(f, \sigma), \tau; z) = M(f, \sigma; z) \tag{19}$$

for $z \in \tilde{R}_{n/2}$ and $\sigma, \tau < 1/(2n)$. But an application of Fubini's theorem shows that the left sides of (18) and (19) are equal; for, if $\lambda = \mu + i\nu$,

$$M(M(f, \tau), \sigma; z) = \iint_{|\zeta|<\sigma} s_\sigma(\zeta) d\xi d\xi \iint_{|\lambda|<\tau} s_\tau(\lambda) f(z + \zeta + \lambda) d\mu d\nu$$

$$= \iint_{|\lambda|<\tau} s_\tau(\lambda) d\mu d\nu \iint_{|\zeta|<\sigma} s_\sigma(\zeta) f(z + \zeta + \lambda) d\xi d\eta = M(M(f, \sigma), \tau; z).$$

Thus (17) is established.

We observe next that according to Property 4 of M,

$$\lim_{\sigma\to 0} \iint_{R_{n/2}} | M(f, \sigma; z) - f(z) | dxdy = 0.$$

Since $M(f, \sigma; z) = M(f, \tau; z)$ for fixed τ as $\sigma \to 0$, we see that

$$\iint_{R_{n/2}} | M(f, \tau; z) - f(z) | dxdy = 0$$

and finally $f(z)=M(f, \tau; z)$ almost everywhere in $\tilde{R}_{n/2}$. This tells us that f is almost everywhere equal to a holomorphic function in $\tilde{R}_{n/2}$, and since n is arbitrary, the result holds for R.

When f is Lebesgue integrable in R, then for each $z_0 \in R$ and any disk $|z-z_0| \leq \rho$ contained in R, we know from Fubini's theorem [3] that the integral

$$\int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta \tag{20}$$

exists for almost all r in the interval $0 \leq r \leq \rho$. Let $E_\rho(z_0)$ denote the subset of $0 \leq r \leq \rho$ for which (20) exists. Then $E_\rho(z_0)$ has linear Lebesgue measure equal to ρ. We may now state the following corollary to Theorem 2.

COROLLARY 2.1. *Let f be a single-valued function in a region R such that f is Lebesgue integrable over any compact subset of R. Assume that for $r \in E_\rho(z_0)$*

$$\lim_{r\to 0} \frac{1}{r} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta = 0 \tag{21}$$

for all $z_0 \in R$. Furthermore, assume that to each compact set $K \subset R$, there correspond M and r_0 such that

$$\frac{1}{r}\left| \int_0^{2\pi} f(z + re^{i\theta})e^{i\theta}d\theta \right| < M \tag{22}$$

for all $z \in K$ and almost all r in the interval $0 \leq r \leq r_0$. Then f is almost everywhere equal to a holomorphic function in R.

It is clear that (21) and (22) imply (13) and (14) respectively. For example, if (21) holds, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta \right| < \epsilon r$$

for almost all $r < \delta$. Then for $\rho < \delta$,

$$\left| \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta dr \right| \leq \left(\int_0^{\rho} \epsilon r dr \right) (\epsilon\rho^2/2),$$

so that

$$\left| \frac{1}{\rho^2} \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta dr \right| < \epsilon/2,$$

which in turn implies (13).

A weaker but interesting consequence of Corollary 2.1 is the following.

COROLLARY 2.2. *Let f be a continuous, single-valued function in R and let $\{U_i\}$ be an arbitrary open covering of R. Assume that*

$$\int_C f(z)dz = 0 \tag{23}$$

for every circle C that lies entirely within at least one open set of the covering $\{U_i\}$. Then f is holomorphic in R.

To prove this, we observe that (23) implies both (21) and (22). Since f is assumed to be continuous, the conclusion of Corollary 2.1 may be replaced by the stronger statement that f is everywhere holomorphic in R.

References

1. L. V. Ahlfors, Complex Analysis, New York, 1953.
2. L. O. Friedrichs, On differential operators in Hilbert spaces, Amer. J. Math., vol. 61, 1939, pp. 523–544.
3. P. R. Halmos, Measure Theory, New York, 1950.
4. H. Looman, Über eine Erweiterung des Cauchy-Goursatschen Integralsatzes, Nieuw Arch. Wisk., vol. 14, 1924, pp. 234–239.
5. W. F. Osgood, Lehrbuch der Funktionentheorie, Leipzig, 1912.
6. H. Rademacher, Bemerkung zu den Cauchy-Riemannschen Differentialgleichungen und zum Moreraschen Satz, Math. Z., vol. 4, 1919, pp. 177–185.
7. T. Rado, Subharmonic Functions, Berlin, 1937.
8. J. Ridder, Über den Cauchyschen Integralsatz für reele und komplexe Funktionen, Math. Ann., vol. 192, 1930, pp. 132–156.
9. S. Saks, Theory of the Integral, Warsaw, 1937.
10. J. Wolff, Über die Loomensche Erweiterung eines Satzes von Pompéiu, Nieuw Arch. Wisk., vol. 14, 1924, pp. 337–339.

ON FLETCHER'S PAPER "CAMPANOLOGICAL GROUPS"

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In the article *Campanological Groups* that appeared in this MONTHLY, T. J. Fletcher [1] considered, among other things, Thompson's [3] solution to the question of whether the full peal of Grandsire Triples could be generated by the use of plain and bob leads alone. Fletcher continues: "Thompson shows this to be impossible by proving that plaining or bobbing any Q-set always results in the loss or gain of an even number of round blocks. . . . The beauty of the proof is marred by the fact that the stage showing that the number of round blocks lost or gained is always even, is carried out by a long and tedious process of enumeration. But it is very difficult to see any means by which this could have been avoided. The enumeration of the cosets of a group of large order is inevitably tedious, and modern processes do not seem to offer any way of reducing

Thompson's labors to any marked extent."

It is the purpose of this note to supply that portion of the proof that Thompson and Fletcher were seeking. The notation used is that of Fletcher [1]. The proof is derived from a theorem of Rankin [2] who proved a more general result, of which this is a special case.

Suppose that we have a decomposition R_1 of the 360 leads into round blocks and a Q-set $\{x(PB^{-1})^i\}$, $i=1, 2, 3, 4, 5$, each of whose members are bobbed. Let S_1 be the substitution on the numbers i such that j is substituted for i if $x(PB^{-1})^j$ is the first member of the Q-set that occurs after $x(PB^{-1})^i$ in that round block of R_1 in which they occur. The number of cycles in the substitution S_1 is obviously the number of round blocks of R_1 that contain elements of the Q-set under consideration.

In R_1, the element that follows $x(PB^{-1})^i$ is $x(PB^{-1})^iB$. Now if $x(PB^{-1})^i$ is plained instead of bobbed, the element following becomes $x(PB^{-1})^iP$ $=x(PB^{-1})^{i+1}(PB^{-1})^{-1}P=x(PB^{-1})^{i+1}(BP^{-1})P=x(PB^{-1})^{i+1}B$. Hence we may form from R_1 a new decomposition R_2 by replacing the succession of $x(PB^{-1})^i$ by $x(PB^{-1})^iB$ by the succession of $x(PB^{-1})^i$ by $x(PB^{-1})^iP=x(PB^{-1})^{i+1}B$ and by letting all the other successions remain fixed. With respect to this new decomposition R_2 which was formed from R_1 by replacing a bobbed Q-set by a plained Q-set, and with respect to this Q-set, let S_2 be defined as S_1 was with respect to R_1.

Now it remains to show that the number of cycles of S_1 differs, if at all, from the number of cycles of S_2 by an even number. It is apparent that S_2 is the cyclical permutation (12345) followed by S_1. If we write the cyclical permutation as a product of transpositions, we have $(12)(13)(14)(15)S_1=S_2$. But multiplication of cycles by the transposition (pq) increases the number of cycles by one if p and q are in the same cycle and diminishes the number of cycles by one if p and q are in different cycles. Since we have four transpositions, the number of round blocks with the chosen Q-set bobbed hence differs by an even number from the number of round blocks that have the Q-set plained.

The smallest round block is that formed by a succession of three bobbed leads. We have shown therefore that the largest round block formed by bobbing and plaining alone cannot exceed 357 leads or 4998 changes. That a touch of this length is actually attainable was shown in 1751 by John Holt [4].

I should like to remark that the literature dealing with change ringing is quite large and that some of the modern publications can be found by consulting the weekly journal of the change ringers [5].

References

1. T. J. Fletcher, Campanological groups, this MONTHLY, vol. 63, 1956, pp. 619–626.
2. R. A. Rankin, A campanological problem in group theory, Proc. Cambridge Philos. Soc., vol. 44, 1948, pp. 17–25.
3. J. W. Snowden, Grandsire, Leeds, 1905.
4. J. A. Trollope, Grandsire, Leeds, 1948.
5. The Ringing World, c/o Woodbridge Press, Ltd., Guildford, Surrey.

In another place, Ernst Straus and the author have given general conditions for the inversive convexity of C', and hence conditions also for the convexity of $C=P^{-1}(C')$.

For the special case considered here, that is when C' has curvature and A is chosen interior to C', one can describe the convexity of C in the following way. Let K denote the circle of curvature of C' at a point Q. As Q moves about C' in a circuit, so does K. So long as the origin, namely the interior pedal point, is within, or on K, then the point R on C, corresponding to Q on C', traverses a convex arc, and when A is outside K then R is on a concave arc of C. Thus, the cusps of Figure 2 correspond to A passing in and out of the curvature circles of C', and one can see from this that there will be an even number of cusps. A standard, pedal curve relationship is that the circle on $\overline{AR}$ as a diameter is always tangent to C' at Q (Figure 1). When R is a cusp point of C, this circle is also a circle of curvature of C'.

FILTERS AND EQUIVALENT NETS

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The purpose of this note is to introduce a natural equivalence relation between nets [**5**] in such a way as to bring filters [**3**, **4**] and classes of equivalent nets into one-to-one correspondence. Our equivalence relation for nets stems from a theorem of Bruns and Schmidt [**2**, p. 184], while our correspondence rests on results of R. G. Bartle [**1**]. We feel that our discussion helps to clarify the concept of subnet [**5**]. The present self-contained version of our note is presented at the suggestion of a referee.

Let us first agree on certain matters of notation and of terminology. The notation $\alpha: S\rightarrow T$ denotes a map α of a set S into a set T, $\alpha(s)$ denotes the image of $s\in S$ under α, and $\alpha(S_1)$ denotes the set of all $\alpha(s_1)$ with $s_1\in S_1$. We write $\alpha\circ\beta$ for the composite of α and $\beta: T\rightarrow U$ so that $(\alpha\circ\beta)(s)=\alpha(\beta(s))$. A relation $\leqq$ *partially orders* a set P in case $\leqq$ is reflexive and transitive. In a partially ordered set P, we put $p^+=[q\in P: q\geqq p]$ for each $p\in P$, and we call P a *directed set* in case $p^+\cap q^+$ is non-empty for every $p, q\in P$. A family $\mathfrak{B}$ of non-empty subsets of a set X which is directed by inverse set-inclusion is called a *filter-base* in X. The totality of filter-bases in X will be denoted by $\mathfrak{G}(X)$. If $\mathfrak{B}, \mathfrak{D}\in\mathfrak{G}(X)$, then $\mathfrak{D}$ *refines* $\mathfrak{B}$ (which we write $\mathfrak{D}\succ\mathfrak{B}$) in case every set in $\mathfrak{B}$ contains some set in $\mathfrak{D}$. When $\mathfrak{D}\succ\mathfrak{B}$ and $\mathfrak{B}\succ\mathfrak{D}$, we call $\mathfrak{B}$ and $\mathfrak{D}$ *equivalent* and write $\mathfrak{B}\sim\mathfrak{D}$. We call $\mathfrak{B}, \mathfrak{D}\in\mathfrak{G}(X)$ *compositive* in case $\mathfrak{B}\vee\mathfrak{D}=[F\cap G: F\in\mathfrak{B}, G\in\mathfrak{D}]$ is a filter base in X. (Thus $\mathfrak{G}(X)$ is a partially ordered set which is not a directed set unless X has only one element. We have borrowed the term *compositive* from the work of E. H. Moore [**7**] since we feel that this important relation between

filter-bases deserves a name.) A *filter* in X is a filter-base in X which contains all supersets of each of its members. A *net* in X is a map $\alpha: A \to X$, where A is a directed set. A net $\beta: B \to X$ is a *subnet* of a net α in case $\beta = \alpha \circ \pi$, where $\pi: B \to A$ is *convergent* in the sense of E. H. Moore [**7**, p. 34], *i.e.*, there is a map $\rho: A \to B$ such that $\pi((\rho(a)^+) \subseteq a^+$ for every $a \in A$.

Each net $\alpha: A \to X$ gives rise to a filter-base $\mathbf{B}(\alpha) = [\alpha(a^+): a \in A]$ in X. We write $\mathbf{F}(\alpha)$ for the filter based on $\mathbf{B}(\alpha)$, *i.e.*, $\mathbf{F}(\alpha)$ is the family of all supersets of members of $\mathbf{B}(\alpha)$. If $\mathfrak{B} \in \mathcal{B}(X)$, we may define $A(\mathfrak{B}) = [(x, F): x \in F \in \mathfrak{B}]$, $(x, F) \leqq (x_1, F_1)$ in case $F \subseteq F_1$, and $\beta((x, F)) = x$ to obtain a net $\beta = \mathbf{N}(\mathfrak{B})$ in X. (This is the substance of the footnote on p. 554 of [1] as well as of 2.$L(f)$ of [6].) It is easy to verify that $\mathbf{B}(\mathbf{N}(\mathfrak{B})) = \mathfrak{B}$ for every $\mathfrak{B} \in \mathcal{B}(X)$.

A *topology* in a set X is specified by a map $\tau: X \to \mathcal{B}(X)$ such that $x \in U$ for every $U \in \tau(x)$. A point $x_0 \in X$ *τ-adheres to a filter-base* $\mathfrak{B}$ in X in case $\tau(x_0)$ and $\mathfrak{B}$ are compositive. A point $x_0 \in X$ *τ-adheres to a net* α in X in case x_0 τ-adheres to $\mathbf{F}(\alpha)$, or, equivalently, to $\mathbf{B}(\alpha)$. We shall call a net α in X *as fine as* a net β in X and write $\alpha \geqq \beta$ in case every point of X which τ-adheres to β also τ-adheres to α for every topology τ of X.

LEMMA. *For nets α, β in X, we have $\alpha \geqq \beta$ if and only if* $\mathbf{F}(\alpha) \subseteq \mathbf{F}(\beta)$.

Proof. Let us assume that $\alpha \geqq \beta$ and suppose that some $F \in \mathbf{F}(\alpha)$ is not in $\mathbf{F}(\beta)$. Then $F' \cap G \neq \phi$ for every $G \in \mathbf{F}(\beta)$, since $F' \cap G = \phi$ yields $G \subseteq F$, $F \in \mathbf{F}(\beta)$, a contradiction. We obtain a topology in X by setting $\tau(x) = [[x]]$ for every $x \in F$ and $\tau(x) = [F' \cap G: G \in \mathbf{F}(\beta)]$ for $x \in F'$. But then every x in F' τ-adheres to β, while no x in F' τ-adheres to α, contrary to our assumption, $\alpha \geqq \beta$. We have proved that $\alpha \geqq \beta$ implies $F(\alpha) \subseteq F(\beta)$. The converse is trivial. The proof of the lemma is complete.

It is now easy to see that $\alpha \geqq \beta$ if and only if $\mathbf{B}(\beta) \succ \mathbf{B}(\alpha)$. It is also clear that if $\alpha \geqq \beta$ and $\eta: X \to Y$ is a map of X into Y, then $\eta \circ \alpha$ and $\eta \circ \beta$ are nets in Y such that $\eta \circ \alpha \geqq \eta \circ \beta$. Thus the convergent maps of E. H. Moore are just those maps $\pi: B \to A$ for which $e_A \geqq \pi$, where e_A is the identity map of A onto A. The subnets $\beta = \alpha \circ \pi$ of α then satisfy $\alpha \geqq \beta$ but, even for finite X, it is possible that $\alpha \geqq \beta$ while β is not a subnet of α. (See, however, the remark contained in our final paragraph.)

It seems reasonable, now, to call two nets α, β in X *equivalent* and to write $\alpha \sim \beta$ in case $\alpha \geqq \beta$ and $\beta \geqq \alpha$, *i.e.*, $\alpha \sim \beta$ if and only if $\mathbf{F}(\alpha) = \mathbf{F}(\beta)$, $\alpha \sim \beta$ if and only if $\mathbf{B}(\alpha) \sim \mathbf{B}(\beta)$, since when this is true there is no topology in X which will distinguish between them. Using this notation, we have $\mathbf{N}(\mathbf{B}(\alpha)) \sim \alpha$ for every net α in X because $\mathbf{B}(\mathbf{N}(\mathbf{B}(\alpha))) = \mathbf{B}(\alpha)$. The maps $\mathbf{B}$ and $\mathbf{N}$ establish a one-to-one correspondence between the classes of equivalent nets in X and the classes of equivalent filter-bases in X; while the maps $\mathbf{F}$ and $\mathbf{B}$ serve the same purpose for filters in X and classes of equivalent nets in X.

Let us add one further observation concerning subnets. In the present notation, Kelley's fundamental lemma on subnets [**5**, p. 278] can be given a slightly

more precise form. Let α be a net in X, $\mathfrak{B}$ a filter-base in X compositive with $\mathrm{B}(\alpha)$. Let $\gamma((x, a, B))=x$ for $a\in A$, $B\in\mathfrak{B}$, and $x\in\alpha(a^+)\cap B$. Define $(x, a, B) \leqq (y, a_1, B_1)$ in case $a\leqq a_1$ and $B_1\subseteq B$. Then it is easy to see that γ is a subnet of α such that $\mathrm{B}(\gamma)=\mathfrak{B}\vee\mathrm{B}(\alpha)$. When $\alpha\geqq\beta$ and $\mathfrak{B}=\mathrm{B}(\beta)$, we see that γ is a subnet of α such that $\gamma\sim\beta$. If we permit ourself the luxury of identifying equivalent nets, we may regard the expressions "$\alpha\geqq\beta$" and "β is a subnet of α" as synonymous. In the course of the preparation of this note, we have become aware (through conversation) of an unpublished theory of set-nets developed by B. J. Pettis. Pettis gives a generalization of the notion of subnet which is equivalent for point-nets (that is, for nets) to the relation $\alpha\geqq\beta$ of the present note. The discussion of "ultimate" concepts in the present notation is left to the interested reader.

References

1. R. G. Bartle, Nets and filters in topology, this MONTHLY, vol. 62, 1955, pp. 551–557.
2. G. Bruns and J. Schmidt, Zur Äquivalence von Moore-Smith Folgen und Filtern, Math. Nachr., vol. 13, 1955, pp. 169–186.
3. H. Cartan, Théorie des filtres, C. R. Acad. Sci. Paris, vol. 205, 1937, pp. 595–598.
4. ———, Filtres et ultrafiltres, *ibid.*, vol. 205, 1937, pp. 777–779.
5. J. L. Kelley, Convergence in topology, Duke Math. J., vol. 17, 1950, pp. 277–283.
6. ———, General Topology, New York, 1955.
7. E. H. Moore and R. W. Barnard, General Analysis, Mem. Amer. Philos. Soc. I, 1939, Part II.

MATHEMATICAL NOTES

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CONVERGENCE OF SERIES WITH POSITIVE TERMS*

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1. Introduction. It is the purpose in what follows to prove necessary and sufficient conditions for convergence of series with positive terms that serve as a general framework for short proofs of the sufficient conditions of many of the known tests for convergence or divergence of such series (see [1], [2], [3]).

It will be understood that each series considered will be a series with positive terms unless the contrary is stated.

2. Necessary and sufficient conditions. As the basis of the conditions to be proved, we suppose as known the general criterion that a series with positive

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terms is convergent if and only if its partial sums are bounded.

THEOREM 1. *A necessary and sufficient condition that a series* $\sum a_n$ *with positive terms converge is that there exist positive numbers* p_n *and a nonnegative integer* k *such that* $p_n - p_{n+1} \geq a_{k+n}$ *for each* $n=1, 2, \cdots$.

Proof. The condition is necessary since, for $k=0$, we may take $p_n = \sum_{i=n}^{\infty} a_i$. On the other hand, the condition implies $p_1 > p_1 - p_{n+1} \geq \sum_{i=1}^{n} a_{k+i}$. Hence the partial sums of the series are bounded above by $p_1 + s_k$, where s_k is the kth partial sum. Therefore the series is convergent by the general criterion and the condition is sufficient.

The condition is necessary but its use in practice will be due to its sufficiency. For this reason, the following theorem is included for use in tests for divergence.

THEOREM 2. *A necessary and sufficient condition that a series* $\sum a_n$ *with positive terms diverge is that there exist an unbounded set of positive numbers* p_n *and a nonnegative integer* k *such that* $0 < p_{n+1} - p_n \leq a_{k+n}$ *for each* $n=1, 2, \cdots$.

Proof. The condition is necessary since, for $k=1$, we may take $p_n = \sum_{i=1}^{n} a_i$. On the other hand, the condition implies $p_{n+1} - p_1 \leq \sum_{i=1}^{n} a_{k+i}$. Hence the partial sums s_{k+n} are not bounded above since the set of numbers p_{n+1} are increasing and unbounded. Therefore the series is divergent by the general criterion and the condition is sufficient.

3. Tests for convergence and divergence. In order to set up a test for convergence, we only need to specify a nonnegative integer k and a set of positive numbers p_n and require that the inequality in Theorem 1 be satisfied. For example, if $\sum c_n$ is a convergent series, a set of positive numbers is defined by the equation $p_n = \sum_{i=n}^{\infty} c_i$ and $p_n - p_{n+1} = c_n$. Hence, for $k=0$, the inequality of Theorem 1 requires that $c_n \geq a_n$, which is the familiar comparison test. Similar remarks hold for tests for divergence based on Theorem 2.

Because of the principle explained in the preceding paragraph, we will list the choices of k and p_n, which, when associated with the inequality in Theorem 1 or Theorem 2, immediately give many of the familiar tests. The tests for convergence below have reference to Theorem 1 while those for divergence have reference to Theorem 2.

Comparison test.

Convergence: $p_n = \sum_{i=n}^{\infty} c_i$, $k=0$; $\sum c_n$ convergent.
Divergence: $p_n = \sum_{i=1}^{n} d_i$, $k=1$; $\sum d_n$ divergent.

Root test.

Convergence: $p_n = \theta^n(1-\theta)^{-1}$, $0<\theta<1$, $k=0$.
Divergence: $p_n = n$, $k=0$.

Ratio comparison test.

Convergence: $p_n = (a_1/c_1)\sum_{i=n}^{\infty} c_i$, $k=0$; $\sum c_n$ convergent.
Divergence: $p_n = (a_1/d_1)\sum_{i=1}^{n} d_i$, $k=1$; $\sum d_n$ divergent.

Remark 1. The last test will appear in its familiar form when we observe that, for convergence, the condition is satisfied if we require $a_{n+1}/a_n \leq c_{n+1}/c_n$; while, for divergence, the condition is satisfied if we require $a_{n+1}/a_n \geq d_{n+1}/d_n$.

Ratio test.
Convergence: $p_n = a_n/\theta$, $\theta > 0$, $k = 1$.
Divergence: $p_n = na_1$, $k = 1$.

Remark 2. For divergence, the condition is satisfied if $a_n \leq a_{n+1}$.

Raabe's test.
Convergence: $p_n = na_n/\theta$, $\theta > 0$, $k = 1$.
Divergence: $p_n = a_1 \sum_{i=1}^{n} 1/i$, $k = 1$.

Remark 3. For divergence, the condition is satisfied if $na_n \leq (n+1)a_{n+1}$. Here the divergence of the harmonic series has been assumed. Later it will be shown to diverge as a consequence of Theorem 2.

Integral test.
Convergence: $p_n = \int_n^\infty f(x)dx$, $f(n) = a_n$, $k = 1$.
Divergence: $p_n = \int_1^n f(x)dx$, $f(n) = a_n$, $k = 0$.

Remark 4. For convergence, it is assumed that the improper integral converges; for divergence, that it diverges. Each condition is satisfied if $f(x)$ is monotonely decreasing. The test, as given, is more general than Cauchy's integral test since it does not require that $a_n \geq a_{n+1}$ nor that $f(x)$ decrease monotonely.

Kummer's test.
Convergence: $p_n = d_n a_n/\theta$, $\theta > 0$, $d_n > 0$, $k = 1$.
Divergence: $p_n = a_1 d_1 D_n$, $d_1 > 0$, $D_n > 0$, $k = 1$.

Remark 5. It is easy to see that these lead to necessary and sufficient conditions, since they are essentially a restatement of Theorems 1 and 2 with a change in notation for the set of positive numbers p_n. The necessity of Kummer's conditions seems not to have been known previously (see [1], [2], [3]). For divergence, the condition is satisfied if we require the divergence of $\sum 1/d_n$ and that $a_n d_n \leq a_{n+1} d_{n+1}$, since then there exists an unbounded set D_n such that $0 < D_{n+1} - D_n < 1/d_{n+1}$.

Abel-Dini-Pringsheim test. Let $\sum d_n$ be a divergent series and D_n be its nth partial sum. Let $\theta = 1 + p > 1 + 1/m$, where m is a positive integer. Then the choice $p_n = m/D_{n-1}^{1/n}$ and the inequalities

$$m/D_{n-1}^{1/m} - m/D_n^{1/m} \geq D_n - D_{n-1}/D_n D_{n-1}^{1/m} \geq d_n/D_n D_{n-1}^{p} \geq d_n/D_n^{\theta},$$

together with Theorem 1, prove the convergence of the series $\sum d_n/D_n^\theta$, $\theta > 1$, as well as a series with any one of the terms in the inequalities as general term.

Similarly, the choice $p_n = D_n^{1-\theta}$, $\theta \leq 1$ leads to the divergence of the series $\sum d_n/D_n^{\theta}$.

Remark 6. Cauchy's condensation test and Ermakoff's test can be derived as special cases of those already given. It should be clear now that the list of tests can be enlarged with ease by the proper choice of the numbers p_n. Rather than this, we will develop the theory along another line in the next section.

4. Convergent and divergent series. The following theorem is useful to obtain specific convergent series.

THEOREM 3. *If $f(x)$ and $-f'(x)$ are positive functions and $-f'(x)$ is monotonely decreasing for $x \geq 1$, then the series $\sum -f'(n)$ converges.*

Proof. The hypothesis and the mean value theorem imply $f(x) - f(x+1) = -f'(x+\theta) \geq -f'(x+1)$, where $0 < \theta < 1$. Hence it follows from Theorem 1 that the series $\sum -f'(n+1)$ (and thus also $\sum -f'(n)$) converges.

Note that $-f'(x)$ is monotonely decreasing provided $f''(x) > 0$. This immediately implies the convergence of the following series:

$$\sum br^n, \qquad 0 < r < 1; \qquad f(x) = ar^x, \qquad b = -a \log r,$$

$$\sum bn^{-1-\theta}, \qquad \theta > 0; \qquad f(x) = ax^{-\theta}, \qquad b = a\theta,$$

$$\sum b(n \log n \cdots \log_{p-1} n)^{-1}(\log_p n)^{-1-\theta}, \qquad \theta > 0, \qquad f(x) = a(\log_p x)^{-\theta},$$

where $\log_p n$ denotes $\log\log \cdots \log n$ with p operations and $b = a\theta$. This list could be enlarged without difficulty.

THEOREM 4. *If $f(x)$ is a positive unbounded function and $f'(x)$ is a positive, monotonely decreasing function for $x \geq 1$, then the series $\sum f'(n)$ diverges.*

The proof follows immediately from the mean value theorem and Theorem 2. From this theorem, it is seen that the following series diverge:

$$\sum br^n, \qquad r > 1; \qquad f(x) = ar^x, \qquad b = a \log r \neq 0,$$

$$\sum bn^{-1+\theta}, \qquad 0 < \theta \leq 1; \qquad f(x) = ax^{\theta}, \qquad b = a\theta \neq 0,$$

$$\sum b(n \log n \cdots \log_{p-1} n)^{-1}(\log_p n)^{-1+\theta}, \qquad 0 < \theta \leq 1, \qquad f(x) = a(\log_p n)^{\theta},$$

where $b = a\theta \neq 0$. Note that the last case includes the divergence of the harmonic series for $a = \theta = p = 1$.

The last two theorems may be restated so as to be used directly as tests. This is done in the next theorem for Theorem 3.

THEOREM 5. *If an indefinite integral of $f(x)$ is a negative function and $f(x)$ is a positive, monotonely decreasing, and continuous function for $x \geq 1$, then the series $\sum f(n)$ converges.*

It is interesting to observe that the last theorem is an "integral test" without a direct reference to an improper integral.

References

1. T. J. I'a. Bromwich, An Introduction to the Theory of Infinite Series, London, 1926.
2. K. Knopp, Theory and Application of Infinite Series, London-Glasgow, 1928.
3. E. G. Phillips, A Course of Analysis, Cambridge, 1930.

POLYNOMIALS WITH THE BINOMIAL PROPERTY

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Let us call $\{b_n(x)=\sum_{i=0}^{n} b_{ni}x^i,\ b_{nn}\neq 0\}$ a binomial set of polynomials if the relation

$$b_n(x+y)=\sum_{i=0}^{n}\binom{n}{i}b_i(x)b_{n-i}(y),\qquad n=0,1,\cdots, \tag{1}$$

is satisfied. Two examples of binomial sets are the powers, $\{x^n\}$, and the factorial polynomials $\{x^{(n)}=x(x-1)\cdots(x-n+1)\}$. It seems worth while to point out that binomial sets are (except for constant multipliers) the basic sets of polynomials of type zero introduced by I. M. Sheffer.* His basic polynomials are defined by his theorem:

Let $J(B)=c_1B'+c_2B''+\cdots$ (c_n=constant, $c_1\neq 0$). To each operator J there corresponds one and only one polynomial set $\{B_n(x)\}$ (which we call the basic set) such that

$$B_0(x)=1,\quad B_n(0)=0,\quad n>0,\qquad J(B_n(x))=B_{n-1}(x).$$

The assumption, c_n=constant, classifies J as a type zero operator. For the case where the operator is $\sin D$,

$$J(B)=B'-\frac{1}{3!}B'''+\frac{1}{5!}B^{\mathrm{v}}-\cdots,$$

and the polynomial sets are

binomial set: $1,\ x,\ x^2,\ x^3+x\ \ x^4+4x^2,\ x^5+10x^3+9x,\cdots,$

basic set: $1,\ x,\ \frac{1}{2!}x^2,\ \frac{1}{3!}(x^3+x),\ \frac{1}{4!}(x^4+4x^2),\ \frac{1}{5!}(x^5+10x^3+9x),\cdots.$

To show the correspondence between basic and binomial sets, we start with

THEOREM 1. *If $\{B_n(x)\}$ is a type zero basic set, then $\{n!B_n(x)\}$ is a binomial set.*

This result follows at once from Sheffer's relation $e^{xH(t)}=\sum_{i=0}^{\infty}B_i(x)t^i$ by equating powers of t in

$$e^{xH(t)}e^{yH(t)}=e^{(x+y)H(t)}=\sum_{n=0}^{\infty}t^n\sum_{i=0}^{n}B_i(x)B_{n-i}(y)=\sum_{n=0}^{\infty}B_n(x+y)t^n.$$

Before proceeding to the converse theorem, we note

THEOREM 2. *If $\{b_n(x)\}$ is a binomial set, then $b_0(x)=1$, $b_n(0)=0$, $n>0$.*

* I. M. Sheffer, Some properties of polynomial sets of type zero. Duke Math. J., vol. 5, 1939.

The first two equations of (1) are

$$b_0(x+y) = b_0(x)b_0(y), \qquad b_1(x+y) = b_0(x)b_1(y) + b_1(x)b_0(y).$$

Thus the constant $b_0(x)=1$. Setting $x=y=0$ in $b_1(x+y)$, we get $b_1(0)=0$ and an induction gives $b_n(0)=0$, $n>0$.

THEOREM 3 (*converse of Theorem* 1). *If* $\{b_n(x)\}$ *is a binomial set, there exists a type zero operator* J *whose basic set is* $\{(1/n!)b_n(x)\}$.

Having Theorem 2, it suffices to produce an operator J such that

$$J(b_n(x)) = nb_{n-1}(x), \qquad J(y) = c_1y' + c_2y'' + \cdots (c_1 \neq 0).$$

Simple algebraic manipulation will procure a few terms of J. If $b_0(x)=1$, $b_1(x)=ax$, $b_2(x)=a^2x^2+bx$, $\cdots$, the operator is $J(y)=(1/a)y'-(b/2a^3)y''+\cdots$. A step-by-step process will produce the remaining terms uniquely. Suppose that the operator J_{n-1} has the properties

$$J_{n-1}(b_k(x)) = kb_{k-1}(x), \qquad k = 1, \cdots, n-1,$$
$$J_{n-1}(y) = \frac{1}{a}y' + \cdots + c_{n-1}y^{(n-1)}.$$

This operator possesses a unique basic set

$$b_0(x), \cdots, \frac{1}{n-1!}b_{n-1}(x), \frac{1}{n!}b_n^*(x), \cdots,$$

whose first n terms coincide with the given binomial set (except for the constant multipliers). From Theorem 1, $b_0(x), \cdots, b_{n-1}(x), b_n^*(x), \cdots$, must also form a binomial set. A relation connecting $b_n(x)$ and $b_n^*(x)$ can be obtained from (1):

$$\sum_{i=1}^{n-1}\binom{n}{i}b_i(x)b_{n-i}(y) = b_n(x+y) - b_n(x) - b_n(y) = b_n^*(x+y) - b_n^*(x) - b_n^*(y).$$

If $b_n(x) = \sum_{i=1}^{n} r_ix^i$, the expansion of the above expression is

$$b_n(x+y) - b_n(x) - b_n(y) = [nr_nx^{n-1} + \cdots + 2r_2x]y + \text{terms in } y^2.$$

Since the expression obtained from $b_n^*(x)$ must be identical with this, the two nth degree polynomials can differ only in their coefficients of x, *i.e.*, $b_n(x)=b_n^*(x)+cx$, $c=r_1-r_1^*$. Let $J_n=J_{n-1}-(c/ar_nn!)D^n$. Then

$$\begin{aligned} J_n(b_n(x)) &= J_{n-1}(b_n^*(x)) + J_{n-1}(cx) - (c/ar_nn!)D^n[b_n^*(x) + cx] \\ &= nb_{n-1}(x) + (c/a) - (c/ar_nn!)n!r_n = nb_{n-1}(x). \end{aligned}$$

Thus $(1/n!)b_n(x)$ is a basic polynomial of the operator J_n, and $\{(1/n!)b_n(x)\}$ is the basic set of the operator $J\equiv J_\infty$.

SOME INEQUALITIES INVOLVING HERMITE POLYNOMIALS

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Problem 4215 [1946, 470], this MONTHLY, indicates that

$$\Delta_n(x) = H_n^2(x) - H_{n+1}(x)H_{n-1}(x) = (n-1)! \sum_{i=0}^{n-1} H_i^2(x)/i!,$$

where $H_n(x) = (-1)^n e^{x^2/2} d^n(e^{-x^2/2})/dx^n$ is the Hermite polynomial of degree n. From this we obtain immediately the inequality

$$\Delta_n(x) > 0, \text{ all } x,\ n \geq 1. \tag{1}$$

Mukherjee and Nanjundiah in [1] establish the identity $n\Delta_n(x) = [H_n'(x)]^2 - H_n(x)H_n''(x)$, with which (1) can also be proved. Sharper estimates of (1) may be found in [2] and [3]. The corresponding inequality with the derivatives of Hermite polynomials is treated in [4] and [5].

Toscano in [6] establishes the identity

$$\delta_n(x) = H_{n+1}(x)H_{n+2}(x) - H_n(x)H_{n+3}(x) = 2xn! \sum_{i=0}^{[n/2]} H_{n-2i}^2(x)/(n-2i)!$$

from which follows

$$\delta_n(x) \begin{cases} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{cases}, \qquad n \geq 0. \tag{2}$$

Toscano also shows that $\Delta_n'(x) = (n-1)\delta_{n-2}(x)$. Hence

$$\Delta_n'(x) \begin{cases} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{cases}, \qquad n \geq 2. \tag{3}$$

In this paper we establish similar inequalities. First, (1) has two immediate generalizations of a different nature:

$$[H_n(x)]^{2(2r+1)} - [H_{n+1}(x)]^{2r+1}[H_{n-1}(x)]^{2r+1} > 0, \tag{4}$$

r a positive integer, all x, $n \geq 1$.

Simple examples indicate that if $2r+1$ is replaced by an even integer, the inequality is no longer valid.

$$H_n^2(x) - kH_{n+1}(x)H_{n-1}(x) > 0, \quad 0 \leq k \leq 1, \quad \text{all } x,\ n \geq 1. \tag{5}$$

One can show that for k outside this range, the expression on the left changes sign.

Next we show that

(6) $(n+1)H_n^2(x) - (n-1)H_{n+2}(x)H_{n-2}(x) \geq 0$, all x, $n \geq 2$, with equality for $x = 0$ only.

Using the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \tag{7}$$

the left-hand side of (6) can be written as $x\delta_{n-1}(x)$, and the result follows from (2). $H_2^2(x) - H_0(x)H_4(x) = 4x^2 - 2$ shows that $n+1$ and $n-1$ cannot be replaced by unity in (6).

We now prove

$$H_n(x)H_n'(x) - H_{n+1}(x)H_{n-1}'(x) \begin{Bmatrix} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{Bmatrix}, \qquad n \geq 1. \tag{8}$$

Using the relation

$$H_n'(x) = nH_{n-1}(x), \tag{9}$$

the left-hand side of (8) becomes $nH_n(x)H_{n-1}(x) - (n-1)H_{n+1}(x)H_{n-2}(x)$. Using (7) to replace $H_{n-1}(x)$ and $H_{n-2}(x)$ in this expression, yields $H_n(x)H_n'(x) - H_{n+1}(x)H_{n-1}'(x) = x\Delta_n(x)$, and the result follows from (1).

Next we show that

$$[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) \leq 0, \text{ all } x,\ n \geq 1. \tag{10}$$

With the use of (9) and the differential equation

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0, \tag{11}$$

we obtain

$$[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) = n^2H_{n-1}^2(x) - H_{n-1}(x)[xH_{n+1}'(x) - (n+1)H_{n+1}(x)].$$

The use of (7) and (9) to eliminate $H_{n+1}(x)$ and $H_{n+1}'(x)$ respectively then yields $[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) = -nH_{n-1}^2(x)$, from which the result follows.

Now, we show

(12) $[H_n'(x)]^2 - H_{n+1}(x)H_{n-1}''(x) \geq 0$, all x, $n \geq 1$, equality for $x = 0$ and n odd only.

Multiplying by n both sides of the inequality in (6), we obtain

$$n(n+1)H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0.$$

Then

$$(n+1)H_n^2(x) + n(n+1)H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0,$$

or

$$(n+1)^2 H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0,$$

with equality for $x=0$ and n odd only. Using (9) leads to the result.

Next, we have

(13) $[H_n''(x)]^2 - H_n(x)H_n^{(\mathrm{iv})}(x) \geq 0$, all x, $n \geq 2$, with equality for $x = 0$ and n odd only.

With the use of (9), the left-hand side of (13) becomes $n^2[H_{n-1}'(x)]^2 - n(n-1)H_n(x)H_{n-2}''(x)$ and the result follows by (12) since $n(n-1)/n^2 < 1$.

Next we establish the identity

$$n\Delta_n''(x) = [H_n''(x)]^2 - H_n(x)H_n^{(\mathrm{iv})}(x). \tag{14}$$

With the use of (7) and (9), we have

$$n\Delta_n(x) = nH_n^2(x) - xH_n(x)H_n'(x) + [H_n'(x)]^2.$$

Using (11), we can write

$$n\Delta_n'(x) = x[H_n'(x)]^2 - H_n(x)H_n'(x) - xH_n(x)H_n''(x).$$

Then $n\Delta_n''(x) = xH_n'(x)H_n''(x) - 2H_n(x)H_n''(x) - xH_n(x)H_n'''(x)$. From (11) we obtain $H_n^{(\mathrm{iv})}(x) = xH_n'''(x) + (2-n)H_n''(x)$, which with the above yields the result.

(15) $\Delta_n''(x) \geq 0$, all x, $n \geq 2$, with equality for $x = 0$ and n odd only.

In general, the derivatives of $\Delta_n(x)$ of order greater than 2 do not remain of one sign in any half line.

References

1. B. N. Mukherjee, and T. S. Nanjundiah, On an inequality relating to Laguerre and Hermite polynomials, Math. Student, vol. 19, 1951, pp. 47–48.

2. Otto Szász, Identities and inequalities concerning orthogonal polynomials and Bessel functions, J. Analyse Math., vol. 1, 1951, pp. 116–134.

3. Giuseppe Palamà, Su di un limite inferiore della distanza di due zeri consecutivi di $H_n(x)$ e su di una limitazione di $H_n^2(x) - H_{n+1}(x)\ H_{n-1}(x)$, Boll. Un. Mat. Ital., (3), vol. 7, 1952, pp. 311–315.

4. H. Skovgaard, On inequalities of the Turán type, Math. Scand., vol. 2, 1954, pp. 65–73.

5. Arthur E. Danese, Explicit evaluations of Turán expressions, Ann. Mat. Pura Appl., (4), vol. 38, 1955, pp. 339–348.

6. Letterio Toscano, Su una disuguaglianza relativa ai polinomi di Hermite, Boll. Un. Mat. Ital., (3), vol. 7, 1952, pp. 171–173.

A GENERAL FORMULA FOR CIRCULAR PERMUTATIONS

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The general formula for circular permutations does not seem to have been discussed in textbooks. In this note, we establish a general formula for the number of circular permutations of a set of n objects containing similar elements. We first prove the following:

THEOREM 1. *Let the set of n objects contain r_1 elements a_1, r_2 elements $a_2, \cdots, r_p$ elements a_p. Let h be the highest common factor of r_ν $(\nu=1, \cdots, p)$. Then, if h is a prime, the number P of circular permutations of the n objects is given by*

$$(1) \qquad P = (1/n)\left\{\frac{n!}{r_1!\cdots r_p!} - \frac{(n/h)!}{(r_1/h)!\cdots(r_p/h)!}\right\} + \frac{((n/h)-1)!}{(r_1/h)!\cdots(r_p/h)!}.$$

Proof. Since h is a prime, we can divide the n objects into h similar sections, each containing r_1/h elements $a_1, \cdots, r_p/h$ elements a_p. If each section is denoted by $[C]$, the n objects may be arranged as

$$(2) \qquad [C] + [C] + \cdots + [C] \qquad \text{(to } h \text{ terms).}$$

Take, for instance, a special permutation $[C_\mu]$ formed by permuting the elements of $[C]$. If we arrange the given n objects as $[C_\mu]+\cdots+[C_\mu]$ (to h terms), then this is clearly one of the permutations of the n objects. This permutation, as a circular permutation, is unchanged if we shift the first element to the end, the second to the end, and so on, and at last, the n/hth element to the end. Thus, the linear permutations of the n objects formed by linking h similar sections of $[C]$ is in an n/h: 1 correspondence with the circular permutations of the n objects formed by the same method of linkage. Since $r_1/h, \cdots, r_p/h$ have no common factor, we cannot divide into further sub-sections of elements. Now, the number of linear permutations obtained by linking h of the sections $[C]$ of the same form of permutation is equivalent to the number of the linear permutations formed by permuting the elements of $[C]$. If k denotes the number of linear permutations of each $[C]$, it is known that

$$(3) \qquad k = \frac{(n/h)!}{(r_1/h)!\cdots(r_p/h)!}.$$

It is also known that the number K of linear permutations of the n objects is given by $K=n!/(r_1!\cdots r_p!)$. It is evident that there is an n: 1 correspondence between the linear permutations and the circular permutations of the n objects, except for the above permutations formed by linking h similar sections $[C]$. Therefore, it follows that $P=(1/n)(K-k)+k/(n/h)$, and this is the result given in (1).

We obtain the following corollary by taking $h=1$ in Theorem 1, since the number of circular permutations cannot be a fraction.

COROLLARY. *If* $r_1, \cdots, r_p$ *have no common factor, then* $(n-1)!/(r_1! \cdots r_p!)$ *is an integer.*

By means of Theorem 1 we can prove the following:

THEOREM 2. *In the notation of Theorem* 1, *let* $h=h_0h_1 \cdots h_m$, *where* $h_0=1$ *and* $h_1, \cdots, h_m$ *are primes, and let* $H_i=h_0h_1 \cdots h_i$. *Then*

$$
\begin{aligned}
P = &\sum_{i=0}^{m-1} \frac{1}{n/H_i}\left\{\frac{(n/H_i)!}{(r_1/H_i)! \cdots (r_p/H_i)!} - \frac{(n/H_{i+1})!}{(r_1/H_{i+1})! \cdots (r_p/H_{i+1})!}\right\} \\
&+ \frac{((n/h)-1)!}{(r_1/h)! \cdots (r_p/h)!}.
\end{aligned}
\tag{4}
$$

Proof. We divide the n objects into h_1 similar sections $[C_1]$, each containing r_1/H_1 elements $a_1, \cdots, r_p/H_1$ elements a_p. Next, we divide $[C_1]$ into h_2 similar sections $[C_2]$, each containing r_1/H_2 elements $a_1, \cdots, r_p/H_2$ elements a_p. By successive similar operations, we arrive at $[C_m]$, which contains r_1/H_m elements $a_1, \cdots, r_p/H_m$ elements a_p. From the given conditions it is seen that $r_1/H_m, \cdots, r_p/H_m$ have no common factor. By Theorem 1, we know that the relation between the linear permutations formed by linking h_1 similar sections $[C_1]$ and the circular permutations for the same arrangement is an n/H_1: 1 correspondence, the relation between the linear permutations obtained by linking h_2 similar sections $[C_2]$ and the circular permutations for the same arrangement is an n/H_2: 1 correspondence, and so on. If we write

$$k_i = \frac{(n/H_i)!}{(r_1/H_i)! \cdots (r_p/H_i)!} \qquad (i = 0, \cdots, m),$$

then, from the arguments of Theorem 1, it is easily seen that

$$P = \sum_{i=0}^{m-1} \frac{1}{n/H_i}(k_i - k_{i+1}) + \frac{k_m}{n/h},$$

and this is the result given in (4).

BEST FITTING INTEGRAL CURVES OF LINEAR DIFFERENTIAL EQUATIONS

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1. We are given a set of points $(x_k, \bar{y}_k)$ $k=0, \cdots, m$. These points are number pairs, the numbers representing measurements obtained by physical experiment. They differ from a set of points (x_k, y_k) which lie on an integral curve of some linear differential equation of known order and known driving function by the random error usually present in any set of physical measurements. We wish to determine the coefficients of the differential equation together with a set of initial conditions so that the resulting integral curve will be a best fitting

curve for the given set of data. The term "best fitting curve" will be defined after some preliminary theory has been developed. We proceed to a more special problem.

2. Given a differential equation

$$b_0 y + b_1 y' + \cdots + b_m y^{(m)} = D(x), \tag{1}$$

where $D(x)$ is a function such that

$$\overline{M}_k = \int_0^\infty x^k D(x)dx$$

are finite constants not all 0 for all $k=0, \cdots, 2m$ and the b_k are disposable parameters. Let $y=F(x)$ be a function with this same property that

$$M_k = \int_0^\infty x^k F(x)dx$$

exist for $k=0, \cdots, 2m$.

THEOREM. *If there exist a set of numbers b_k such that $y=F(x)$ is a solution of* (1), *then this set is uniquely determined by M_k and $\overline{M}_k$ provided the determinant $|C_{i,j}| \neq 0$ where $i=1, \cdots, m+1, j=1, \cdots, m+1$, and $C_{i,j}=C_k=(-1)^k M_k/k!$, $k=i+j-2$.*

Proof. Let the Laplace transform of $F(x)$ be $f(s)=\mathcal{L}[F(x)]=\int_0^\infty F(x)e^{-sx}dx$. Differentiating gives the expressions

$$f^{(k)}(0) = (-1)^k M_k \tag{2}$$

for the first $2m$ derivatives of $f(s)$ at $s=0$, their existence being assured by the assumption that M_k exist. Similarly if $g(s)$ is the Laplace transform of $D(x)$, then $g^{(k)}(0)=(-1)^k\overline{M}_k$, $k=0, \cdots, 2m$. Then

$$f(s) = \frac{g(s) + a_0 + a_1 s + \cdots + a_{m-1}s^{m-1}}{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}, \tag{3}$$

where

$$a_k = \sum_{k+1}^{m} b_i F^{i-k-1}(0). \tag{4}$$

Equations (2) and (3) will now be used to obtain $2m+1$ equations for determining the $2m+1$ parameters a_k and b_k. The equations are all linear in a_k and b_k and may be found by the following device.

Expand $f(s)$ and $g(s)$ by MacLaurin's series. Then the coefficients c_k and d_k, respectively, are given by

(5) $$c_k = f^{(k)}(0)/k! = (-1)^k M_k/k!, \qquad d_k = (-1)^k \overline{M}_k/k!.$$

Multiplying both sides of equations (3) by $b_0 + b_1 s + \cdots + b_m s^m$ and equating the coefficients of like powers of s gives

(6) $$a_k = c_0 b_k + c_1 b_{k-1} + \cdots + c_k b_0 - d_k, \qquad k = 0, \cdots, m-1;$$

(7) $$c_k b_m + c_{k+1} b_{m-1} + \cdots + c_{k+m} b_0 = d_{k+m}, \qquad k = 0, \cdots, m.$$

Under the conditions of the theorem equations (6) and (7) have a unique solution and the theorem is proved.

The homogeneous case can be solved by setting $D(x) \equiv 0$. If $|C_{i,j}| \neq 0$, then each $b_k = 0$ by the previous theorem and there exists no homogeneous differential equation which has $y = F(x)$ as its solution. However, if the rank of the matrix $\|C_{ij}\|$ is m and $y = F(x)$ satisfies such a differential equation it is unique since equations (7) now determine uniquely the ratios of the b_k's.

3. The theorem of the preceding paragraph is true under much weaker conditions on $F(x)$ and $D(x)$. We demand only that there exist a $p \geqq 0$ such that

$$M_k = \int_0^\infty x^k F(x) e^{-px} dx \quad \text{and} \quad \overline{M}_k = \int_0^\infty x^k D(x) e^{-px} dx$$

exist for $k = 0, \cdots, 2m$. For if we transform (1) by the transformation $Z = e^{-px} y$, the new equation becomes

(8) $$B_0 Z + B_1 Z' + B_2 Z'' + \cdots + B_m Z^{(m)} = e^{-px} D(x).$$

If $y = F(x)$ is a solution of (1), then $Z = e^{-px} F(x)$ is a solution of (8). By the above theorem the B_k are determined uniquely and the inverse transformation will produce equation (1) uniquely.

4. *Definition*: *Let $y = G(x)$ be a curve such that*

$$\mu_k = \int_0^\infty x^k G(x) e^{-px} dx$$

exist for some $p \geqq 0$ and for $k = 0, \cdots, 2m$. A curve $y = F(x)$ is a best fitting integral curve of a linear differential equation (1) *with respect to $y = G(x)$ and for a given p if it satisfies* (1) *and if*

$$M_k = \int_0^\infty x^k F(x) e^{-px} dx$$

exist and are equal to μ_k.

We return to the problem of the first paragraph. Let $y = G(x)$ be an arbitrary curve containing the point set $(x_k, \bar{y}_k)$. Compute the moments μ_k and then compute d_k and c_k, replacing M_k by μ_k. Equation (7) is solved for the values b_k. Equations (6) and then (4) are used to compute the initial conditions. An

analogue computer may now be used to draw $y=F(x)$, or equation (1) with computed initial conditions may be used to solve for $y=F(x)$.

If $y=G(x)$ does not approximate an integral curve of (1), the best fitting integral curve may be of little value. However, the method may be very useful for fitting curves which approach the x-axis asymptotically where polynomial curves are impractical. Many such curves approximate integral curves of linear homogeneous differential equations. When applicable, the method provides an excellent means of preparing tabular data for use in an analogue computer.

A SUBSET OF THE COUNTABLE ORDINALS

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Let Λ be the space of all countable ordinals with the order topology. There are two obvious types of uncountable subsets of Λ:

(1) *sets which contain some uncountable closed set,*
(2) *sets which are contained in the complement of some uncountable closed set.*

The purpose of this paper is to show that there is a third type of uncountable subset of Λ:

(3) *sets which intersect every uncountable closed set but which contain no uncountable closed set.*

This was surprising to the author primarily because of the following fact:

(A) *the intersection of any countable family of sets of type* (1) *is again a set of type* (1).

Proof of the existence of sets of type (3). Let f be a one-to-one transformation of Λ onto a subset of a line L. For each positive integer n, let X_n be a countable collection of intervals covering L of length less than $1/n$ such that, for $n>1$, X_n is a refinement of X_{n-1}. For x in X_n, let $\lambda(x)$ be the set of all α in Λ such that $f(\alpha)$ is in x. We will show that, for some x, $\lambda(x)$ is of type (3).

Let us assume that there is no n and x in X_n such that $\lambda(x)$ is of type (3). We will now show that there are intervals $x_1\supset x_2\supset x_3\supset \cdots$ such that, for each n, x_n belongs to X_n and $\lambda(x_n)$ is of type (1). By (A), $\bigcap_{n=1}^{\infty}\lambda(x_n)$ is uncountable, but this is impossible since $\bigcap_{n=1}^{\infty} x_n$ is a single point and f is one-to-one.

The complement of every countable set and of every set of type (2) is of type (1). Therefore, by (A), at least one term x_1 of X_1 is such that $\lambda(x_1)$ is of type (1). Similarly, if x_n has been defined as a term of X_n such that $\lambda(x_n)$ is of type (1), then by the same argument there is a subinterval x_{n+1} of x_n belonging to X_{n+1} such that $\lambda(x_{n+1})$ is of type (1).

CLASSROOM NOTES

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ON THE THEOREMS OF CEVA AND MENELAUS

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The following combined proof, based initially on purely projective methods, of the two theorems may be of some interest.

Let ABC (Figure 1) be a triangle lying in a real extended Euclidean plane and P any point in the plane not on a side of the triangle. Let AP meet BC in X, BP meet CA in Y and CP meet AB in Z. Suppose DEF is a finite straight line in the plane through none of the points A, B, C, or P, and let it meet BC in D, CA in E, AB in F. Since not all of the non-collinear points X, Y, Z can be on DEF we can assume without loss of generality that X does not. Let DEF meet AP in Q, BQ meet AC in Y' and CQ meet AB in Z'. For the present we will consider all the elements of the configuration to be real.

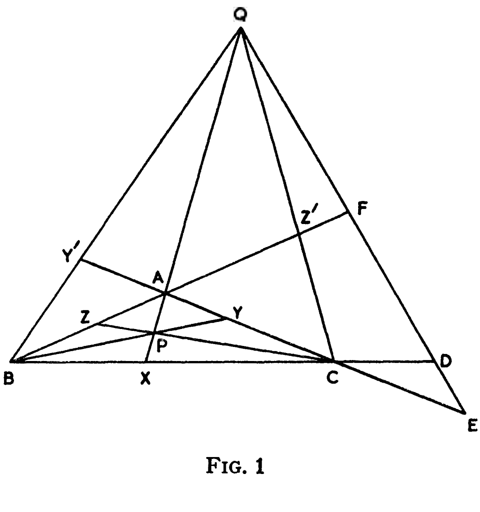

FIG. 1

Projecting the points Q, D, E, F from A onto BC, from B onto CA and from C onto AB establishes the equality of the cross-ratios $\{XD, CB\}$, $\{Y'C, EA\}$, $\{Z'B, AF\}$. Further

$$\{XD, CB\} \times \{XC, BD\} \times \{XB, DC\} = \frac{XC \cdot DB}{XB \cdot DC} \times \frac{XB \cdot CD}{XD \cdot CB} \times \frac{XD \cdot BC}{XC \cdot BD} = -1.$$

Since $\{XD, CB\}=\{Y'C, EA\}$ it follows that $\{XC, BD\}=\{Y'E, AC\}$, and since $\{XD, CB\}=\{Z'B, AF\}$ that $\{XB, DC\}=\{Z'F, BA\}$. Hence

$$\{XD, CB\} \times \{Y'E, AC\} \times \{Z'F, BA\} = -1. \tag{1}$$

Projecting the points Q, P, A, X from B onto CA and from C onto AB

$$\{Y'Y, AC\} = \{Z'Z, AB\}. \tag{2}$$

By means of a real projection, project DEF into the line at infinity of an

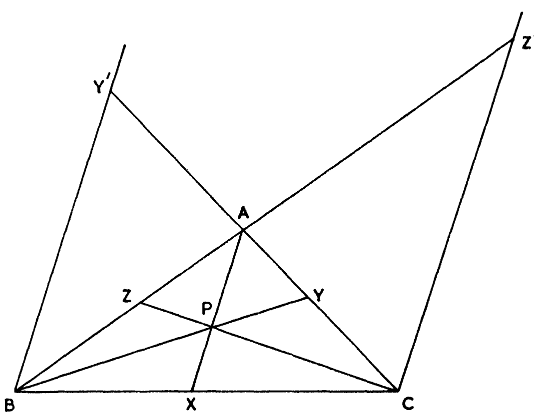

FIG. 2

extended Euclidean plane to obtain Figure 2. Equations (1) and (2) give in Figure 2 the relationships

$$\frac{XC}{XB}\cdot\frac{Y'A}{Y'C}\cdot\frac{Z'B}{Z'A} = -1,$$

$$\frac{Y'A\cdot YC}{Y'C\cdot YA} = \frac{Z'A\cdot ZB}{Z'B\cdot ZA} \quad \text{or} \quad \frac{Y'A}{Y'C}\cdot\frac{Z'B}{Z'A} = \frac{YA}{YC}\cdot\frac{ZB}{ZA}.$$

Hence we deduce Ceva's theorem in Figure 2

$$\frac{XC}{XB}\cdot\frac{YA}{YC}\cdot\frac{ZB}{ZA} = -1,$$

where the point of concurrency, P, is any finite point in the plane not on a side of the triangle.

The proof is still valid if, in Figure 1, DEF passes through Y but not through

Z or if through Y and Z. The resulting formulas in Figure 2, $XC\cdot ZB/XB\cdot ZA = -1$ and $XC/XB = -1$, merely express a relationship deducible directly from ratios formed by parallels and the bisection by a diagonal of a diagonal of a parallelogram respectively. If, relaxing an initial restriction, P lies on DEF the proof requires only a simple modification. The points Q, Y', Z' then coincide with P, Y, Z respectively in Figure 1. Equation (2) is then non-existent. Equation (1) is still valid and after the projection to Figure 2 leads at once to $XC\cdot YA\cdot ZB/XB\cdot YC\cdot ZA = -1$. P is then a point at infinity in Figure 2 and the formula can also be obtained from the ratios of parallels. In all of these three cases the usual proof of Ceva's theorem implies infinite areas and is not valid.

Returning to Figure 1, the concurrency of the lines AX, BY', CZ' in Q leads to the Ceva relation

$$\frac{XC}{YB}\cdot\frac{Y'C}{Y'A}\cdot\frac{Z'A}{Z'B} = -1.$$

The expanded form of equation (1) is

$$\frac{XC\cdot DB}{XB\cdot DC}\times\frac{Y'A\cdot EC}{Y'C\cdot EA}\times\frac{Z'B\cdot FA}{Z'A\cdot FB} = -1.$$

Combining the two expressions we have at once Menelaus' theorem

$$\frac{DB}{DC}\times\frac{EC}{EA}\times\frac{FA}{FB} = 1.$$

(We note here that if Ceva's theorem only is required, the restriction that DEF is a finite line is not needed. Also the restriction that P shall not lie on DEF is only required for procedure to Menelaus' theorem.)

We now discuss the question of unreal elements. Following the development given by Coolidge in his treatise *The Geometry of the Complex Domain*, unreal elements are fundamentally in the projective field. Some care however is needed for the relationships between points lying on an isotropic line. Isotropic lines are so called from their property that their behavior is the same with respect to all rectangular axes through their real point (Greek ἴσος equal, τρόπος direction or course. Compare the use of the word also in Physics and Biology). In the following discussion we shall slightly vary from Coolidge's notation.

Consider points on the line $y = R\iota x$ where the axes and R are real. We denote the points by the symbols 1, 2, 3, etc. and their x coordinates by $x_r + \iota x_r'$ ($r = 1, 2, 3$ etc.) where x_r, x_r' are real. The distance function for points 1, 2 is then ${}_1d_2 = \sqrt{1-R^2\{(x_2-x_1)^2 - (x_2'-x_1')^2 + 2\iota(x_2-x_1)(x_2'-x_1')\}}$, or $\sqrt{1-R^2}\{(x_2+\iota x_2') - (x_1+\iota x_1')\}$. Hence if $R \neq \pm 1$, the ratio between the distance functions ${}_1d_2$ and ${}_3d_4$ is independent of R. Varying R is effectively projecting the points from one line to other lines with vertex the point at infinity on the y-axis. This projection not only gives the usual invariant projective properties

among a range of points but also the invariance of the ratio between any two corresponding distance functions. For the isotropic lines through the origin $R=\pm 1$ and the distance function is always zero. If we accept the viewpoint that the invariances persist in these limiting cases, the restriction as to the reality of the elements of the configuration in the real plane can be completely relaxed.

A two-way linkage can also be obtained as follows, but the configuration does not contain an independent projective proof of either theorem. With the triangle ABC and concurrent lines APX, BPY, CPZ as before, let YZ meet BC in U, ZX meet CA in V and XY meet AB in W. Since the triangles ABC, XYZ are in perspective they are also coaxal and hence U, V, W are in a straight line. From the quadrangle $AYPZ$ the cross-ratio $\{XU, CB\} = -1$, and similarly for $\{YV, AC\}$, $\{ZW, BA\}$. Multiplying these together and rearranging in expanded form

$$\frac{XC}{XB}\cdot\frac{YA}{YC}\cdot\frac{ZB}{ZA} = -\frac{UC}{UB}\cdot\frac{VA}{VC}\cdot\frac{WB}{WA}$$

and the truth of Menelaus' theorem for the line UVW implies that of Ceva's theorem for any point P not on a side of the triangle. Similarly, using the dual (reciprocal) figure and proof, the truth of Ceva's theorem leads to that of Menelaus for any line not through a vertex of the triangle.

GRAPHICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS*

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1. Introduction. Lately,† graphical solutions have appeared for the linear differential equation

$$y' + P(x)\cdot y = Q(x). \tag{1}$$

The following procedure has certain practical and theoretical advantages over those that have appeared.

2. The method. First write the equation in the form

$$y' = (B(x) - y)/(A(x) - x), \tag{2}$$

where $A(x) = x + 1/P(x)$, $B(x) = Q(x)/P(x)$.

It is easily seen from equation (2) that y' is the slope of the straight line through the two points $(A(x), B(x))$ and (x, y). This fact makes use of the "parallel ruler" unnecessary.

Set up a coordinate system showing the initial point, $P=(x_0, y_0)$, through which the solution must pass. Rule off a number of strips, parallel to the y-axis,

* Presented to a meeting of the mathematicians of the Northwest, Vancouver, B. C., April 2, 1939.

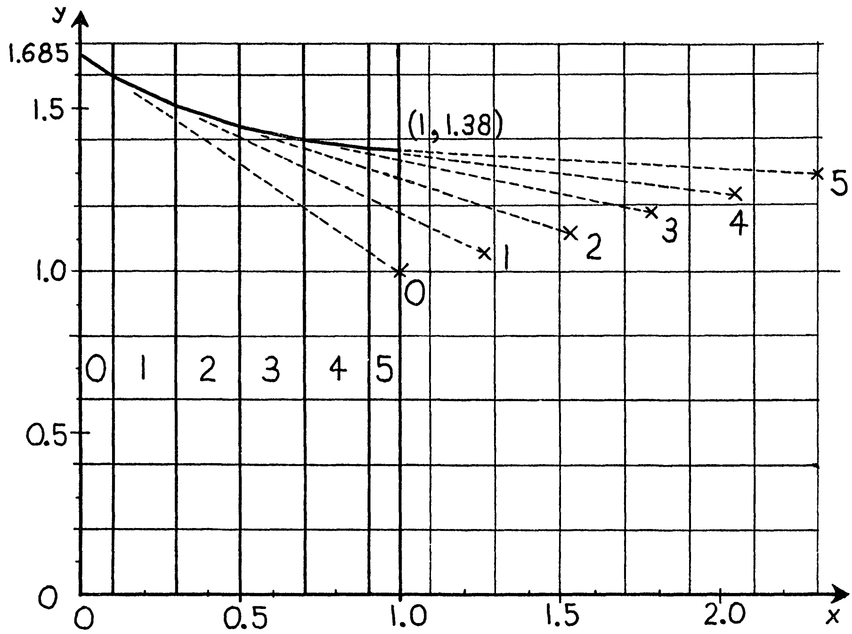
† M. E. Levenson, On a graphical solution, this MONTHLY, vol. 63, 1956, pp. 115–116.

and of width w. Let the initial strip, the one numbered 0, have its center along the line $x=x_0$. The center of the jth strip will be the line $x=x_j=x_0+jw$. To save ruling lines, use a graph paper on which the scale is so chosen that the necessary rulings are already on the paper.

To each strip will belong a point, (A, B). The point belonging to the jth strip is (A_j, B_j), where $A_j=A(x_j)$ and $B_j=B(x_j)$. Let $y=f(x)$ be any solution of differential equation (1). Then the tangent to $y=f(x)$ drawn at the center of any strip passes through the point belonging to that strip. It is helpful to number the strips and the corresponding points.

3. The solution. When the strips and corresponding points have been laid off, the solution for any given initial conditions, $x=x_0$, $y=y_0$, is immediate. For simplicity, let $P_0=(x_0, y_0)$ be at the center of strip 0. Start with your ruler passing through P_0 and (A_0, B_0). Guided by the ruler, draw the straight line through P_0 to the nearer edge of Strip 1. Taking care not to remove the pencil from the paper, turn the ruler so that it still rests against the pencil, but now passes through (A_1, B_1). Now extend the line you have started, completely across Strip 1. Similarly, as you cross Strip 2, the ruler is in line with the point (A_2, B_2) belonging to that strip.

It takes only a few seconds to extend the "solution" across each strip, so that the entire solution is very quickly found, even if the strips are narrow and numerous.



4. Example. $y'+(1+0.3x)^{-1}y=1$.

Solution. First put the differential equation in the form (2),

$$y' = ((1 + 0.3x) - y)/((1 + 1.3) - x).$$

Thus, $A(x)=1+1.3x$ and $B(x)=1+0.3x$.

The diagram shows Strips 1, 2, 3, 4, each of width 0.2, the last half of Strip 0 and the first half of Strip 5. This will take the solution up to $x=1$.

The values of $A(x)$ and $B(x)$ are then tabulated:

n	0	1	2	3	4	5
x	0.0	0.2	0.4	0.6	0.8	1.0
$A(x)$	1.00	1.26	1.52	1.78	2.04	2.30
$B(x)$	1.00	1.06	1.12	1.18	1.24	1.30

The points (A_j, B_j) are then plotted and numbered.

The diagram shows how the solution, really a broken line, starts at (0, 1.685), with each link in line with the corresponding point (A_j, B_j). It ends at about (1, 1.38).

The theoretical solution is slightly complicated,

$$y = (10/13)(1 + 0.3x) + 0.91577(1 + 0.3x)^{-10/3},$$

and for $x=1$, $y=1.3819$.

I have found this method very useful, not only for solving linear differential equations, but also for plotting such curves as $y=Ke^{rx}$ and $y=Cx^n$ for given values of K, r, C, and n. Here $y=Ke^{rx}$ is a solution of $y'=ry=(0-y)/((x-1/r)-x)$, and $y=Cx^n$ is a solution of $y'=ny/x=(0-y)/((1-1/n)x-x)$. In each case, $B(x)\equiv 0$ and $A(x)$ is a simple expression.

THE NAMES OF THE CURVE OF AGNESI*

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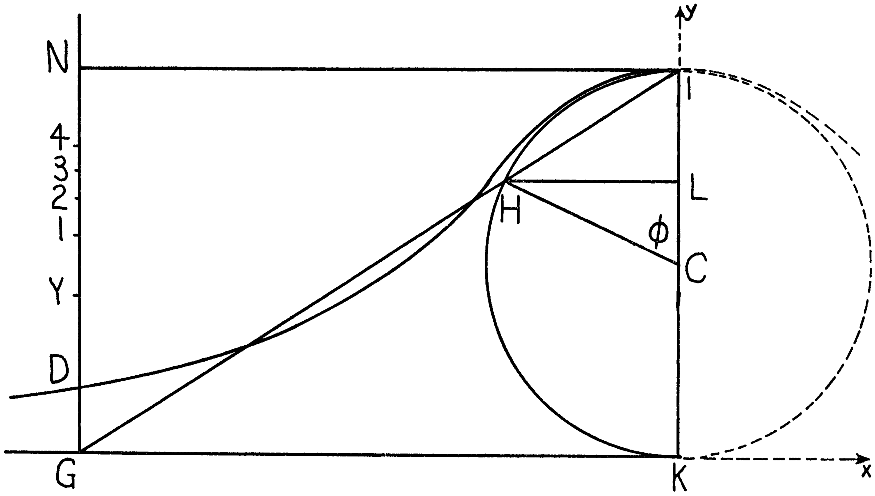
As Vacca first noted [1] it is to the Camaldolese monk, Guido Grandi, that we owe the naming of the locus $x^2y=a^2(a-y)$ which is associated with the name of Maria Gaetana Agnesi (1718–1799). Grandi, in his book ([2], p. 15), gave the name *Scala*, the scale curve, to this locus because it can serve as a measure of light intensity, and in the same work (p. 5, Propositio III) he wrote, "Given a semicircle of diameter IK, $\cdots$ the tangent KG, $\cdots$ (and) IG intersecting

* Presented to the Louisiana-Mississippi Section of the Mathematical Association of America, February 17, 1956.

the periphery at H. This determines the sine HL. Let $(GK)^2$ be to $(KI)^2$ as the diameter is to YN, and this to $1N$. In this way is had the infinity of terms $2N$, $3N$, $4N$, etc. I affirm that the sum of all the differences of these terms taken



alternately $Y1, 23, 45$, etc. equals the versed sine IL of the intercepted arc IH." Taking $\angle ICH=\phi$, $KI=a$, $KG=x$, with $GD=LI=y$ ([2], p. 7, Propositio IV), this means $YN=a^3/x^2$, $1N=a^5/x^4, \cdots$; $Y1=a^3/x^2-a^5/x^4, \cdots$; and $IL=a^3/(x^2+a^2)=a[(a/x)^2-(a/x)^4+(a/x)^6- \cdots]$

$$= (a/2)(1-\cos\phi) = (a/2)\text{ vers }\phi. \tag{1}$$

(If $a=2$ we have the unit circle representation of vers ϕ.)

In 1718, Grandi returns to this curve, now as a "scale of velocities . . . that curve which I describe in my book of quadratures, proposition 4, derived from the versed sine, which I am wont to call the *Versiera* but in Latin (is) *Versoria*." [3]. In (1) Grandi had sufficient justification for this terminology. I find no evidence that he was motivated by the fact that in literary Latin *versoria* denotes "a rope that guides a sail," and *sinus* may mean "the bend or belly of a sail swollen by the wind."

The work of Grandi (and earlier mention by Fermat, and later by Newton) did not attract much interest to the curve. It came to the notice of mathematicians principally through the influential *Instituzioni analitiche* (1748) of Agnesi, the first volume of which was translated into French in 1775. A complete English translation was made by J. Colson of Cambridge in 1801. It is due to this important work of Agnesi that her name became associated with the curve [4].

What is the origin of the name "Witch of Agnesi"? The substantive *versiera*

is a synonym for the substantive adjective *versoria*, "turning in every direction," a word derived from the Latin *vertere*, "to turn." In the course of time *versiera* took on another meaning in this way. The Latin words *adversaria*, and by aphaeresis, *versaria*, signify a female who is contrary, an adversary; and in Ecclesiastical Latin one of the added meanings was a female who is contrary to God. Thus, even in the literary Latin of the Middle Ages, the word *versiera* came to be applied, although comparatively rarely, to the one par excellence who is contrary to God, that is, the devil: "a female fiend or goblin," "the devil's grandmother," and other related meanings, the equivalent of the English word witch.

Although Colson was presumably the first to use the word witch in connection with this curve, it should not be supposed that the sinister implication consequent on the use of this word is to be attributed to a mistaken or facetious translation. In his *Analytical Institutions* of Agnesi (1801, vol. 1, p. 222) we read: " . . . the curve to be described, which is vulgarly called the Witch." This translation departs from the original which has: " . . . which is called *Versiera*" [5]. Thus it appears that as early as 1801 the technical meaning of *Versiera*, intended by Grandi and Agnesi, had a competitor in Italy among mathematicians in the nontechnical, ecclesiastical connotation of the term.

When we advert to the sinister implication of the word witch, especially when associated with the name of a particular woman, its inappropriateness, even as a mathematical term, is apparent. Far from honoring Agnesi, the ominous term would seem rather to discredit the life and work of a most remarkable and honorable woman—a truly remarkable linguist, mathematical author, correspondent and translator of scholars, serious student of philosophy and theology, and zealous hospital nun.

It is suggested that American and English mathematicians, in their desire to honor the memory of Agnesi, abandon the use of the term "Witch of Agnesi," adopting instead the practice of the French (courbe d'Agnesi) and the Germans (Agnesische Kurve), writing simply "the curve of Agnesi."

The author gratefully acknowledges the generous assistance of the referee and of the Rev. J. F. Moore, S.J.

References

1. G. Vacca, Sulla versiera, Bolletino di bibliografia e storia delle scienze matematiche, Turin, vol. 4, 1901, pp. 33–34.

2. G. Grandi, Quadratura Circuli, et Hyperbolae Per Infinitas Hyperbolas, et Parabolas Quadrabiles Geometrice exhibita et demonstrata, 2nd ed. Pisa, 1710, p. 15. (I understand that this material is also in the first edition (1703) of the work.)

3. ———, Opere di Galileo Galilei; Note al Trattato del Galileo del moto naturalmente accelerato, Florence, 1718, Tome 3, p. 393, Corollario VI: " . . . nata da'seni versi, che da me suole chiarmarsi la *Versiera* in latino però *Versoria*."

4. J. L. Coolidge, Six female mathematicians, Scripta Math., vol. 17, 1951, pp. 20–31.

5. M. G. Agnesi, Instituzioni analitiche ad uso del la gioventù italiana, Milan, 1748, Tome 1, p. 280, Problema III. 238: " . . . alla curva da discriversi, che dicesi la Versiera."

6. Hugo Beran, The witch of Agnesi, Scripta Math., vol. 8, 1941, p. 135.

ON DEGENERATE CONICS

J. W. LASLEY, JR., University of North Carolina

Introduction. One learns in elementary analytic geometry that the real conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

is elliptic, hyperbolic, parabolic, according to whether the discriminant $D = ab - h^2$ of the quadratic terms is positive, negative, or zero.

One learns further that the conic is composite (degenerate) if, and only if, the discriminant of the conic

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is, or is not, zero.

Metric transformations. It is pointed out that all conics except the proper parabola have at least one center, and that by translation to a center the second degree terms are preserved and the linear terms are deleted. By following this translation by a suitable rotation, one may then remove the product term. Every conic, except the proper parabola, may in this manner be reduced to the simple (canonical) form

$$ax^2 + by^2 + c = 0 \tag{2}$$

of a modified sum of squares.

Projective transformations. In projective geometry it is pointed out that by the mere device of choosing a self-polar triangle as the triangle of reference, all conics, including the proper parabola, are capable of being transformed into form (2) in this way.

An orthogonal transformation. The transformation employed in the projective case is not usually orthogonal. This paper purports to review the case for the degenerate conics, and to show that an orthogonal transformation can be devised to reduce the degenerate conic to form (2) by a single transformation, which will provide a little used criterion for the determination of the sort of degenerate conic it is.

Let us ask whether we can find a transformation

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z', \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z', \end{aligned} \tag{3}$$

where the conditions

$$(4)\qquad \begin{aligned} a\lambda + h\mu + g\nu &= k\lambda, \\ h\lambda + b\mu + f\nu &= k\mu, \\ g\lambda + f\mu + c\nu &= k\nu, \end{aligned}$$

hold for each set $(\lambda_i, \mu_i, \nu_i)$, $(i=1, 2, 3)$ and an appropriate $k_i=1, 2, 3$, one for each set.

In order to address ourselves to this question, let us consider the conic in the homogeneous form

$$(5)\qquad F = ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$$

and solve the characteristic equation

$$(6)\qquad \begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix} = 0$$

for $k = k_1, k_2, k_3$. These values k satisfy (6) and make (4) consistent. The solutions $(\lambda_i, \mu_i, \nu_i)$ from (4), one set for each k_i, may or may not make (3) orthogonal. If they do not, they can be made to do so by dividing each set by the square root of the sum of the squares of the three numbers in the set, and taking the numbers so obtained for the coefficients in (3).

Invariants. Under transformation (3), now orthogonal, F is an invariant (covariant). So also is $G=x^2+y^2+z^2$. The linear combination $F-kG$ is a further covariant, which because there is no trace of (3) in k, makes the coefficients of the characteristic equation

$$(7)\qquad k^3 - \sum a\cdot k^2 + \sum A\cdot k - \Delta = 0$$

invariant also. We thus have

$$(8)\qquad \sum a' = \sum a, \qquad \sum A' = \sum A, \qquad \Delta' = \Delta.$$

But the transformation (3) applied to (5) reduces it to

$$(9)\qquad a'x'^2 + b'y'^2 + c'z'^2 = 0$$

where a', b', c' are the solutions k of (7). This applied to (8) gives

$$(10)\qquad \sum A = b'c' + a'c' + a'b', \qquad \Delta = a'b'c'.$$

Degenerate conics. If our conic is degenerate, $\Delta=0$. If $c'=0$, then $\sum A = a'b'$ and (9) becomes

$$(11)\qquad a'x'^2 + b'y'^2 = 0$$

where a' and b' are solutions of

$$(12)\qquad k^2 - \sum a\cdot k + \sum A = 0.$$

It follows at once that if $\sum A$ is positive, a' and b' have the same sign and the lines given by (11) are conjugate complex. If $\sum A$ is negative, the lines given by (11) are real and distinct; if $\sum A$ is zero, the lines are real and coincident.

Now because of the symmetry present in (5), (8), and (9), the case $b'=0$ (or $a'=0$) is not essentially different from the foregoing case in which $c'=0$. For example, if $b'=0$, $\sum A=a'c'$, equation (11) becomes $a'x'^2+c'z'^2=0$. The conclusions stated at the end of the preceding paragraph still hold; only this time in metric cases we have conjugate complex parallel lines, coincident ideal lines, and real and distinct parallel lines as new features—geometric loci hardly obtainable as conic sections, but certainly possible graphs of equation (5), if not of equation (1). Thus, for the conventional conics we take $c'=0$. In this case there is no distinction between D and $\sum A$, since both are $a'b'$. Symmetrically, if $b'=0$, $\sum A$ is the same as B; if $a'=0$, $\sum A$ is the same as A. In all cases, $\sum A$ plays the role of the discriminant of a quadratic form in two variables. In all cases, the conclusions reached above obtain: If $\sum A$ is positive, the lines are conjugate complex; if negative, the lines are real and distinct; if zero, the lines are coincident. This with the understanding that equations of the form $px+qy+rz=0$ represent a straight line, provided p, q, r are constants and $(p, q, r) \neq (0, 0, 0)$.

Summary. Thus we see that for the determination of the type of degenerate conics we have three criteria. The discriminant of the quadratic form $D=ab-h^2$ for telling the conics from which the degenerate conics degenerate. If D is positive, we have elliptic lines; for example, $x^2+y^2=0$. If D is negative, we have hyperbolic lines; for example, $xy=0$. If D is zero we have parabolic lines; for example, $x^2=0$, $x^2-1=0$, $x^2+1=0$.

The rank r of the conic tells us whether the lines are distinct (rank 2) or coincident (rank 1); for example, $xy=0$, $x^2+y^2=0$, $x^2-1=0$, $x^2+1=0$ are distinct lines. No considerations of reality are made here. If $r=1$, the lines are coincident; for example, $x^2=0$.

In the case of the criterion $\sum A$, the trace of the adjoint matrix of the conic: if $\sum A$ is positive, we have a pair of complex lines; for example, $x^2+y^2=0$, $x^2+1=0$. If $\sum A$ is negative we have a pair of real and distinct lines; for example, $xy=0$, $x^2-1=0$. If $\sum A$ is zero, we have a repeated real line; for example, $x^2=0$.

The trace criterion thus distinguishes conjugate complex lines, for which $\sum A$ is positive, from real and distinct lines, for which $\sum A$ is negative. It distinguishes distinct lines, for which $\sum A \neq 0$, from repeated lines, for which $\sum A=0$. Moreover, it distinguishes conic sections—those which may be actually cut from the cone; for example, $xy=0$, $x^2+y^2=0$, $x^2=0$, from the graphs of the equation of the second degree—to which must be added degenerate conics such as $x^2-1=0$, $x^2+1=0$, not obtainable by cutting a cone.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1266. *Proposed by D. C. B. Marsh, Colorado School of Mines*

Solve

$$a^3 - b^3 - c^3 = 3abc, \qquad a^2 = 2(b + c)$$

simultaneously in positive integers.

E 1267. *Proposed by Ivan Niven, University of Oregon*

The divergence of the harmonic series $\sum 1/n$ is often established by comparison with the obviously divergent series $\sum f(n)$ where $f(n) = 2^{-k}$, the integer k being defined by the inequality $2^k \geqq n > 2^{k-1}$. Establish the convergence or divergence of the series $\sum(1/n - f(n))$.

E 1268. *Proposed by A. J. Goldman and P. S. Wolfe, Princeton University*

Evaluate the determinant D_n which has $(1, 2, \cdots, n)$ as first row, $(2, 3, \cdots, n, 1)$ as second row, *etc.*

E 1269. *Proposed by Frank Kocher, Pennsylvania State University*

Prove that the area under one arch of the curve generated by a vertex of a regular polygon rolling on a straight line is equal to the area of the polygon plus twice the area of its circumscribed circle.

E 1270. *Proposed by Leo Moser, University of Alberta*

What is the smallest positive even integer n such that in both n and $n+1$ dimensions the regular simplex of edge 1 will have a rational number as its content. (Dedicated to Professor H. S. M. Coxeter.)

SOLUTIONS

A Linear Diophantine Equation

E 1236 [1956, 664]. *Proposed by Hazel E. Evans, University of Pittsburgh*

For $a > b$ and $N < ab$ find the maximum value of N for which the equation

$$ax + by = N$$

has a solution in non-negative integers.

Solution by E. D. Schell, Remington Rand Univac, New York. We assume that it is intended that a and b represent positive integers. Now suppose $K=\max N$ subject to the stated conditions. Then $ax+by=K$, and (a, b) divides K. But the largest $N<ab$ for which this could be true is $ab-(a, b)$.

Set $-(a, b)=\alpha a-\beta b$, where $\alpha<b$, $\beta<a$, and $\alpha, \beta>0$, by using the Euclidean algorithm. Add ab to each side, obtaining $ab-(a, b)=\alpha a+(a-\beta)b$. Then $x=\alpha$ and $y=a-\beta$ are non-negative solutions for K.

Also solved by D. A. Breault, J. C. W. De la Bere, Underwood Dudley, A. R. Hyde, Sidney Kravitz, D. C. B. Marsh, J. B. Muskat, E. N. Nilson, W. L. Ostrowski, Azriel Rosenfeld, D. J. Schaefer, and the proposer. Late solutions by J. W. Harter, J. H. Hodges, and R. H. Hou.

Editorial Note. If a and b are taken to represent *any* integers, then the following facts can be established: (1) if $a>b=0$, there is no solution; (2) if $a>0>b$, then $K=ab-(a, b)$; (3) if $0=a>b$, then $K=b$; (4) if $0>a>b$, then $K=0$.

A Pair of Line Integrals

E 1237 [1956, 664]. *Proposed by Viktors Linis, University of Ottawa*

Let E be an ellipse, r_1 and r_2 focal radii, α the angle between the focal radii, and ds the element of arc. Evaluate the integrals

$$\int_E ds/(r_1r_2)^{1/2} \quad \text{and} \quad \int_E (\cos \alpha/2)ds.$$

Solution by Chih-yi Wang, University of Minnesota. Let the parametric representation of E be $x=a\cos\theta$, $y=b\sin\theta$, $a>b>0$, $0\leqq\theta<2\pi$. Then we have

$$ds = [a^2 - (a^2 - b^2)\cos^2\theta]^{1/2}d\theta,$$
$$r_1 = a - (a^2 - b^2)^{1/2}\cos\theta,$$
$$r_2 = a + (a^2 - b^2)^{1/2}\cos\theta,$$

whence

$$\int_E ds/(r_1r_2)^{1/2} = \int_0^{2\pi} d\theta = 2\pi.$$

For the second integral, we make use of the cosine law and the half angle formula to obtain

$$\cos\alpha/2 = b/[a^2 - (a^2 - b^2)\cos{}^2\theta]^{1/2},$$

whence

$$\int_E (\cos\alpha/2)ds = \int_0^{2\pi} b\,d\theta = 2\pi b.$$

Also solved by J. C. W. De la Bere, David Freedman, A. R. Hyde, J. B. Johnston, M. S. Klamkin, C. S. Ogilvy, C. D. Olds, L. A. Ringenberg, Jeff Ritterman, Azriel Rosenfeld, Nathan Shklov, A. V. Sylwester, David Zeitlin, and the proposer.

Three Consecutive Powers of 3

E 1238 [1956, 665]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine integral values of $n>0$ such that 3^n, 3^{n+1}, 3^{n+2} all have the same number of digits in their denary expansions.

Solution by Joe Lipman, University of Toronto. If n is an integer such that 3^n, 3^{n+1}, 3^{n+2} all have the same number of digits in their denary expansions, then

$$\underbrace{10000\cdots}_{k\text{ digits}} < 3^n < \underbrace{11111\cdots}_{k\text{ digits}}.$$

Now the mantissa of $\log 11111\cdots$ is $0.0457574\cdots$. If the inequality is satisfied, $n\log 3=$ an integer $+$ a decimal fraction between zero and 0.0457574. But $\log 3=0.477121256$, which is just slightly greater than 1/21. Therefore we can expect the n's to recur at intervals of 21 or 23. Thus we have

n	$n\log 3$
21	10.019546376
42	20.039092152
65	30.01288163
86	41.03242802
109	$52.00621\cdots$
130	$62.02575\cdots$
151	$72.04529\cdots$
174	83.0190985

A comparison of 174 and 21 shows that the corresponding mantissae differ by only 0.0004478. This is because $153\log 3=72.9995522$. Thus any number of the form $21+153k$, where $k<19546376/4478=43.8\cdots$ will be one of the required n's. For $k=44$, the resulting mantissa is 0.9998432. Subtracting $0.9542425=2\log 3$, we get the mantissa 0.0456007. So instead of $21+153(44)$, use $21+153(44)-2=6751$. Now using the sequence $6751+153k_1$, where $k_1<456007/4478$, repeat the above process to get more n's. Then derive the new sequence $22355+153k_2$, and so on. In this way arbitrarily large n's can be determined as long as tables of sufficient accuracy are available.

Also solved by J. C. W. De la Bere, Monte Dernham, Hazel E. Evans, Michael Goldberg, A. R. Hyde, I. M. Isaacs, Sidney Kravitz, D. C. B. Marsh, Herbert Nadler, C. S. Ogilvy, D. S. Passman, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, G. W. Walker, and the proposer.

Two Related Quadrangles

E 1239 [1956, 665]. *Proposed by Josef Langr, Prague, Czechoslovakia*

Let $Q' \equiv A'B'C'D'$ be the quadrangle formed by the orthocenters A', B', C', D' of triangles BCD, CDA, DAB, ABC of a given convex quadrangle $Q \equiv ABCD$. Show that: (1) the vertices of Q and Q' lie on a common equilateral hyperbola, (2) Q and Q' have equal areas.

Solution by D. C. B. Marsh, Colorado School of Mines. The vertices of Q determine an equilateral hyperbola, which may be taken as $xy=1$ by superimposing a properly scaled coordinate system upon it. We label coordinates as A: $(a, 1/a)$, B: $(b, 1/b)$, C: $(c, 1/c)$, D: $(d, 1/d)$. It is a simple matter to find the orthocenters A': $(-1/bcd, -bcd)$, B': $(-1/acd, -acd)$, C': $(-1/abd, -abd)$, D': $(-1/abc, -abc)$, which are obviously co-hyperbolic with A, B, C, D, and (1) is established.

Assuming A, B, C, D are the vertices of Q in order, the area of Q is given by the sum of the absolute values of

$$(1/2)\begin{vmatrix} a & 1/a & 1 \\ b & 1/b & 1 \\ c & 1/c & 1 \end{vmatrix} \quad \text{and} \quad (1/2)\begin{vmatrix} c & 1/c & 1 \\ d & 1/d & 1 \\ a & 1/a & 1 \end{vmatrix}.$$

Multiplying the first columns of both determinants by $-1/abcd$ and the second columns by $-abcd$ does not change the numerical value, but the form becomes that of the area of Q', demonstrating (2).

Also solved by K. W. Crain, J. C. W. De la Bere, C. S. Ogilvy, O. J. Ramler, Sister M. Stephanie, and the proposer.

Editorial Note. The vertices of a convex quadrangle determine a *nondegenerate* equilateral hyperbola unless the line through one pair of vertices is perpendicular to the line through the other pair. This exceptional case is easily treated either on its own merits or as a limiting situation of the general case treated above.

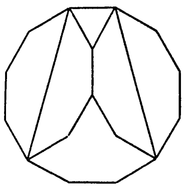
Two Six-piece Dissections

E 1240 [1956, 665]. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Find six-piece dissections of a regular dodecagon into a square and into a Greek cross.

Solution by the proposer. It is readily verified that a chord subtending four sides of a regular dodecagon is equal to a side of the equivalent square. There are numerous six-piece dissections based on this fact. Those shown in Figures

1 and 2 are perhaps the neatest.



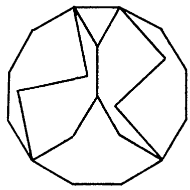
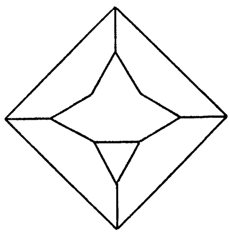
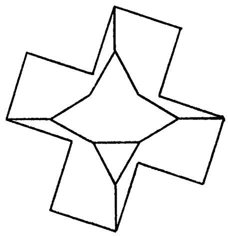


FIG. 1 FIG. 2





These dissections were found by a general method described in *The Australian Mathematics Teacher*, vol. 7, 1951, pp. 7–10, vol. 9, 1953, pp. 17–21, 64.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4738. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

If, for all positive x, $\sum_{k=1}^{\infty} |F(kx)| < \infty$ and $\sum_{k=1}^{\infty} F(kx) = 0$, then $F(x)$ vanishes identically.

4739. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose C is a closed convex subset of the Euclidean space E^3 whose boundary is a regular octahedron, and that C_1, C_2, and C_3 are translates of C (i.e.,

$C_i = C + x_i$ for some $x_i \in E^3$). Then, if each of the intersections $C_1 \cap C_2$, $C_2 \cap C_3$, and $C_3 \cap C_1$ is non-empty, must $C_1 \cap C_2 \cap C_3$ be non-empty?

4740. *Proposed by R. J. Dickson, Lockheed Aircraft Corporation, Burbank, California*

Is every locally schlicht analytic mapping of the complex plane onto itself a schlicht mapping?

4741. *Proposed by L. A. Rubel, Institute for Advanced Study*

Prove or disprove the statement: If a metric space S is homeomorphic to its completion, then S is complete.

4742. *Proposed by Joshua Barlaz, Rutgers University*

Evaluate the Cesàro first order mean for the series $\sum_{n=2}^{\infty} (-1)^n \log n$.

SOLUTIONS

Functions Restrained by an Integral Inequality

4660 [1955, 659]. *Proposed by E. M. Wright, University of Aberdeen, Scotland*

For all $x \geqq 1$, $f(x)$ and $\phi(x)$ are non-negative functions, bounded and integrable in any finite interval. They satisfy the inequality

$$xf(x) \leqq \int_1^x f(t)dt + \phi(x).$$

(i) If $\int^\infty \phi(t)t^{-2}dt = \infty$, find an $f(x)$ such that $f(x) \to \infty$ as $x \to \infty$.

(ii) If $\phi(x)/x \to 0$ and $\int^\infty \phi(t)t^{-2}dt < \infty$, show that

$$g(x) = \frac{1}{x}\int_1^x f(t)dt$$

tends to a limit as $x \to \infty$ and that $\lim g(x) = \lim\sup f(x)$.

(iii) Show that, whatever restriction we may impose on the order of $\phi(x)$ as $x \to \infty$, we cannot thereby ensure that $f(x)$ tends to a limit.

Solution by R. O. Davies, University College, Leicester, England. Let (E) denote the inequality.

(i) If $f(x) = \phi(x)x^{-1} + \int_1^x \phi(t)t^{-2}dt$, then $f(x) \to \infty$, and integration by parts shows that (E) holds with equality.

(ii) Integration by parts shows that

$$g(x) = \int_1^x \left[x^{-1}f(x) - x^{-2}\int_1^x f(t)dt\right]dx.$$

Hence, for $x_1<x_2$, using (E) multiplied through by x^{-2}, we have

$$(1)\qquad g(x_2)-g(x_1)=\int_{x_1}^{x_2}\left[x^{-1}f(x)-x^{-2}\int_1^x f(t)dt\right]dx \leqq \int_{x_1}^{x_2}\phi(t)t^{-2}dt.$$

Consequently, $g(x)$ is bounded above (as well as below, by zero). It now follows that $g(x)$ tends to a limit. For otherwise by choosing a large value of x_1 for which $g(x_1)$ was near its lower limit and a larger value of x_2 for which $g(x_2)$ was near its upper limit we could obtain from (1) a contradiction to the convergence of $\int\phi(t)t^{-2}dt$.

That $\lim g(x)\leqq\limsup f(x)$ is a standard result, and the reverse inequality follows from (E), since $\phi(x)x^{-1}\to 0$.

(iii) Let $f(x)=x^{-1}+1$. For all large x we have

$$xf(x)=1+x<\log x-1+x=\int_1^x f(t)dt,$$

and so (E) will be satisfied with a $\phi(x)$ which is zero for all large x. Without violating (E) we may now destroy the convergence of $f(x)$ by changing its value to zero for (say) all large integer values of x; or, if we wish, throughout small intervals surrounding them, since there is strict inequality in (E).

Also solved by R. P. Boas, Jr., and the proposer.

Zeros in a Triple Diagonal Matrix

4681 [1956, 191]. *Proposed by Jack Klugerman, Evans Signal Laboratory, Belmar, N. J.*

Given a real symmetric matrix A which is triple diagonal, *i.e.*, it has a diagonal, an upper diagonal, a lower diagonal, and the remaining elements are zero; if c is the eigenvalue with highest multiplicity m, then there must be at least $m-1$ zeros in the upper diagonal.

Solution by N. J. Fine, University of Pennsylvania. Since A is real symmetric, its eigenvalues span R^n, so the subspace V corresponding to c has dimension m. We can find a basis $v_1, \cdots, v_m$ for V such that $(v_i, e_k)=0$ for $k\leqq k_i$, $(v_i, e_{k_i+1})\neq 0$, where the e_k are unit vectors and $1\leqq k_2<k_3<\cdots<k_m$. We then have, for $i=2,\cdots,m$,

$$\begin{aligned} 0 &= c(v_i, e_{k_i}) = (Av_i, e_{k_i}) = (v_i, Ae_{k_i}) \\ &= (v_i, a_{k_i,k_i-1}e_{k_i-1}+a_{k_i,k_i}e_{k_i}+a_{k_i,k_i+1}e_{k_i+1}) \\ &= a_{k_i,k_i+1}(v_i, e_{k_i+1}). \end{aligned}$$

Thus the $m-1$ upper diagonal elements a_{k_i,k_i+1} are zero. (For a discussion of the concepts used here see, *e.g.*, MacDuffee, *Vectors and Matrices.*)

Also solved by Harley Flanders, Wallace Givens, W. V. Parker, and the proposer.

Non-rectifiable Simple Closed Curve

4687 [1956, 259]. *Proposed by John Wermer, Brown University*

Let Γ be a simple closed curve in the complex plane containing the origin in its interior. Show that if Γ is not rectifiable, then we can approximate the constant 1 uniformly on Γ by functions $\sum_{n=-N}^{N} c_n z^n$, $c_0=0$.

Solution by the proposer. Let ϕ map $|\lambda|<1$ conformally on the interior of Γ with $\phi(0)=0$. Suppose we cannot approximate the constant 1 in the indicated fashion. Then we cannot approximate 1 uniformly on the unit circle by functions $\sum_{-N}^{N} c_n\phi^n(\lambda)$, $c_0=0$. By a classical theorem of F. Riesz, there then exists a measure μ on the unit circle with $\int_{|\lambda|=1}\phi^n(\lambda)d\mu(\lambda)=0$, $n\neq 0$, $\int_{|\lambda|=1}d\mu(\lambda)=1$. The measure $\phi(\lambda)d\mu(\lambda)$ is then orthogonal to all functions $P(\phi(\lambda))$ where P is a polynomial. Since ϕ is schlicht, functions $P(\phi(\lambda))$ approximate uniformly on $|\lambda|=1$ to each function f which is continuous in $|\lambda|\leqq 1$ and analytic in $|\lambda|<1$. Hence $\phi(\lambda)d\mu(\lambda)$ is orthogonal to all such functions. By another theorem of F. (and M.) Riesz, this implies that $\phi(\lambda)d\mu(\lambda)=h(\lambda)d\lambda$, where $h(\lambda)$ is the boundary value of a function $h(z)$ analytic in $|z|<1$ and with $\int_0^{2\pi}|h(re^{i\theta})|d\theta$ bounded as $r\to 1$.

On the other hand, if ϕ' is the derivative of ϕ, then

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\phi^n(\lambda)\phi'(\lambda)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=r}\frac{d}{d\lambda}\left\{\frac{\phi^{n+1}(\lambda)}{n+1}\right\}d\lambda = 0, \qquad n\neq -1$$

and

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\phi^{-1}(\lambda)\phi'(\lambda)d\lambda = 1$$

for each $r<1$. Now also

$$\int_{|\lambda|=r}\phi^n(\lambda)h(\lambda)d\lambda = 0, \quad n\neq -1, \qquad \int_{|\lambda|=r}\phi^{-1}(\lambda)h(\lambda)d\lambda = 1.$$

Hence for all n

$$\int_{|\lambda|=r}\phi^n(\lambda)\left\{h(\lambda)-\frac{1}{2\pi i}\phi'(\lambda)\right\}d\lambda = 0.$$

But the functions $\{\phi^n(\lambda)\}_{-\infty}^{\infty}$ are uniformly dense on $|\lambda|=r$ by a theorem of Walsh. Hence $h(\lambda)=\phi'(\lambda)/2\pi i$ on $|\lambda|=r$. This is true for each $r<1$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|\,rd\theta = \int_0^{2\pi}|h(re^{i\theta})|\,rd\theta$$

is bounded as $r\to 1$. Let Γ_r denote the image under ϕ of $|\lambda|=r$. Then the lengths of the Γ_r are uniformly bounded as $r\to 1$ by the preceding and, also, Γ_r converges to Γ as $r\to 1$. Hence Γ is of finite length. Hence, if Γ is not rectifiable, the approximation must be possible.

Sums of Distinct Divisors

4688 [1956, 346]. *Proposed by A. H. Clifford, Tulane University*

What positive integers n have the property that every positive integer less than n is expressible as the sum of distinct divisors of n?

I. *Solution by Virginia S. Hanly, Ohio State University.* Let the prime factorization of n be $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, $p_1<p_2<\cdots<p_r$. In order that every positive integer less than n be expressible as a sum of distinct positive divisors of n, it is necessary and sufficient that $p_1=2$, $p_{i+1}-1=\sigma(p_1^{a_1}p_2^{a_2}\cdots p_i^{a_i})$ for $i=1, 2, \cdots, r-1$, where $\sigma(k)$ denotes the sum of all positive divisors of k. The necessity is obvious.The proof of sufficiency is by induction on r, observing first that if b and c are positive integers, $c\geqq b-1$, then the variable $x_0+x_1b+\cdots+x_kb^k$, $0\leqq x_i\leqq c$, assumes each of the values $0, 1, \cdots, c(b^{k+1}-1)/(b-1)$. Now let the proposed set of conditions be valid for $q_r=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$. We assert that every positive integer not greater than $\sigma(q_r)$ is expressible as a sum of distinct divisors of q_r. The assertion is clearly true for $r=1$. Any number $x_0+x_1p_r+\cdots+x_{a_r}p_r^{a_r}$, where each x_i is a sum of distinct divisors of q_{r-1}, is a sum of distinct divisors of q_r. Thus, if our assertion is true for $r-1$ it is true for r.

II. *Note by Bernard Jacobson, Michigan State University.* If positive divisors only are to be used, then the complete solution is given by B. M. Stewart (*Sums of distinct divisors*, Amer. J. Math. vol. 76, 1954, pp. 779–785, Corollary 1). If it is permitted to use also negative divisors, then a similar analysis will show that the numbers n have the prime factorization

$$n = 2^b3^c\prod_{i=1}^{k} p_i^{a_i}, \qquad 3 < p_1 < \cdots < p_k,$$

subject to the conditions $p_1-1\leqq 2\sigma(2^b3^c)$, $p_j\leqq 2\sigma(2^b3^c\prod_{i=1}^{j-1}p_i^{a_i})$ for $j=2, \cdots, k$, and b and c are not both zero. (This result was communicated to the American Mathematical Society, 1956. See *Abstract* 407, Bull. Amer. Math. Soc. vol. 62, 1956, p. 351.)

Also solved by A. S. Davis, J. P. Mayberry, D. C. B. Morse, P. P. Saworotnow, and the proposer.

Limit of a Class of Sums

4689 [1956, 346]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Let $f(x)$ be any function such that $f'''(x)\geqq 0$, $f(n)\sim f(n+1)$. Prove that $\sum_{n=0}^{\infty}(-1)^n t^{f(n)}$ tends to $\frac{1}{2}$ as $t\to 1^-$.

Indications by the proposer. Assuming $f(0)=0$ and $f'(0), f''(0)\geqq 0$, the problem is equivalent to showing that

$$\lim_{t\to 1^-} \{t^{f(0)} - 2t^{f(1)} + 2t^{f(2)} - \cdots\} = 0.$$

The last expression (the series being absolutely convergent for $|t| < 1$) may be written in the form

$$(t^{f(0)} - 2t^{f(1)} + t^{f(2)}) + (t^{f(2)} - 2t^{f(3)} + t^{f(4)}) + \cdots = \sum_{n=0}^{\infty} \Delta^2 t^{f(n)},$$

where $\Delta^2 F(n) = F(2n) - 2F(2n+1) + F(2n+2)$. From the mean value theorem we have $\Delta^2 F(n) = F(2n+\theta_n)$, $0 \leqq \theta_n \leqq 2$. Therefore

$$\sum_{n=0}^{\infty} \Delta^2 t^{f(n)} \leqq \log^2 \frac{1}{t} \sum [f'(2n+2)]^2 t^{f(2n)} + \log \frac{1}{t} \sum t^{f(2n)} f''(2n+2),$$

and the last two sums approach zero as $t \to 1^-$.

Two Tetrahedrons and an Invariant

4690 [1956, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given a tetrahedron $ABCD$ and the tetrahedron $A_1B_1C_1D_1$ obtained by passing planes through A, B, C, D parallel to the opposite faces of $ABCD$, show that

$$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 - 2\overline{PD}^2 - \overline{PD}_1^2$$

is a constant independent of the position of point P. Extend this property to a skew polygon of n vertices.

Solution by N. A. Court, University of Oklahoma. Considering the general case first, let A_i $(i=1, \cdots, n)$ be n given points in space having G for their centroid, and let k denote the sum of the squares of the $n(n-1)/2$ segments determined by the n given points. If P is any point in space, we have:

$$\text{(1)} \qquad \sum_{i=1}^{n} \overline{PA}_i^2 = \sum_{i=1}^{n} \overline{GA}_i^2 + n\overline{PG}^2 \qquad [1;\text{ p. 316, art. 274}],$$

$$\text{(2)} \qquad k = n \sum_{i=1}^{n} \overline{GA}_i^2 \qquad [1;\text{ p. 321, art. 280}],$$

whence

$$\text{(3)} \qquad \sum_{i=1}^{n} \overline{PA}_i^2 = n\overline{PG}^2 + k/n.$$

If E is a point on the line GA_n, and $EG:GA_n = t$, both in magnitude and in

sign, we have, by Stewart's theorem [2]:

$$PE^2 \cdot GA_n + PG^2 \cdot A_nE + PA_n^2 \cdot EG + GA_n \cdot A_nE \cdot EG = 0. \tag{4}$$

Now $EG = tGA_n$, and $A_nE = -(t+1)GA_n$, hence (4) becomes, after division by GA_n,

$$PE^2 - (t+1)PG^2 + tPA_n^2 - t(t+1)GA_n^2 = 0. \tag{5}$$

Eliminating PG^2 between (3) and (5), the result may be put in the form

$$\sum_{i=1}^{n} PA_i^2 - n(PE^2 + tPA_n^2)/(t+1) = k/n - ntGA_n^2. \tag{6}$$

The right hand side of (6) is a constant, independent of the position of P, and this constant is the value of the left hand side, which proves the proposition in the general case. Observe that this proposition is valid in Euclidean space of any number of dimensions.

In the case of the tetrahedron $(T) = ABCD$, the vertex D_1 of the anticomplementary tetrahedron $(T_1) = A_1B_1C_1D_1$ of (T) lies on the line GD, where G is the centroid of (T), and $D_1G = 3GD$ [**3**: p. 53, art. 176]. Thus the vertex D_1 may in the present case play the role of the point E of the general case, and we have: $=3$, $n=4$, and (6) becomes

$$PA^2 + PB^2 + PC^2 + PD^2 - 4(PD_1^2 + 3PD^2)/4 = k/4 - 12GD^2,$$

or

$$PA^2 + PB^2 + PC^2 - 2PD^2 - PD_1^2 = k/4 - 12GD^2,$$

where k is the sum of the squares on the six edges of (T).

References

1. L. N. M. Carnot, *Geometrie de Position*, Paris, 1803.
2. Nathan Altshiller-Court, *College Geometry*, 2nd ed., New York, 1952.
3. ———, *Modern Pure Solid Geometry*, New York, 1935.

Also solved by G. B. Robison, and the proposer.

Minimal Weakly Prime Ideal

4691 [1956, 347]. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring R is any ideal I having the property that either $aR \subset I$ or $Ra \subset I$ implies that a is in I. Give an element-wise characterization of the unique minimal weakly prime ideal of R.

Solution by Alfredo Jones, Instituto de Matematica y Estadistica, Montevideo, Uruguay. Given any ideal I, let: $W(I) = \{a: R^naR^m \subset I \text{ for some } m, n > 0\}$. $W(I)$

is obviously an ideal. $W(I)$ is weakly prime because if $AR\subset W(I)$, $R^naRR^m = R^naR^{m+1}\subset I$, so $a\in W(I)$, and similarly if $Ra\subset W(I)$. And if I is weakly prime $W(I)=I$, so we thus obtain all weakly prime ideals. But if $I_1\subset I_2$ then $W(I_1) \subset W(I_2)$. Therefore the minimal weakly prime ideal is: $W(0)=\{a: R^naR^m=0$ for some $m, n>0\}$.

Also solved by D. S. Kahn, and the proposer.

Topological Space with Unique Limits

4694 [1956, 426]. *Proposed by R. W. Bagley, University of Kentucky*

There are simple examples which show that an uncountable topological space in which limits are unique (hence T_1) is not necessarily Hausdorff. Are there such examples for countable spaces? Here "limit" is used in the usual sense of limit of a sequence where the directed set is the positive integers rather than limit of a generalized sequence as defined by Kelley and others. With this general definition (where the directed set is allowed to vary), Kelley proved that a space is Hausdorff if, and only if, limits are unique. (See *Convergence in topology*, Duke Math. J., 1950, pp. 277–283).

I. *Solution by H. E. Vaughan, University of Illinois.* Frechet, in his book *Les Espaces Abstraits*, pp. 212–213, attributes the following example to Urysohn. Let R consist of the rational numbers belonging to the closed interval $[0, 1]$ together with an irrational number, and let convergence be defined as follows: A sequence of points of R, which, with respect to the usual topology of the real line, converges to a rational number is to converge to the same limit in R; a sequence which ordinarily converges to an irrational number is to converge, in R, to the irrational member of R. It is readily seen that, with this definition of convergence, R is an L-space, and an investigation of the neighborhoods of its points shows that it is also a topological space (in the modern, very restricted, sense of the phrase) which is not a Hausdorff space.

II. *Solution by M. K. Fort, Jr., University of Georgia.* Let R be the set of all rational numbers. We define T to be the set of all subsets X of R such that either X is empty or $R-X$ has at most a finite number of limit points in the real number system relative to the usual topology for the real number system. It is easy to verify that T is a T_1 topology for R. However, T is not Hausdorff since any two non empty members of T have a non empty intersection. Limits of sequences are unique, since a sequence $x_1, x_2, x_3, \cdots$ converges to a point p relative to this topology if and only if $x_n=p$ for all sufficiently large values of n.

Also solved by G. E. Bredon, Helen F. Cullen, L. R. Ford, Jr., Melvin Henriksen, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics of Business, Accounting, and Finance. By K. L. Trefftzs and E. J. Hills. Harper, New York, 1956. 591 pp. $4.50.

This rather long book is designed to meet the needs of the average and sub-average student beginning a commercial education and "may provide the only college training in mathematics that many students receive." The first 116 pages is a review of the arithmetic usually presented in the first eight grades. The next 134 pages reviews most of the first year's work in high school algebra. The remainder is devoted to elementary business problems, mathematics of finance, and insurance.

The aim of the first two parts seems to be to provide a sufficient amount of drill necessary to make a student proficient in the fundamental arithmetic operations and in basic algebraic techniques. Very little new material is included, the authors preferring the time honored "high school" method for obtaining square roots to Hero's method. No attempt is made to develop the real number system. In fact an illustration of the use of a rule for multiplying rounded numbers (p. 73) implies that the real numbers are not dense and at the same time disproves the rule. In the algebra part only a half page is devoted to fractional and negative exponents, and the binomial theorem and progressions are omitted entirely, as are functions and graphs. There are, however, a large number of problems, but the student is not trusted with his own analysis. Problems in algebra are classified, each type carefully analyzed, and the steps necessary for their solution enumerated.

The student is given little opportunity for analysis in the remainder of the book. The analysis of problems of various types, the statement of the rule to be followed, the enumeration of the steps necessary to produce a solution is expertly done by the authors instead. The student is not asked to use time diagrams, although they occasionally appear to assist the authors in the development of a rule or formula. Equations are written (p. 293) which imply the equivalence of dated payments at a simple interest rate. Since this is not a true equivalence relation, one wonders if students solving problems by this technique may not get different answers if different focal dates are selected. The mere statement (p. 353) of transitivity for equivalence at a compound interest rate seems hardly sufficient for students to master this important concept. "Finding the unknown time" (p. 339) by interpolation in the compound interest table is treated at a time when $(1+i)^n$ is defined only for integral values of n, and the resulting answer is called an approximation of the time when in actuality it is the exact time necessary for P to accumulate to S by a later rule (p. 345). Since

no knowledge of progressions is assumed the students must accept the formula for the amount of an annuity on faith (p. 395). Omitted from the section is the concept of equivalence of interest rates, all general annuities, and finding the interest rate for an annuity. Installment buying is treated without annuity symbols, which leads to an unnecessarily complicated method of determining the interest rate. The financial tables used have an unusually attractive format.

This book would not be adequate for the above-average student. It would be a pity for future leaders of business and industry to obtain the view that mathematics is so mechanical and lacking in concepts at a time when recent advances in mathematical theory and the use of electronic computing equipment holds such promise for the future.

C. L. SEEBECK, JR.
University of Alabama

Integral Transforms in Mathematical Physics. By C. J. Tranter. Wiley, New York, 1956. 133 pp. $2.00.

This is another fine little book in Methuen's Monographs on Physical Subjects. The emphasis is on the use of integral transforms in partial differential equations with chapters also on evaluation of integrals, combined use of relaxation methods and transforms, and a new chapter in this second edition on dual integral equations of the type arising from physical problems with one set of "mixed" boundary conditions.

The first four chapters deal with Laplace, Fourier, Hankel, and Mellin transforms, with their inversion formulae, and with a number of applications. The sixth chapter contains a discussion of finite transforms with applications of sine, cosine, Hankel, and Legendre transforms.

A fairly strong background in analysis is required for appreciation of the book as this analysis is naturally not presented in such a small book. A course in complex variables should suffice. Some knowledge of boundary value problems in mathematical physics is also a necessary prerequisite.

The examples or exercises are adequate for enhancing the understanding. The book should serve as a valuable supplement in or as a reference book for one who wishes to see whether he can apply integral transform methods to some particular problem.

R. B. DEAL
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Physics and Mathematics, Series I, Volume I. Progress in Nuclear Energy. Edited by R. A. Charpie, et al. McGraw-Hill, New York, 1956. x+398 pp. $12.00.

This volume is an outgrowth of the United Nations Conference on the Peaceful Uses of Atomic Energy held at Geneva in 1955. It is devoted primarily to summarizing in great detail such information concerning neutron physics and fissionable nuclei as is needed for the design of nuclear reactors. Most of this

data is here published for the first time, the internationally practised policy of withholding information on this subject for fifteen years having been relaxed specifically on the occasion of this conference. In their attempt to bridge this gap in the literature the authors of the eleven chapters in this book are in the anomalous position of correlating and distilling a literature which has been inaccessible to most readers. There are numerous descriptions of apparatus, empirical curves and tables. Yet the volume is not self contained. There is no introductory chapter to define the problems and indicate in what way the chapters in this book are addressed to them. Although numerous formulae occur in the first 250 pages, they are given only for comparison with empirical data. Neither their derivation nor the principles on which they are based are to be found here. Only in the ensuing 100 pages, in the chapters entitled "The Physics of Fast Reactors" and "Heterogeneous Methods for Calculating Reactors" does one find some attention to mathematics. In the former, a survey is given of mathematical methods used in calculating neutron transport phenomena. The equations considered in various approximations are the Boltzmann equation and the diffusion equation. The latter chapter, by S. M. Feinberg, of the U.S.S.R., seems to stand alone as a contribution of the Russian group in that no indication of related work done elsewhere is given. The exposition is correspondingly more self-contained and is of interest in that the theory is explicitly formulated for reactors with a 3-dimensional lattice structure.

The book appears to be of most value to designers of nuclear reactors. For others, it is most likely to be of interest only insofar as the text is a guide to the bibliography.

D. L. Falkoff
Brandeis University

Fundamental Concepts of Algebra. By Claude Chevalley. Pure and Applied Mathematics Series, volume VII, Academic Press, New York, 1956. viii +241 pp. $6.80.

Fundamental Concepts offers a pronounced Bourbaki flavor. Indeed in a sense, the book may be considered as a welcome abridgment into the English language from Bourbaki's multi-volumed *Éléments de Mathématique*. The author, making no pretense to cover all the basic ideas in algebra, omits such topics as the theory of fields in order to elucidate modules and exterior algebras. He devotes approximately three-quarters of the pages in the volume to the third and fifth chapters, entitled respectively "Rings and Modules" and "Associative Algebras." The other three chapters discuss monoids, groups, and (very briefly) algebras.

The exposition is very closely knit, leaving the reader little opportunity to skip. The logical development advances steadily with a dynamic quality, so the reader also has little desire to skip. Yet the reader is not pampered. He must understand and assimilate the concepts quickly, lest he be unprepared to comprehend the subsequent sections. Within a section the arrangement of material

is typically as follows: introductory description of new ideas, theorem, proof, theorem, proof, more new concepts, theorem, proof. Paragraphs are frequently very lengthy. Long proofs require high concentration by the reader; rarely is there found a hint of the future path for the argument or a summary of the portion of the assertions already proved. An excerpt from the preface concerning motivation is highly indicative of the style of writing: "what the student may learn here is not designed to help him with problems he has already met but with those he will have to cope with in the future; it is therefore impossible to motivate the definitions and theorems by applications of which the reader does not know the existence as yet." With this philosophy of presentation, the exposition displays considerable austerity. Examples are cited when certain algebraic structures are introduced, but illustrations for various significant theorems or major results are not attempted. The theory is relentlessly pushed onward.

A highlight of the book is the exercise list concluding each chapter. The problems, of which there are more than a hundred, seem unusually well chosen to offer the solver useful information and a broadening outlook on the theory.

There is a variety of errata which can easily be removed in a next printing and which, for the most part, will delay the reader only momentarily.

The preface suggests use of the text in a first graduate course. The desirability of every serious graduate student's learning the book's content at an early stage should not be challenged. Nevertheless the teacher of a beginning graduate student should warn him that studying this volume demands intensive work. The dividends paid will be well worth the effort, many times over.

R. A. Good
University of Maryland

BRIEF MENTION

Publications of potential interest to mathematicians, but which are more properly reviewed in other periodicals, are described below.

Mathematics for Electronics with Applications. By Henry M. Nodelman and Frederick W. Smith. McGraw-Hill, 1956. vii+391 pp. $7.00.

This interesting book is one which is apt to be overlooked by mathematicians. It is quite possible that the misuse of technical mathematical vocabulary will so mitigate against this book that the important modern applications of mathematics to electronics will be lost. Statements such as, "a rectangular matrix is *equivalent to* a square matrix" (page 156) and "the determinant of a rectangular matrix is always zero" (page 159) and the implication on page 215 that the word "isomorphic" is synonymous with the phrase "one-to-one correspondence" will justifiably raise the ire of mathematicians, and make this book unsuitable as a mathematical text. Nevertheless, competent mathematicians may well wish to examine it for the many current applications which it contains. It is a sad commentary that the same publishing house which carefully prepared *Modern Mathematics for the Engineer* by Beckenbach would fail to correct the

misuse of technical mathematical vocabulary in the volume here under discussion. This text could well have helped bridge the current gap between electrical engineering and mathematics, were it not for the misuse of certain important mathematical words.

One may shudder at the inclusion of the "cross-hatch" method of evaluating three by three determinants or wonder why, in today's world, more work on actual circuit design using Boolean algebra is not included. Still the collection of "up-to-date problems based on current engineering practice" should be welcomed by teachers seeking to teach mathematics to engineering students.

Brief Analytic Geometry. By Thomas E. Mason and Clifton T. Hazard. Ginn, 1957. 229 pp. \$3.50.

The authors state that "no major changes in subject matter or methods of presentation have been made" in this third edition of *Brief Analytic Geometry.* "Changes have been made in the numerical data of many of the exercises. Several new exercises have been added. Few of the illustrative examples have been changed." This well-known book needs no additional review other than to mention a new edition has been prepared.

Calculus Refresher for Technical Men. By A. A. Klaf. Dover 1956. 431 pp. \$1.95.

This paperback calculus refresher is in no way comparable to the excellent paperbacks by Oakley or by Graesser and Petersen.

Trigonometry Refresher for Technical Men. By A. A. Klaf. Dover Publications Inc., 1956. 629 pp. \$1.95.

The Icosahedron and the Solutions of Equations of the Fifth Degree. By Felix Klein. Dover, 1956. xvi+289 pp. \$1.85.

American geometers will indeed welcome the inexpensive paperback reprint of the English translation of Klein's historical volume on the Icosahedron, originally published in 1884 and translated into English in 1888. This work was reviewed extensively on pages 45–61, Vol. 9 (1887) *American Journal of Mathematics* by F. N. Cole.

Table of the Fresnel Integral To Six Decimal Places. By T. Pearcey. Cambridge University Press. 63 pp. \$2.50.

These six and seven figure tables of Fresnel integrals will be welcomed by persons interested in diffraction theory. The clear type and adequate margins are a relief.

Intermediate Algebra. By Paul K. Rees and Fred W. Sparks. McGraw-Hill, 1956. x+306 pp. \$3.90.

The authors state that the features of the first edition have been "all preserved" and that "the chief purpose in the preparation of the second edition of

Intermediate Algebra was to provide a selection of problems greater in number and more carefully graded than those in the first edition." Many teachers will undoubtedly welcome the new edition of this old favorite.

Mathematics Magic and Mystery. By Martin Gardner. Dover, 1956. xii+176 pp. $1.00.

This collection of mathematical and near mathematical tricks, puzzles and games is a welcome low cost addition to the library of anyone interested in such pastimes, and who isn't?

Electronic Computers. Edited by T. E. Ivall. Philosophical Library, 1956. 163 pp. $10.00.

Electronic Computers is not an advanced text for experts, but instead is definitely for the non-expert. Still it contains much of interest to mathematicians who are trembling on the brink of modern machine computation, both analogue and digital. While most of the machines mentioned are English, the American counterparts are well-known and the circuit designs and principles are international. A thoroughly enjoyable book giving general principles rather than specific "cook-book" directions.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

CONFERENCE ON MATRIX COMPUTATIONS

A Conference on Matrix Computations will be held at Wayne State University on September 3–6, 1957. The purpose of this conference is to bring together those persons who are concerned with the mathematical methods used in computing centers and who can communicate both in the technical language of digital computers and in the symbolism of matrix algebra.

Morning sessions will be devoted largely to invited addresses. Methods now being used to solve systems of linear equations, to compute the inverse of a matrix and to find characteristic values and characteristic vectors will be described. Papers suggesting new methods for the solution of standard problems will be solicited and an especial effort will be made to bring to attention new problems demanding the efforts of mathematicians. Smaller groups with well defined common interests will form discussion panels in the

afternoon. It is expected that a report of methods in use at some of the main European centers will be given.

There is no tuition fee for the conference. Individuals who wish to present papers or to suggest speakers should contact Professor Wallace Givens, Chairman, Department of Mathematics, Wayne State University, Detroit 2, Michigan.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1957.

Catholic University of America, June 26 to August 9: Dr. Ramler, college geometry, analytic projective geometry, ordinary differential equations; Dr. Moller, higher algebra I; Dr. Wiegmann, higher algebra II, introduction to matrix theory; Dr. Taam, advanced calculus I; Dr. Saworotnow, advanced calculus II; Dr. Finan, basic concepts of mathematics.

Columbia University, July 3 to August 16: Dr. Taft, introduction to higher algebra; Dr. Mendelson, differential equations; Mr. Gordon, probability; Professor Chevalley, fundamental concepts of mathematics, higher algebra; Professor Taylor, theory of functions of a real variable; Professor Feldman, theory of functions.

Syracuse University, July 1 to August 9: Professor Gelbart, analysis and applications I (differential equations); Professor Davis, an intermediate course in algebra, teaching high school mathematics; Professor Hemmingsen, history of mathematics; Professor Exner, analysis of elementary mathematics; Professor Gilchrist, programming for digital computers. August 12 to September 13: Professor Kostenbauder, analysis and applications II (vector analysis).

University of California, Berkeley, Department of Statistics, June 17 to July 27 and July 29 to September 7: Professor Neyman, individual research; Professors Neyman, Fix and Smith (University of Cambridge, England), research seminar in statistical problems of health. This course will be given in cooperation with Drs. Brooke, Hall, Serfling, and Willis (Communicable Disease Center, Public Health Service, Atlanta) and Dr. Mantel (National Institutes of Health, Bethesda). They will present the medico-biological side of practical problems preceding the statistical discussions.

University of California, Los Angeles, June 24 to August 2: Professor Horn, functions of a complex variable; Professor Straus, theory of relativity; Professor Bell, fundamental mathematical concepts.

University of Colorado, June 17 to August 23: Dr. Zirakzadeh, foundations of geometry; Professor McKelvey, topology; Professor Bunt, teaching of secondary school mathematics, mathematics workshop in teaching problems; Professor Magnus, history of mathematics, foundations of analysis; Mr. Householder, mathematical statistics; Professor Rogers, finite mathematics.

University of Wisconsin, July 1 to August 24: Professor F. B. Jones (University of North Carolina), topics in topology, advanced calculus; Dr. Walker (American Optical Co.), differential geometry; Dr. Artzy (Israel Institute of Technology), advanced topics in algebra, theory of probability; Professor Fadell, elementary topology, higher analysis; Professor Wagner, determinants and matrices; Dr. Kruskal, differential equations; Dr. Payne, foundations of algebra; Mr. Evey, theory and operation of computing machines.

PERSONAL ITEMS

Dr. F. E. Browder, Yale University, has been awarded a National Science Foundation Postdoctoral Fellowship.

Dr. C. E. Shannon, a professor at Massachusetts Institute of Technology and a mathematical consultant in Bell Telephone Laboratories research department, has received the 1956 Research Corporation Award for his work in information theory.

New Jersey State Teachers College, Montclair: Associate Professor B. E. Meserve has been appointed Chairman of the Department of Mathematics; Dr. D. R. Davis, Chairman of the Department, has retired.

University of Minnesota, College of Science, Literature and Arts: Visiting Associate Professor Bjarni Jónsson, University of California, Berkeley, has been appointed Associate Professor; Associate Professor M. D. Donsker has been promoted to Professor; Assistant Professor W. S. Loud has been promoted to Associate Professor; Dr. G. E. Baxter and Dr. J. M. Slye have been promoted to Assistant Professors; Assistant Professor Ella Thorp has retired with the title Assistant Professor Emeritus.

Wayne State University: Dr. Karl Zeller, Tübingen University, Germany, has been appointed Associate Professor; Professor G. G. Lorentz is on leave of absence and has been appointed Visiting Professor at the University of Michigan. The School of Business Administration and the Computation Laboratory announce a new Master's Program in Automatic Data Processing.

Professor D. B. Ames, University of New Hampshire, has accepted a position as research mathematician with Hughes Aircraft Company, Culver City, California.

Dr. R. W. Bagley, Associate Research Scientist, Lockheed Aircraft Corporation, Sunnyvale, California, is on leave for the year to work on an operations research project at Stanford University.

Mr. H. W. Becker has been elected Secretary-Treasurer of the Omaha-Lincoln Section, Institute of Radio Engineers.

Assistant Professor Kurt Bing, Rensselaer Polytechnic Institute, has been promoted to Associate Professor.

Assistant Professor W. E. Briggs, University of Colorado, has been appointed Director of the University Academic Year Institute for Secondary School Teachers of Science and Mathematics sponsored by the National Science Foundation.

Professor Arthur Erdélyi, California Institute of Technology, is on leave and has been appointed Visiting Professor at Hebrew University, Jerusalem, Israel.

The annual award of the Duodecimal Society of America for 1956 has been given to Jean Essig, Inspector General of Finances for France.

Dr. F. G. Fisher, Navy Electronics Laboratory, San Diego, California, has accepted a position as a consulting mathematician with the U. S. Navy, Bureau of Ordnance, Washington, D. C.

Mr. J. L. Freier, is now a mathematician with Project Cyclone, Reeve Instrument Company, New York, New York.

Mr. R. M. Gordon has been appointed Supervisor, Customer Education, ElectroData Division, Burroughs Corporation, Pasadena, California.

Assistant Professor R. P. Gosselin, University of Connecticut, has been awarded a grant from the National Science Foundation for research in Fourier series.

Mr. F. D. Grogan, Quality Surety Office, Rocky Mountain Arsenal, Denver, Colorado, has a position as a systems analyst, Flight Controls Group, Glenn L. Martin Company, Denver.

Assistant Professor W. T. Guy, Jr., University of Texas, has been promoted to Associate Professor.

Mr. H. N. Hadley, Naval Powder Factory, Indian Head, Maryland, has accepted a position as senior reliability analyst with the AVCO Manufacturing Company, Lawrence, Massachusetts.

Mr. J. L. Hatfield, Mary Washington College, has been appointed Assistant Professor at College of William and Mary in Norfolk.

Associate Professor L. S. Hill, Hunter College, has been promoted to Professor.

Associate Professor R. C. James, Haverford College, has been appointed Professor

and Chairman of the Department of Mathematics, Harvey Mudd College, effective September, 1957.

Mr. B. V. Lachapelle, Cornell University, has been appointed a research associate at the University of Montreal.

Professor Harry Langman, Detroit Institute of Technology, has been appointed Professor at Ohio Northern University.

Mr. J. G. Leghorn, University of Colorado, has accepted a position as an engineer with the Glenn L. Martin Company, Denver, Colorado.

Dean A. E. Meder, Jr., Rutgers University, is on leave of absence and has been appointed Executive Director of the Commission on Mathematics.

Mr. J. W. Mettler, Teacher, Trenton Central High School, New Jersey, has been appointed Assistant Professor at Pennsylvania State University.

Mr. George Millman, Analytical Statistician, Office of the Quartermaster General, Army Department, Washington, D. C., is now a mathematician at the Evans Signal Laboratories, Army Signal Corps, Ft. Monmouth, New Jersey.

Dr. M. E. Muller, Senior Mathematician, Scientific Computing Center, International Business Machines Corporation, New York City, is on leave of absence as a research associate at Princeton University.

Dr. R. Z. Norman, Princeton University, has been appointed Assistant Professor at Dartmouth College.

Mr. S. E. Puckette, Yale University, has been appointed Assistant Professor at the University of the South.

Professor Emeritus L. L. Silverman, Dartmouth College, has been appointed Visiting Professor at the University of Houston.

Dr. G. H. Swift, Duke University, has accepted a position as applied science representative with International Business Machines Corporation, Seattle, Washington.

Dr. E. D. Watters, Jr., Senior Mathematician, Bendix Research Laboratories, Detroit, Michigan, is an engineer at Westinghouse Electric Corporation, Baltimore, Maryland.

Dr. E. S. Wolk, University of Connecticut, has been promoted to Assistant Professor.

Mr. J. T. Yamada, University of Toronto, has been appointed Lecturer at McGill University.

Mr. Elmer Latshaw, a mechanical engineer, Naval Air Materiel Center, Philadelphia, Pennsylvania, died on January 18, 1957. He was a member of the Association for thirty-seven years.

Mr. J. M. Pellegrino, Mathematician, Electric Boat, Groton, Connecticut, died on March 19, 1956.

Miss Audrey I. Richards, Utica College of Syracuse University, died on September 15, 1956.

Professor A. C. Schaeffer, Chairman of the Department of Mathematics, University of Wisconsin, died on February 2, 1957. He was a member of the Association for nine years.

Brigadier General R. H. Somers died on January 22, 1957. He was a charter member of the Association.

Professor John von Neumann, Institute for Advanced Study, died on February 8, 1957. He was a member of the Association for twentv-four years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The nineteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, January 12, 1957. Professor H. L. Alder, Chairman of the Section, presided at the morning and afternoon general sessions. Two concurrent sessions were also held in the afternoon—one on the teaching of mathematics and one on research. Professor Alder presided at the former and Professor Harley Flanders, Vice-Chairman of the Section, at the latter. There were 156 persons in attendance at the meeting including 87 members of the Association.

Following Professor Blakeslee's report on the 1956 high school contest, the section voted unanimously to endorse the proposal that the Association sponsor a nationwide high school mathematics contest.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Harley Flanders, University of California, Berkeley; Vice-Chairman, Professor B. J. Lockhart, U. S. Naval Postgraduate School; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College.

By invitation of the section, Professor David Blackwell, University of California, Berkeley, delivered an address at the morning session entitled *Statistical Prediction of Sequences.* Abstract of this address follows:

A method of prediction of successive elements in an infinite sequence of zeros and ones, based on observation of previous elements, is described. The method has the property that, applied to any sequence, the proportion of correct predictions for the first n elements of the sequence will be at least max $(p_n, 1-p_n)-\epsilon$ for all sufficiently large n, where p_n is the proportion of ones in the first n elements. An extension to more general statistical decision problems is indicated.

The following papers were presented:

1. *An approximation to the equally tempered musical scale,* by Professor I. J. Schoenberg, University of Pennsylvania and Stanford University.

The author presented some of the interesting historical researches of Professor J. M. Barbour on the equally tempered scale, contained in Barbour's article *A geometric approximation of the roots of numbers,* this MONTHLY, vol. 64, 1957, pp. 1–10.

2. *The Mathematical Association of America high school contest,* by Professor D. W. Blakeslee, San Francisco State College.

The procedures and results of the 1956 Annual Contest for high school students sponsored by the Northern California Section were reviewed. It was felt that the contest was highly successful; seventy-three schools and over 2300 students having participated.

3. *The visiting-lectureship program for high schools,* by Professor H. L. Alder, University of California, Davis.

Announcement is made of a program sponsored by the Northern California Section of the Mathematical Association of America whereby mathematicians from seven universities and colleges in Northern California are available to give lectures in high schools on a topic of general interest in mathematics at either a mathematics class, a special meeting arranged for the purpose, a school assembly, or (if a fairly good attendance can be assured) a mathematics club meeting. Invitation to avail themselves of this opportunity will be sent initially to about 70 high schools.

4. *Reflecting chess bishops*, by Professor S. Stein, University of California, Davis.

A reflecting chess bishop is a chess bishop which moves along a diagonal until it hits the border of the chess-board and then reflects off like a ray of light (if it hits a corner it reflects directly back). THEOREM: *Two reflecting bishops can be placed on an* $m\times n$ *chess-board to cover all squares if and only if* $(m-1, n-1)=1$.

5. *On Picture-Writing*, by Professor G. Polya, Stanford University.

This paper appeared in this MONTHLY, vol. 63, 1956, pp. 689–698.

6. *Factors of Fermat numbers*, by Professor R. M. Robinson, University of California, Berkeley.

Fermat believed that the numbers $F_m=2^{2^m}+1$ are all prime, but Euler showed in 1732 that F_5 has the factor 641. Actually, no Fermat prime has ever been identified except $F_0=3$, $F_1=5$, $F_2=17$, $F_3=257$, and $F_4=65537$. The number F_m is now known to be composite in twenty-nine cases, namely for $m=5$, 6, 7, 8, 9, 10, 11, 12, 15, 16, 18, 23, 36, 38, 39, 55, 63, 73, 117, 125, 144, 150, 207, 226, 228, 268, 284, 316, 452. In fourteen of these cases ($m>38$, $m\neq 73$), the compositeness was first established in 1956, using a high-speed computer (SWAC) and a program coded by the author. For example, F_{207} was found to have the prime factor $3\cdot 2^{209}+1$.

7. *An eigenvalue problem for ordinary differential equations*, by Professor S. P. Diliberto, University of California, Berkeley.

Let $W(t, \theta_2)$ be the matrix solution of the real linear (matrix) ordinary differential equation

$$dW/dt = A(t, t+\theta_2)W \tag{*}$$

determined by the initial condition $W(0, \theta_2)=I$, where A and W are n-square, and $A(\theta_1, \theta_2)$ is continuous in θ_1, θ_2 with period ω_i in $\theta_i (i=1, 2)$.

Let S denote the space of all continuous ω_2-periodic n-vector functions of θ_2, *e.g.*, $\alpha(\theta_2)\in S$ implies $\alpha(\theta_2)=(\alpha_1(\theta_2), \cdots, \alpha_n(\theta_2))$, where each $\alpha_i(\theta_2)$ is C' and has period ω_2 in θ_2. A transformation T: $S\to S$ is defined by: $\gamma(\theta_2)\in S$, $(T\gamma)(\theta_2)\in S$, where $(T\gamma)(\theta_2)=W(\omega_1, \theta_2-\omega_1)\gamma(\theta_2-\omega_1)$. The eigenvalues of T are studied and shown to characterize the limit ($t\to+\infty$) behavior of the solutions of (*).

8. *A note on Rouche's theorem*, by Professor C. L. Clark, Oregon State College.

The usual condition of Rouche's theorem that $|g(z)|<|f(z)|$ on a simple closed curve C lying in a simply connected region within which $g(z)$ and $f(z)$ are analytic can be replaced by the condition that $f(z)$ and $f(z)+g(z)$ lie in the same component of the mapping space $(Z-O)^C$, *i.e.*, the space of all continuous mappings of C into $Z-O$, Z being the complex plane and O the origin. This generalization of Rouche's theorem is obtained by elementary use of the index function as developed by G. T. Whyburn and C. Kuratowski and is also an improvement of other generalizations. The results suggest further questions concerning zeros of functions.

9. *The teaching of interpolation*, by Professor H. A. Arnold, University of California, Davis.

Even in elementary courses it is important to teach the retention of extra "guard figures" in numbers used in numerical calculations. These numbers include input data and the results of interpolations. Teachable numerical examples are given to show this is feasible.

10. *A heuristic outlook in checking*, by Professor C. M. Larsen, San Jose State College.

Most teachers have observed students who, when asked to check their work, make wrong answers come out "right." Such students may be conditioned to force the checks when teachers say, for example, "To *check* a solution, put it into the given equation and *make sure* the equation is

satisfied." An alternative attitude, aimed at discovering error, rather than checking correctness, may be encouraged by advising, "To *test* a solution, put it into the given equation and *hunt for discrepancies.*" Some evidence was presented indicating that such advice may lead students to think more critically and constructively about their work.

11. *Some observations on teaching mathematics for the computer age*, by Professor Irving Sussman, University of Santa Clara.

A digest is made of diverse competent opinions on the impact which the emergence of electronic digital computers will have, or should have, on the teaching of undergraduate and preparatory mathematics. Although the expert opinions vary in details, there is general agreement that important changes both in course content and teaching methods have become necessary—this even in the pure mathematics curricula. The question of how such redirection of emphasis is to be brought about in view of the loss of such a large percentage of potential teaching personnel to industry, and in the face of traditional academic inertia, is posed as an unsolved problem.

ROY DUBISCH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.
ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.
INDIANA, May 4, 1957.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Carleton College, Northfield, May 11, 1957.
MISSOURI
NEBRASKA
NEW JERSEY, Fall, 1957.
NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.
NORTHERN CALIFORNIA, January 18, 1958.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA, November 28, 1957.
ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.
SOUTHEASTERN
SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.
WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

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RALPH D. JAMES, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

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MATHEMATICAL EDUCATION IN THE U.S.S.R.*

B. V. GNEDENKO,† University of Kiev, U.S.S.R.

During my brief stay in the United States in September, 1956, I was able to visit a number of universities, to meet many American colleagues, and to discuss informally various scientific and pedagogical topics. At some universities I delivered lectures; among these, at Stanford University, I delivered an address on the topic that serves as the title of this article. I got the impression that the lecture provoked a definite interest and that American scientists and teachers wished to know more about mathematical life in the Soviet Union. What I have said explains why I was gratified when Professor R. D. James approached me with the proposal‡ that I write this article for the American Mathematical Monthly. I realize that in a brief article it is not possible to deal fully with all questions that might interest the reader. In order that all who are interested in details of one kind or another, or in general considerations dealt with all too briefly by me, shall be in a position to write to me for fuller explanation, I give my home address.§ As far as possible I shall endeavor to give the necessary information by letter, and, in case I am not competent, I shall refer such enquiries to specialist colleagues, who can give more satisfactory answers. I shall be pleased if my article promotes such correspondence, because a lively exchange of opinions offers to both sides great opportunities for the understanding of various aspects of the organization of teaching methods.

1. General aspects. In present day general education as well as in special education, mathematics occupies an important place. Children of pre-school age encounter it in its simplest form. Later, from the first to the last school year, pupils study in succession arithmetic, elementary algebra, geometry, plane trigonometry, and also the elements of analytic geometry and of mathematical analysis (differential and integral calculus). The transition to special schools (pedagogical, technical, agricultural, *etc.*) and also to higher educational institutions means, for the overwhelming majority of young people, a continuation of mathematical education. Moreover, the completion of higher education and graduation from a university with a diploma does not at all mean the end of further acquaintance with mathematics. Many engineers, teachers, economists, and other specialists are compelled from time to time to turn to mathematics for help, and must study on their own those parts necessary for their immediate needs. Frequently, courses of different types and lectures for specialists who already have their diplomas are organized in the universities of the Soviet Union. All the large cities have institutions for improving the qualifications of

* Translated by J. St. Clair-Sobell and W. H. Simons.

† Regular member of the Academy of Sciences of the Ukrainian Soviet Socialist Republic.

‡ I am indebted to Professor I. J. Schoenberg for recommending this to me. Ed.

§ Sverdlova Str. 14, app. 29, Kiev 3, U.S.S.R.

teachers in the secondary schools. In these institutions, teachers can get advice both on scientific and methodological questions; they can work in the offices and laboratories; they can follow the lectures of specialists.

However, contemporary society cannot be satisfied merely with the transmission of knowledge accumulated by mankind in the past. Each day brings new problems for the solution of which it is necessary not only to avail oneself of methods developed long ago but to create new ones. The present day development of physics and technology makes numerous demands, in the first place, on mathematics, which only a mathematician capable of creating new ideas can satisfy. But also, for the purposes of instruction, persons are required who view their subject as a living and developing discipline and not as a conclusively formed and ossified system of knowledge. This essential idea must pervade all instruction from its first stages to its conclusion. This idea must be central in importance not only in the universities but also in the elementary schools. In fact, the person now being taught in school will become an independent member of society only after some years and, after this, it will be necessary for him to continue working for many more years. What demands life will present to him, what mathematics he will have to use, it is impossible to foresee in school, even along the most general lines. Therefore, even in the first school years, it is necessary to develop in children flexibility of the intellect, to inculcate the thought that in the future it will be necessary to add many more forms of knowledge to the scientific foundation that is laid in the school and, among these, mathematical ones.

In order that this idea should prevail in the schools, it is indispensable to educate teachers in a corresponding spirit. For this purpose, there is in the Soviet Union an extensive network of higher educational institutions in which mathematics serves as a fundamental subject of instruction. These educational institutions are naturally divided into two groups: special faculties of the universities (the mechanical-mathematical faculties of the universities of Moscow, Leningrad, Kiev, Lvov, Saratov, and Tomsk; and the physico-mathematical faculties in the other thirty-one universities) and the physico-mathematical faculties of the pedagogical institutes. There are in the Soviet Union approximately two hundred pedagogical institutes. The programs of these institutes are based on a four-year period of instruction (at present they are in the process of changing to a five-year period) and in these a great deal of attention is devoted to training students in pedagogical skills. In this article I shall not dwell on questions of mathematical education in the pedagogical institutes, since a comparatively recent article by me* is devoted to this subject.

Mathematical instruction in the universities is directed to a considerable degree towards the development of research habits in the students. For this reason, great attention is paid to special courses, special seminars, and course

* Über die Ausbildung der Mathematik und Physiklehrer in der Sowjetunion, Mathematik und Physik in der Schule, vol. 2, 1955, pp. 489–497.

reports. However, some time is devoted also to the development of methodical habits. There will be more about this later when we discuss the universities in greater detail. The university program is based on a five-year period of instruction.

At the completion of university training, persons showing ability for scientific research are able to continue their education in a graduate school. Graduate training continues for three years. During this time the student is obliged to pass several examinations and to prepare a satisfactory serious independent research paper which serves as a dissertation for the advanced degree of Candidate of the Physico-Mathematical Sciences.

In the Soviet Union there is another advanced degree, that of Doctor of the Physico-Mathematical Sciences. To receive this degree it is necessary to develop a new theory and to obtain substantial mathematical results.

After acquiring the degree of Candidate, it is possible to claim the status of docent and, after the defense of the doctoral dissertation, that of professor.

Mathematical education in the Soviet Union has made successful advances in the past forty years. It is possible to judge this, if only by external evidence: the increase in the number of mathematical investigations carried out in the country, the increased publication of books on mathematics, the number of young people who choose mathematics as their special subject, and so on. However, we see not a few defects in our work. Such defects are present both in the organization of instruction in the secondary schools and also in university education. For a long time many problems have been the subject of discussion by the general public, and other problems still await wide discussion. In these discussions, both scholars and teachers and also the parents of the students take part.

2. The content of the school course in mathematics. A secondary education in the U.S.S.R, presupposes the completion of the ten-year school; children are accepted into the schools at the age of seven. The general ten-year course of education is, at present, carried out in all cities and in a number of rural communities. It is proposed that, in the next four or five years, the general ten-year program will be instituted everywhere. An overwhelming number of schools have a completely uniform structure and educational program; only a small number have special characteristics. Such are the secondary schools for musically and artistically gifted children, and the schools with English, German, or French as the language of instruction. In these schools, beginning with the first year, together with the teaching of the general program there are additional lessons in painting, music, or a foreign language. In schools with a foreign language of instruction, learning the language begins in the first year and, starting with the fifth year, the teaching of a number of disciplines in the selected foreign language begins. All the differences just mentioned are in no way reflected in the extent of mathematical instruction, since the program in mathematics is identical for all secondary schools.

During the first five years, the pupils study arithmetic and also receive instruction in elementary notions of geometry. The purpose of instruction in arithmetic is obviously one and the same for all countries of the world: to teach pupils to operate rapidly, confidently, and intelligently with integers and fractions; to apply this knowledge to the solution of problems, including those of a practical nature. In addition to the solution of a very large number of problems and examples in class, the learner must, for each lesson, carry out independently certain assignments at home. Besides this, from time to time, the pupil must formulate his own arithmetical problems. Elementary instruction in geometry has as its aim to acquaint the learner with the simplest geometrical concepts; in particular, with the concepts of the line, the plane figure, the geometrical solid, the length of a line, the area of a surface, and the volume of a solid. The pupils gain knowledge of the metric system of units and they perform the simplest measurements. At this time, they draw up plans of the classroom, the surroundings, home, apartment, *etc.*, and also learn the simplest approximate calculations.

Systematic courses in algebra and geometry are begun in the sixth year and continue to the tenth year. The course in algebra, in addition to an introduction to literal symbols and a considerable training with the object of obtaining experience in the performance of algebraic operations, also includes the solution of equations of the first and second degree, elements of the theory of inequalities, an elementary course in the theory of irrational numbers, the concept of variable and limit, progressions and logarithms, complex numbers, permutations and combinations, and the binomial theorem. In the algebra course, the pupils are systematically instructed in the idea of functional dependence. They become acquainted with the idea of a system of coordinates and they construct graphs of the simplest functions point by point. In recent years, in pedagogical circles, a prolonged discussion has been going on about the expediency of introducing the elements of mathematical analysis into the program of the secondary schools. Many teachers and scholars have expressed the opinion that the elements of differential and integral calculus should become a constituent and permanent part of the school curriculum. As a result of the discussions, it has been decided to work out gradually that volume of knowledge which it is expedient to communicate to the pupils.

The role of geometry in general education is beyond doubt; without question, it contributes more to logical thinking than does any other mathematical discipline of the school course and, at the same time, it develops spatial concepts; in addition, geometrical knowledge has great indirect practical importance. The content of the course in geometry does not present any peculiarities in comparison with what is done in other countries. Geometry is completed in the tenth year with the study of regular polyhedra and the simplest solids of revolution.

Instruction in trigonometry begins in the eighth year and continues to the tenth year. The course includes the theory of the solution of triangles, the basic

formulas for the trigonometric functions, reduction formulas, inverse trigonometric functions, and trigonometric equations.

Throughout the whole period of instruction, along with class work, the pupils have a very considerable amount of homework (the independent solution of problems, the recitation of theorems, the preparation of models, *etc.*). Problems in this connection are selected both of a purely instructional type and also of substantial content, among them, problems associated with school subjects in the natural sciences.

In the instruction, it is strongly advised to avoid all elements of formalism and aridity, to obtain vitality, lucidity, and, at the same time, logical rigor and harmony. It is recommended that considerable attention be given to the general cultural importance of mathematics in all ages, to its many connections with life in the past and at the present time, and that emphasis be placed on the idea of the continuous development of mathematics. In an explanatory note to the program of the secondary schools, the necessity is pointed out for talks with pupils during classes about various important events in the history of our science.

In conversation with me, American colleagues have expressed interest in how well we in the Soviet Union have succeeded in inculcating into all pupils a conscious mastery of all the material of elementary mathematics. It seems necessary for me to say a few words about this here. I think that our schools are still far from a satisfactory solution of this important problem. Even now there is still a serious percentage of students to whom the spirit of mathematical thinking remains alien. Frequently, such pupils even learn the contents of the course fairly well, but their knowledge in this connection remains formal, and a comparatively insignificant change in the conditions of the problem lands them in serious difficulties. Besides the completely obvious diversity in the inclinations of the pupils and the presence of pupils for whom mathematics is a heavy burden, a certain number of students fall into such a situation because of various mistakes in methods made by the teacher, arising out of the lack of skill in instilling interest in the discipline being taught. Moreover, I think that there is a certain amount of formalism in exposition in some textbooks as well. Thus, for example, for the first five years a pupil is occupied with arithmetic and during these years an enormous number of problems are solved. The manuals of methodology attempt to classify rigorously the diversified problems constantly encountered into groups and, for each group, to elaborate specific methods of solution. This in itself is not bad, but what is bad is that a proportion of the teachers elevate these methods and the application of the recommended procedures into sacred canons. As a result, the students mechanically learn these rules and their thinking is not prepared for a conscious approach to each problem. The tendency to maintain mathematical purity in the solution of arithmetical problems, in my view, also complicates the understanding. At the same time, the majority of problems of this type are completely elementary and psychologically clear, and can be solved by the reduction to one or more equations

of the first degree. The introduction into arithmetic of elements of an algebraic approach will simplify the learning of material, relieve students from complicated and artificial methods of solution, and make the approach to arithmetical problems more natural. What is curious is that later, in the seventh year, it is necessary to wean the students from the very methods they learned with difficulty. Moreover, purely arithmetic methods of solution are almost never used in actual practice.

Soviet mathematical circles see a number of defects in the school curriculum. Particularly obvious are the defects in present textbooks. In this connection, a number of prominent mathematicians have taken part in revision of old textbooks and the writing of new ones and also in resolving various questions of the school curriculum. For example, the textbook for arithmetic has been subjected to a significant revision by Professor A. Y. Khinchin; a textbook on algebra has been written by P. S. Alexandrov and A. N. Kolmogorov; L. A. Luisternik and A. F. Bermant have written a textbook on trigonometry; V. L. Goncharov, in a number of books, has worked out methodological questions of the development of functional thinking in students of various ages. It is not possible here to touch on all questions connected with the Soviet school textbook literature; a satisfactory clarification of these questions requires a special article. It is beyond doubt that the fundamental role in the pedagogical process is played by the teacher. A teacher who reduces his task to the point that he only communicates to the pupils that sum of knowledge specified in the curriculum and merely teaches the pupils to deal with routine problems, rarely achieves any success. From the teacher is demanded enthusiasm for his subject and the conviction that his subject is one of the most important affairs of the nation. From the teacher it is demanded that he implant in the students a love for mathematics and a conviction of the creative power of his pupils, and that he describe in general outline before their intellectual gaze, the impressive picture of the uninterrupted development of mathematics with its limitless connections with technology, the natural sciences, and other manifestations of human activity.

3. School mathematics circles. Mathematical Olympiads. The love of mathematics, the encouragement of an inclination to read mathematical books, and the expanding of the mathematical horizon beyond the limits of the compulsory program cannot be developed seriously exclusively during class periods. Once an interest in the science has appeared, it is necessary to develop and to stimulate it daily. In the Soviet Union, a number of steps are being taken which help, in a considerable degree, to bring to light mathematical capabilities in the students and to help them consolidate the mathematical interests that arise. Among steps of this kind belong, first of all, the school mathematics clubs, mathematics circles for school pupils at the universities and other institutions of higher learning, and the school mathematical Olympiads. All these are com-

pletely voluntary both for the pupils and for the teachers. Nevertheless, they attract great interest among the pupils and among a large number of school teachers, instructors in the higher educational institutions, and students and graduate students of the universities, who give willingly of their energy and time to this extra work with young mathematical enthusiasts.

The school mathematics clubs do not have established programs but each works in the direction of the interests of teachers and pupils. As a rule, these clubs pay close attention to anniversaries and observe those of outstanding mathematicians by special evening meetings and by issuing mathematical posters. For example, last year the hundredth anniversary of the death of Gauss was observed and, at the beginning of this year, that of the death of N. I. Lobachevski. At the moment, in many schools, papers are being prepared on the life and creative work of Euler in connection with the imminent two hundred-fiftieth anniversary of his birth. It is understood that, in addition to this purely historical work, the students in the clubs solve complicated problems, examine the most simple logical paradoxes, prepare models, and familiarize themselves with some questions that go beyond the limits of the curriculum. To help the school mathematical circles, several pamphlets have been written and published. By way of example, I refer to the following: A. Y. Khinchin, *Three Pearls in the Theory of Numbers*; Y. I. Perelman, *Living Mathematics*; B. V. Gnedenko and A. Y. Khinchin, *Elementary Introduction to the Theory of Probability*; L. A. Luisternik, *Geodesic Lines*, B. L. Kardemski, *Mathematical Know-How*. Unfortunately, these booklets for the pupils are not arranged in a series, but each is issued separately. In addition to these, there is a series of pamphlets under the general title, *Popular Lessons in Mathematics*, in which there is a considerable amount of well-selected material for the work of the school circles.

We ascribe great importance in the development of an interest in mathematics to a program begun in 1934 by Leningrad University which subsequently spread throughout the Soviet Union and, to some degree, even beyond its borders. I have in mind the so-called mathematical Olympiads. The Olympiads consist of three different parts: (1) the work of the mathematical circles attached to universities or to other higher educational institutions; (2) a series of lectures on various mathematical topics given by university professors; (3) a written contest in the solving of problems.

In the mathematical circles attached to the universities, problems of unusual type that develop the mathematical training of the participants are presented. As a rule, university students and graduate students direct these circles and the pupils are eager to visit and to participate actively in them. As a result of many years of work by such circles attached to Moscow University, a series of original mathematical monographs for pupils has appeared under the general title, *Library of the Mathematical Circle*. Eight books of this series have appeared up to the present time, with the following titles:

Selected problems and theorems of elementary mathematics

1. *Part 1. Arithmetic and algebra* (D. O. Shklyarskii, G. M. Adelson-Velskii, N. N. Chentsov, A. M. Yaglom, I. M. Yaglom).
2. *Part 2. Plane geometry* (D. O. Shklyarskii, N. N. Chentsov, I. M. Yaglom).
3. *Part 3. Solid geometry* (D. O. Shklyarskii, N. N. Chentsov, I. M. Yaglom).
4. *Convex figures* (I. M. Yaglom, V. G. Boltyanskii).
5. *Non-elementary problems discussed in an elementary way* (A. M. Yaglom, I. M. Yaglom).
6. *Mathematical discussion* (E. B. Dynkin, V. A. Uspenski).

Geometrical transformations

7. *Motion and transformation analogies* (I. M. Yaglom).
8. *Linear and circular transformations* (I. M. Yaglom).

The school circles generally work once a week in the evenings.

Lectures for the students are given on Sundays once every two weeks or once a month. These lectures attract a large number of listeners. As examples, I quote the titles of several lectures given at different cities: *Mathematical Paradoxes and Sophisms* (Lvov, 1949), *Regular Polyhedra and Euler's Theorem* (Lvov, 1949), *Geometry of Points* (Moscow, 1949), *The Arithmetic of Residues with Certain Applications* (Moscow, 1949), *Why We Study Mathematics* (Stalingrad, 1953), *Random Walks* (Kiev 1954), *What is a Differential Equation?* (Moscow, 1954), ... The titles quoted naturally do not reflect, even to a small degree, the diversity of the topics of the lectures. The lectures are usually given separately for pupils in the seventh and eighth years and for pupils in the ninth and tenth years.

The final part of the Olympiad is a competition in the solution of problems; generally, two rounds are held. Only those participants of the Olympiad who have successfully solved the problems in the first round are admitted to the second round. In order to convey some idea of the interest of students in the Olympiads, I shall call attention to some figures: 800 students took part in Lvov in 1955; 600 to 950 in Kiev every year; 800 in the first round in Moscow in 1948, 1350 in 1953. The Olympiads in other cities have an equally extensive scope. Because of the large number of participants, the competitions are administered separately to each of the seventh, eighth, ninth, and tenth years. Although in Moscow, as was pointed out, 1350 took part in the first round, only 517 were admitted to the second round. The number who successfully completed the second round was 262. Those who particularly distinguished themselves were awarded prizes (sets of mathematical books, valuable gifts). Three students obtained first class, 15 second class, 24 third class, and 69 received certificates of merit. Elegance and originality in the solutions was taken into account in awarding prizes. After each round, an analysis of the problems proposed was made.

Let me add that newspapers and magazines for children and young people publish articles about mathematics and propose to their readers problems that require ingenuity for their solution.

The mathematical Olympiads materially assist in the selection of talented young people for the new generation of Soviet mathematicians. There is already quite a considerable percentage of former members of the school mathematics circles and prize winners of the Olympiads on the staffs of universities. Among those who enthusiastically direct the school mathematics circles are many who were themselves, a few years before, active members of the circles.

4. Sample problems. I think that it is not without interest to quote a number of problems that were proposed for the Olympiad rounds, as well as a number of those that were solved in the mathematics circles in preparation for the competitions. A large number of collections of such problems had been published by local presses—Moscow, Lvov, and other universities, sometimes in the form of thin pamphlets and sometimes even as small mathematical leaflets. Here are some problems presented in preparation for the eleventh Moscow Olympiad:

1. Every man who has lived on earth has made a certain number of handshakes. Prove that the number of men who have made an odd number of handshakes is even.

2. Prove that $m/3+m^2/2+m^3/6$ is an integer for every integer m.

3. One coin out of 80 golden coins is false (lighter). How can you determine the false coin in four weighings on a scale with two pans and without weights?

4. How many times in 24 hours are the hands of a clock at right angles?

5. Prove that, in any triangle, the sum of the lengths of its three medians is less than its perimeter and greater than its semiperimeter.

6. What is the maximum number of parts into which the plane can be divided by means of 15 straight lines?—of 4 circles?

7. Through a given interior point of a circle construct chords (a) of given length, (b) of minimum length.

8. Prove that all numbers in the sequence 100001, 10000100001, 1000010000100001, $\cdots$, are composite.

9. Construct a triangle given the points where its median, an altitude, and a bisector of one of its angles intersect the circumference of its circumscribed circle.

10. What is the maximum number of acute angles that may occur in a convex polygon?

11. Solve the system of equations $x_1+x_2+x_3=6$, $x_2+x_3+x_4=9$, $x_3+x_4+x_5=3$, $x_4+x_5+x_6=-3$, $x_5+x_6+x_7=-9$, $x_6+x_7+x_8=-6$, $x_7+x_8+x_1=-2$, $x_8+x_1+x_2=2$.

12. Solve the equation $(x+a)(x+2a)(x+3a)(x+4a)=b^4$.

13. Prove that the sum $\frac{1}{2}+\frac{1}{3}+\cdots+(1/n)$ cannot be an integer for any integer n.

14. Rationalize the denominator of the fraction $1/(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})$.

15. Given four points A, B, C, D, in space such that AB is perpendicular to CD and AC is perpendicular to BD, prove that AD is perpendicular to BC.

16. Into how many parts can n planes divide space?

17. Factor the expression $a^{10}+a^5+1$.

18. Prove that, if α and β are acute angles and $\alpha<\beta$, then $(\tan\alpha)/\alpha<(\tan\beta)/\beta$ and $(\sin\alpha)/\alpha>(\sin\beta)/\beta$.

19. Through a point A inside an angle there passes a straight line which forms, with the sides of the angle, a triangle of minimum area. Prove that the segment of this straight line contained between the sides of the angle is bisected by the point A.

20. Prove that if a, b, c, d are positive numbers and the system of inequalities $ax-by<0$, $cx+dy<0$, $x>0$, $y>0$, has a solution, then the inequality $ad-bc<0$ holds. The converse is also true, namely, if the latter inequality holds, then the system of inequalities has a solution.

I shall now state problems that were given at the time of the competitions. The following were given to pupils of the seventh grade at the 16th Moscow Olympiad.

First round

1. Prove that, in a trapezoid, the sum of the angles subtended by the smaller base is greater than that subtended by the larger base.

2. Find the least integer, made up of 1's only, that is divisible by the number $3\cdots3$, consisting of one hundred 3's.

3. Bisect a line segment using a try square (with a try square, it is possible to construct straight lines and erect perpendiculars, but it is not permissible to drop perpendiculars).

4. Prove that, for every positive integer n, the integer $n^2+8n+15$ is not divisible by $n+4$.

Second round

1. Prove that the g.c.d. of the sum of two numbers and their l.c.m. is equal to the g.c.d. of the numbers themselves.

2. A quadrilateral is circumscribed about a circle. Its diagonals intersect at the center of the circle. Prove that the quadrilateral is a rhombus.

3. Eleven geared wheels are located in a plane in such a way that the first is in mesh with the second, the second with the third, and so on. Finally, the eleventh is in mesh with the first. Can the wheels in such a system turn?

4. A thousand points are the vertices of a 1000-angled convex figure inside which are 500 other points, no three of which lie on the same straight line. The given 1000-angled figure is divided into triangles in such a way that the 1500 given points are the vertices of triangles and these triangles have no other vertices. How many triangles are obtained by such a division?

5. Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 &= 1, \\ x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 &= 2, \\ x_1 + 3x_2 + 5x_3 + 6x_4 + 6x_5 &= 3, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 8x_5 &= 4, \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 &= 5. \end{aligned}$$

The following problems were given to pupils of the 8th grade in Kishinev in 1953.

First round

1. Prove that the bisectors of the angles of a rectangle, when produced, intersect to form a square.

2. Prove that the product of any four consecutive integers when increased by 1 is a perfect square.

3. Factor $bc(b+c)+ca(c-a)-ab(a+b)$.

4. Prove that two triangles are congruent if two medians and an altitude of one are equal to two medians and an altitude of the other.

Second round

1. Prove that the integer $10\overset{[49]}{\cdots}050\overset{[99]}{\cdots}01$ cannot be the cube of an integer.

2. Prove that, if the roots of the equation $x^2+px+q=0$ are real, then the roots of the equation $x^2+px+q+(x+a)(2x+p)=0$ will be real for all real numbers a.

3. Prove that the straight lines joining the feet of the altitudes of a triangle form a new triangle in which these altitudes of the given triangle bisect the angles of the new triangle.

4. On a given straight line, construct a point such that the difference of its distances from two fixed points is a minimum.

At the Kiev Olympiad in 1954, the following problems were given to pupils of the ninth grade.

First round

1. Determine how many minutes after 4 o'clock it will take for the minute hand to overtake the hour hand.

2. Two cyclists left simultaneously, one traveling from A to B and the other, B to A. Each traveled at a constant speed and, on arrival at the destination point, immediately turned around and returned. They met for the first time 40 km. from B. Eight hours after the first meeting, they met for the second time 20 km. from A. Find the distance from A to B and the speed of each cyclist.

3. The lengths of the sides of a triangle are in arithmetic progression. Prove that the radius of the inscribed circle is one third the length of one of the altitudes.

4. A circle is inscribed in a triangle and a square is circumscribed about the triangle. Prove that more than half the perimeter of the square lies inside the triangle.

5. Let S_m denote the sum of the first m terms of an arithmetic progression. Prove that $3(S_{2m}-S_m)=S_{3m}$.

Second round

1. Prove that the planes passing through the edges of a trihedral angle and bisecting the opposite faces intersect in a single straight line.

2. Rationalize the denominator of the fraction $1/(1-\sqrt[n]{a})$.

3. If, from each of two numbers, one half the smaller is subtracted, then what remains of the larger number is three times what remains of the smaller number. How many times greater than the smaller number is the larger number?

4. In a geometric progression, it is given that $a_{m+n}=A$, $a_{m-n}=B$. Find a_m and a_n.

5. A segment of constant length moves so that its end points are on two straight lines that intersect at right angles. Find the locus of the midpoint of the segment.

The following problems were given to pupils of the tenth grade in the mathematical Olympiad at Ordzhonikidze in 1955.

First round

1. Prove that $n^{n-1}-1$ is divisible by $(n-1)^2$ for an arbitrary integer n.

2. For a triangle with sides a, b, c, and semiperimeter p, prove that $a^3+b^3+c^3+1/a+1/b+1/c \geqq 4p$.

3. Prove the identity $1!1+2!2+ \cdots +n!n=(n+1)!-1$.

4. In an isosceles triangle ABC, $AB=BC=b$, $AC=a$, $\angle ABC=20°$. Prove that $a^3+b^3=3ab^2$.

5. Solve the equation $1/(\sin x)-1/(\cos x)=2\sqrt{2}$.

Second round

1. Prove that all numbers of the form $N=1 \overset{[k]}{\cdots} 12 \overset{[k+1]}{\cdots} 25$ are perfect squares and determine the form of $\sqrt{N}$.

2. Two containers A and B, each of the same weight, contain different amounts of water. The weight of A including the water is $\frac{4}{5}$ the weight of B including the water. If all the water in B were emptied into A, then the weight of A would be eight times the weight of B. Given that the water in B weighed 500 gm. more than the water in A, determine the weights of the containers and the original amounts of water in each. (Give arithmetical and algebraical solutions.)

3. If, in a triangular pyramid, sides a, b, c, are mutually orthogonal, and the altitude from the vertex to the base is h, prove that $1/h^2=1/a^2+1/b^2+1/c^2$.

4. Prove that the inequality $\operatorname{ctn} \alpha \geqq 1+\operatorname{ctn} 2\alpha$ is valid if $0<2\alpha<180°$.

5. Determine the maximum and minimum values of the function $y=(x^2+1)/(x^2-x+1)$ and draw its graph.

A considerable number of reports of the Olympiads, together with sample problems, has been published in the journal *Uspehi Mat. Nauk*. It should be said, however, that reports about the majority of the Olympiads have not been published anywhere.

5. The training of mathematicians in the universities. The training of mathematicians in the Soviet Union is carried out, as I mentioned, in 36 universities and in nearly 200 pedagogical institutes. The function of the pedagogical institutes is clear from the name itself; it is necessary to observe, however, that from them come not only the teachers for the secondary schools, but also a certain

number of mathematical specialists. The mathematical training in the universities is much more diverse and much broader. It is natural that the universities are also the basic source of supply of research mathematicians for the scientific institutions, engineering designing offices and laboratories, and for institutions of higher learning. Experience of recent years has shown that the large majority of mathematicians coming from the universities is swallowed up by research institutes and by industry, whereas the universities release only an insignificant number of their graduates for the secondary schools. It is, therefore, particularly important that young people having capabilities and a liking for mathematics should come to the universities. The very process of selecting individuals with mathematical capabilities is exceedingly difficult and we have not yet reached the point where we can attract to the physico-mathematical faculties all those with mathematical capabilities. Moreover, not all who do enter these faculties have such abilities. Certainly, the Olympiads give us remarkable help in the selection of mathematically gifted young people. To a certain degree, entrance examinations aid in this, and only students who have completed their school training with a gold medal are excused from these entrance examinations (but even these must pass a preliminary individual interview with the principal professors of the faculty). A special pamphlet *The Mathematics Profession*, by the academician A. N. Kolmogorov serves as a guide to young people in deciding on their vocation. This pamphlet has been issued in tens of thousands of copies. Certainly, in addition to the search for persons of exceptional mathematical ability, it is no less important to educate young people of average ability up to the level of good, enthusiastic mathematicians. It is understandable that this task depends entirely on the program of instruction, on the quality of the teaching staff of the university, and on those traditions that have been fostered in the student body. It is beyond doubt that the spirit of scientific enthusiasm, if present in the professorial staff, is also transferred to the students. It is also beyond doubt that, if the traditions of the student body are such that scientific knowledge and the desire to further science are fundamental, then the new students entering the university will take up these traditions and these traditions will exert a tremendous influence on the scientific development of youth. To a considerable degree all these factors are of a psychological nature and, important though they are, I shall not examine them in more detail here. I shall limit myself here to an account of the typical educational plans for mathematicians in the universities.

The special mathematical training in Soviet universities has already begun in the first year. In addition to mathematical subjects, the students take only foreign languages (English, French, or German), philosophy, political economy and the history of the party, with also a small amount of compulsory physical activity (two hours a week in the first and second years). For students who are interested in sports, music, choreography, theatricals, *etc.*, there are faculty and university organizations in which they have the opportunity to become proficient in these supplementary interests. Every institution of higher learning

has its own choir, orchestra, ballet, *etc.*, whose membership consists of both students and instructors.

The compulsory courses in mathematics and physics for students specializing in mathematics are given in the following table:

subject	number of hours												
	total	lec-tures	exer-cises	labora-tory	semester								
					1	2	3	4	5	6	7	8	9
1. Analytical geometry	188	102	86	—	6	5	-	-	-	-	-	-	-
2. Mathematical analysis	476	272	204	—	8	8	6	6	-	-	-	-	-
3. Higher algebra	174	102	72	—	4	3	3	-	-	-	-	-	-
4. Differential geometry	102	68	34	—	-	-	3	3	-	-	-	-	-
5. Differential equations	136	68	68	—	-	-	4	4	-	-	-	-	-
6. Physics	340	136	50	154	-	-	-	4	7	5	4	-	-
7. Equations of mathematical physics	100	70	30	—	-	-	-	-	-	4	2	-	-
8. Theory of functions of a complex variable	100	68	32	—	-	-	-	-	2	4	-	-	-
9. Calculus of variations	30	30	—	—	-	-	-	-	-	-	-	3	-
10. Drawing, elements of descriptive geometry	68	18	50	—	2	2	-	-	-	-	-	-	-
11. Theoretical mechanics	240	150	90	—	-	-	4	4	6	-	-	-	-
12. Astronomy	54	36	—	18	-	-	-	-	3	-	-	-	-
13. Foundations of geometry	64	64	—	—	-	-	-	-	-	-	4	-	-
14. Theory of functions of a real variable and functional analysis	86	86	—	—	-	-	-	-	2	4	-	-	-
15. Integral equations	36	36	—	—	-	-	-	-	-	-	2	-	-
16. Theory of probability	54	54	—	—	-	-	-	-	3	-	-	-	-
17. Theory of numbers	30	30	—	—	-	-	-	-	-	-	-	3	-
18. Elementary mathematics	70	36	34	—	-	-	-	-	-	-	4	-	-
19. Electives	328	200	128	—	-	-	-	-	-	4	4	8	6
20. Mathematical practice, technical work on computers	210	—	36	174	-	-	-	-	4	4	4	3	-
21. Practice teaching	60	—	—	60									
22. Pedagogy	64	64	—	—	-	-	-	-	-	4	-	-	-
23. Mathematical methods	70	40	—	30	-	-	-	-	-	-	4	-	-

Practice teaching is carried out in the eighth semester in secondary schools under the direction of instructors experienced in methodology. The practice teaching lasts six weeks, during which time the students do not have classes at the university.

We should make several observations concerning the program given above. First of all, it must be said that at the end of each semester students are required

to pass examinations in the courses taken. In a large number of courses in which there are exercises, they are also required to make reports to an assistant in which they must give evidence of ability to solve problems of a standard pattern.

In the second, third, and fourth years, students are required to write term papers. As a rule these papers are in the nature of a compilation; they are designed chiefly for developing independence. To produce these papers, students are required to read certain monographs and periodicals and present a written report. In the fifth year each student prepares a graduating thesis. Many of these are in the nature of independent research. The topics of the theses are of great diversity and naturally depend on the scientific interests of the supervisors. The graduating theses are publicly defended by the students, and afterwards each student must pass state examinations. The state examination in mathematics includes the fundamental ideas and results of those mathematics courses taken by the student. The last half-year (tenth semester) is spent, in the majority of cases, in preparing for the state examinations and the graduating thesis.

The topics of the special courses and seminars are exceedingly varied. The student may elect those courses in which he is most interested. Specialization begins with the second year: students choose between mathematics and mechanics; in addition, within mathematics (or mechanics) itself they elect, in the third year, a narrow field of mathematics (or mechanics) such as geometry, topology, algebra, theory of probability, theory of numbers, *etc.* Students elect the majority of special courses in the chosen field. At the same time, the instructors recommend that students attend special courses of a general mathematical nature and also some with a natural science content.

In university circles, discussions are constantly taking place about the nature of the mathematical training of the students. Particularly just now, insistent voices are being heard to the effect that we are lecturing to the students too much and, for this reason, leaving too little time for independent thinking on scientific topics. The proper point of view is put forward that the universities must exert maximal pressure to stimulate active scientific interests in the students. In a number of universities attempts are already being made to decrease the students' work load. Furthermore, the opinion is held that the existing curriculum does not contain a course that would bring mathematical theories together on the basis of the modern status of the science. A number of courses in the curriculum, such as the theory of functions of a real variable, the calculus of variations, and integral equations, are presented to the students as separate and unconnected disciplines from the point of view of ideas, whereas they are, in fact, portions of a single mathematical theory. In this connection, it is proposed to give mathematical analysis in the first and second years and to include, in essence, only the results of the development of mathematics in the 18th and 19th centuries, and to complete analysis with one more course. In relation to this, functional analysis will unite and clarify from a single point of view a series of classical divisions of mathematical analysis. This will enable students to under-

stand the processes of contemporary mathematics more thoroughly, to understand mathematics as a single harmonious whole, and, at the same time, to become acquainted with one of the most important trends in contemporary mathematical creativeness. This idea was tried out as an experiment at Moscow University by Professor A. N. Kolmogorov.

The curriculum for students specializing in the fields of mechanics (general mechanics, theory of elasticity, hydro- and aero-mechanics) is somewhat different from the general curriculum for mathematicians stated earlier. This difference begins in the second year, although at that time, the majority of courses in mechanics are taken together with the mathematicians. The basic division begins in the third year when the greatest number of courses in mechanics are taken separately from the mathematicians. The first ten subjects in the mathematics curriculum coincide exactly with that for mechanics in the number of hours, except that the calculus of variations is taken by students in mechanics in the 7th instead of the 8th semester. We therefore state below only those subjects which have a different number of hours from those of the mathematicians, or those subjects that the mathematicians do not take.

subject	number of hours												
	total	lectures	exercises	laboratory	semester								
					1	2	3	4	5	6	7	8	9
11. Theory of probability	50	36	14	—	–	–	–	–	–	–	–	3	–
12. Mathematical practice. Technique of work with computing machines	104	—	104	—	–	–	–	–	4	2	–	–	–
13. Theoretical mechanics	310	210	100	—	–	–	5	3	6	4	–	–	–
14. Strength of materials	108	60	28	20	–	–	–	–	6	–	–	–	–
15. Hydromechanics. Aeromechanics	170	134	—	36	–	–	–	–	–	4	6	–	–
16. Theory of elasticity	68	68	—	—	–	–	–	–	–	2	2	–	–
17. Electives	336	176	160	—	–	–	–	–	–	–	2	10	10
18. Laboratory practice	176	—	—	176	–	–	–	–	4	2	4	–	–

The mechanics students take their practical work in engineering design offices and in factories. Like the mathematicians, the mechanics students must pass state examinations and, in the tenth semester, complete the writing of the graduating thesis.

The course in mathematical practice is deserving of special mention. The purpose of this course is to compel students to work through, independently, problems requiring for their solution the application of results taken from many mathematical disciplines, the performance of numerical calculations, or the understanding of the physical aspect of the problem. Each student is given a few such problems—five or six a year. We shall give examples of such problems.

Sample problems for the 3rd year

1. Construct a table of values of the function $y(x)$ satisfying the differential equation $y''-xy=0$, with initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0'$. Construct the table for eleven values of x in an arithmetic progression, with first term x_0 and difference 0.02. The calculations of the values must be correct to within 0.001 ($x_0=-3.40$; $y_0=0.1721$; $y_0'=-1.3102$).

2. Given the differential equation $y'=a(x-\alpha)(y-\beta)/y$, where a, α, β, are real numbers and $\alpha<\beta$. (In choosing parameters, one should avoid having $a>10$.) It is required to

1° find the coordinates of singular points and to determine their type;

2° construct integral curves originating at the saddle point, either within the segment $\alpha-1\leqq x\leqq\beta+1$, or up to the points where these integral curves intersect the x-axis;

3° restrict the constructed curves to be within a rectangle having its sides parallel to the coordinate axes; the rectangle should have the property that each of the constructed curves has at most one point in common with the sides of the rectangle; construct one integral curve in each of the regions into which the integral curves originating at the saddle point divide the rectangle.

Note. All graphs should be plotted (a) with such accuracy that the angle between the tangent to the plotted curve and the direction of the field is not greater than 45°, and (b) so that all vertices of the broken curve that approximates the actual curve lie on the same integral curve.

3.1. A telephone station receives three types of calls, (a) ordinary, (b) priority, and (c) "flash." Calls of each type are serviced in the order received. Furthermore, calls of higher category are serviced before calls of lower category. The probability of receiving a call of the ith category during the time dt is $\alpha_i dt$, independently of the distribution of previous calls. The probability that a conversation in progress will end during the time dt is bdt, independent of how long it has been in progress. The system is stationary, *i.e.*, a steady state has been established. Find the mean waiting time for subscribers in each of the categories.

3.2. In this part, the same conditions apply as in 3.1, but there are s telephone lines. If, at the time when a call is received, all the lines are in use by subscribers of the same or higher category, the call is refused. In the contrary case, he is immediately connected, and, furthermore, one of the conversations of a lower category is interrupted if necessary for this purpose. Compute the probability of the refusal of a call in each of the categories.

3.3. Two stations A and B are connected by k lines. During an interval dt, A receives a new call with probability $\alpha_1 dt$ and B, with probability $\alpha_2 dt$. If all lines are busy, the calls received by B are refused and subscribers calling A must wait their turn. The probability of concluding a conversation during the time

dt is βdt. The system is stationary. Compute the probability that an incoming call to A will be rejected. Compute the mean waiting period for a subscriber calling B.

4. A system may be in one of the states numbered $1, \cdots, r$. If, at time t the system is in state i, then, during an interval dt, the system may change to state j with the probability $\alpha_{ij}dt$ (independently of its behavior previously). Assuming that a particle leaves state 1 initially, compute the probability that in time t it will reach state 2 exactly n times.

The examples stated show, to some extent, the nature of the problems in the mathematical practice course. Other problems require of the student, a knowledge of the theory of functions of a complex variable, the calculus of variations, *etc.*

It should be noted that each student is given a different set of values of the parameters that may enter into the conditions of the problem.

6. The graduate school. The students who, during the university course, show the best ability are recommended by their professors for further mathematical study in the graduate school. The purpose of graduate training is the education of mathematicians completely dedicated to scientific work. The overwhelming majority of creatively active Soviet mathematicians have gone through graduate training. Under the regulations, graduate training is continued for three years, during which time the student is supported by a grant from the state. Each graduate student is assigned to a supervisor who is consulted whenever necessary for advice of a scientific nature. The supervisor discusses with the student the mathematical literature in the form of periodicals or monographs which the student has read and also makes arrangements with the student regarding plans for his scientific training. Since the interests of the graduate students are extremely varied, there are no standard graduate programs. There are only very general directives about what each student must prepare. Each student must also pass two comprehensive examinations and make reports to his supervisor on two narrowly specialized subjects. For example, one of my students, who is specializing in the theory of probability, must pass an examination in functional analysis that includes Chapters 14–21 of *Functional Analysis and Semigroups*, by E. Hille, *Thêorie des Distributions*, by Laurent Schwartz, and the theory of Fourier transforms. His second examination consists of subjects in physics—quantum mechanics, and several chapters of statistical physics. His preparation in the theory of probability consists of a study of the literature on the theory of the summation of random variables and of *Mathematical Methods of Statistics* by H. Cramer. A second student of mine also has functional analysis in his first examination (the theory of Fourier transforms being replaced by *Leçons d'analyse fonctionnelle* by F. Riesz and Bela Sz-Nagy). In his second examination he takes differential equations of parabolic type. His special

interests are in the field of Markov processes; this is taken into account in those special subjects that are included in his program.

In addition to the examinations on mathematical subjects, the graduate student is required to pass an examination in one foreign language and in philosophy. A second language is studied on a voluntary basis. Graduate students are not required to follow lecture courses, but actually each student does attend those special university courses that interest him. The fundamental concern of the graduate student is to do independent scientific work in a selected field. Near the end of his training, he must present independent investigations that contain new mathematical results of scientific value.

The dissertation is defended in public in the presence of two official opponents who are specialists in the given field. After a successful defense, the student receives the degree of Candidate of the Physico-Mathematical Sciences. Many students have already prepared serious scientific papers at the beginning of their graduate training. For them the last year of graduate school is in the nature of work at an institute for advanced study.

Naturally, mistakes do occur in the acceptance of candidates for the graduate schools when incompetent persons, or those who cannot find a topic for investigation that interests them and that they can handle, are accepted. Such graduate students, after receiving additional scientific training, finish graduate school without the defense of a thesis and without obtaining the scientific degree.

A large number of persons in engineering design offices, in institutions of higher learning, and in schools, defend their dissertations without preliminary study at a graduate school. Such persons are required to pass the qualifying examinations beforehand.

The requirements which prevail in the Soviet Union for the degree of Candidate correspond approximately to those for the degree of Ph.D.* in the United States.

In order to receive the Soviet degree of Doctor of Physico-Mathematical Sciences, it is necessary to defend an additional dissertation in which there are much higher scientific requirements. Naturally, there are not nearly as many doctoral dissertations defended in the Soviet Union as there are Candidate dissertations.

Graduate work is given in all Soviet universities and also in all scientific research institutes attached to the Academy of Sciences of the U.S.S.R. and to the Academies of Sciences of the Union Republics. Some pedagogical as well as polytechnical institutes with strong scientific resources are also permitted to give graduate work in mathematics.

In conclusion, I should like to reiterate what I have expressed on more than one occasion in this article for various reasons, namely, we consider that our organization of mathematical education is moderately good, but, for just this reason, we think that it needs further improvement.

* The Russian original is "the degree of Doctor of Sciences." This is, presumably, a misunderstanding on the part of the author. Ed.

GENERALIZATIONS OF PASCAL'S TRIANGLE*

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1. Introduction. In a recent paper, Freund† proved the formula

$$N_m(r, k) = \sum_{j=0}^{m} N_m(r - j, k - 1) \tag{1}$$

where $N_m(r, k)$ is the number of distinct ways in which r indistinguishable objects can be distributed in k cells allowing at most m objects to fall in each cell. He also showed that the results obtained from (1) could be displayed in a generalized Pascal triangle where the property which was generalized was the rule of formation

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{2}$$

We wish to generalize a different property associated with the Pascal triangle, show that this approach provides an alternative proof of Freund's formula, and finally establish another identity involving binomial coefficients. The method would seem to have the advantage of directness and might be of use in establishing other identities.

2. Pascal's triangle. The elements of Pascal's triangle

$$\begin{array}{ccccc} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

are, of course, the binomial coefficients $\binom{n}{r}$.

If we think of the elements of the $(n+1)$st row as the coefficients of the expansion of

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r},$$

we see, exchanging x and y, that

* This investigation was supported (in part) by a grant from the National Research Council.

† J. E. Freund, Restricted occupancy theory—a generalization of Pascal's triangle, this MONTHLY, vol. 63, 1956, pp. 20–27.

$$\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \equiv \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

and conclude that $\binom{n}{r} = \binom{n}{n-r}$. The relation (2) follows at once from the fact that $(x+y)^{n+1} = (x+y)(x+y)^n$ where we adopt the convention that $\binom{n}{r} = 0$ if $n < r$ or if $r < 0$. This relation in turn implies that

$$\sum_{x=s}^{n} \binom{x}{s} = \sum_{x=s}^{n} \left\{ \binom{x+1}{s+1} - \binom{x}{s+1} \right\} = \binom{n+1}{s+1},$$

which shows that, if we add down any column of Pascal's triangle, the sum is equal to the entry below and to the right of the term at which we stop. Thus many identities in the binomial coefficients may be expressed in terms of the Pascal triangle and proved with the help of properties of that triangle.

3. A generalization. If we think of the rows of Pascal's triangle as made up of coefficients of polynomials obtained by multiplying $x+y$ by itself, it is clear that many triangles could be constructed by changing the initial polynomial, $x+y$, and the common multiplier. If we replace $x+y$ by $1+x+ \cdots +x^m$, the terms of the $(k+1)$st row will be the coefficients of the combinatorial generating function $(1+x+ \cdots +x^m)^k$. If we associate one factor with each of the k cells in the occupancy problem, and select x^i from the corresponding factor if i objects are assigned to a given cell $(i=0, \cdots, m)$, it is clear that $N_m(r, k)$ is the coefficient of x^r in the expansion of the above generating function. Equation (1) then follows.

4. A second generalization. Our interest was aroused in such generalizations when, in the course of a statistical problem, it became necessary to prove the identity

$$\sum_{s=1}^{u} \binom{u}{s} \left\{ \binom{s-1}{u-s-1} - \binom{s-1}{u-s} \right\} = (-1)^u. \tag{3}$$

The left hand side of the identity is clearly the coefficient of x^u in

$$\begin{aligned} f(x; u) &= (x-1) \sum_{s=1}^{u} \binom{u}{s} x^s \sum_{i=0}^{\infty} \binom{s-1}{i} x^i \\ &= (x^2 - x) \sum_{s=1}^{u} \binom{u}{s} x^{s-1}(1+x)^{s-1} \\ &= (x^2 - x) \sum_{s=1}^{u} \binom{u}{s} x^s (1+x)^s / (x^2 + x) \\ &= (x^2 - x)[(1 + x + x^2)^u - 1]/(x^2 + x) \\ &= \sum_{k=0}^{u-1} (x^2 - x)(1 + x + x^2)^k. \end{aligned}$$

This result suggests forming a generalized Pascal's triangle whose second row consists of the coefficients of $0-x+x^2$, succeeding rows being obtained by multiplication by $1+x+x^2$, and recording the coefficients according to increasing powers of x. The first few rows are

$$\begin{array}{ccccccccc}
1 \\
0 & -1 & 1 \\
0 & -1 & 0 & 0 & 1 \\
0 & -1 & -1 & -1 & 1 & 1 & 1 \\
0 & -1 & -2 & -3 & -1 & 1 & 3 & 2 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

We wish to prove that the sum of the elements in the $(u+1)$st column, down to and including the element in the $(u+1)$st row, is $(-1)^u$. To do so, we develop some of the properties of this generalized triangle. Let the polynomial corresponding to the $(k+2)$nd row be

$$(x^2 - x)(1 + x + x^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^t.$$

Then

$$(x^2 - x)(1 + x + x^2)^{k+1} = \sum_{t=1}^{2k+4} c_{k+2,t}x^t = \sum_{t=1}^{2k+2} c_{k+1,t}(x^{t+2} + x^{t+1} + x^t)$$

and

$$c_{k+2,t} = c_{k+1,t} + c_{k+1,t-1} + c_{k+1,t-2}, \tag{4}$$

where $c_{k+1,t}=0$ if $t<1$ or $t>2k+2$.

We can obtain $(x^2-x)(1+x+x^2)^k$ from

$$(x^2 - xy)(x^2 + xy + y^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^ty^{2k+2-t}$$

by setting $y=1$. If we exchange x and y in this identity, we obtain

$$(y^2 - xy)(x^2 + xy + y^2)^k = \sum_{t=1}^{2k+2} c_{k+1,t}x^{2k+2-t}y^t.$$

Setting $y=1$ in both identities and multiplying the second identity by $-x$, we obtain

$$(x^2 - x)(1 + x + x^2)^k \equiv \sum_{t=1}^{2k+2} c_{k+1,t}x^t \equiv \sum_{t=1}^{2k+2} (-c_{k+1,t})x^{2k+3-t}.$$

Hence

(5) $$c_{k+1,t} + c_{k+1,2k+3-t} = 0, \qquad t = 1, \cdots, 2k + 2.$$

We let

$$f(x; u) \equiv \sum_{t=1}^{2u} d_{u,t}x^t \equiv \sum_{k=0}^{u-1} \sum_{t=1}^{2k+2} c_{k+1,t}x^t.$$

Then

$$d_{u,t} = \sum_{k=(t-2)/2}^{u-1} c_{k+1,t},$$

where, if $(t-2)/2$ is not an integer, k begins with the next largest integer, and we wish to prove that

$$d_{u,u} = \sum_{k=(u-2)/2}^{u-1} c_{k+1,u} = (-1)^u.$$

The proof will be by induction on u. We note that $d_{1,1} = -1$, and $d_{2,2} = 1$. Consider the case where $u = 2v$, $v > 1$. Then

$$d_{2v,2v} = \sum_{k=v-1}^{2v-1} c_{k+1,2v}, \qquad d_{2v-1,2v-1} = \sum_{k=v-1}^{2v-2} c_{k+1,2v-1},$$

and, by (4) and (5),

$$\begin{aligned}
d_{2v,2v} &= c_{2v,2v} + \sum_{k=v-1}^{2v-2} c_{k+1,2v} \\
&= c_{2v-1,2v} + c_{2v-1,2v-1} + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-1} c_{k+1,2v} \\
&= - c_{2v-1,2v-1} + (c_{2v-1,2v-1} + c_{2v-1,2v}) + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-3} c_{k+1,2v} \\
&= - c_{2v-1,2v-1} + c_{2v-1,2v-2} + \sum_{k=v-1}^{2v-3} c_{k+1,2v} \\
&= - \sum_{k=v}^{2v-2} c_{k+1,2v-1} + c_{v+1,2} + c_{v,2v} \\
&= - \sum_{k=v}^{2v-2} c_{k+1,2v-1} + c_{v,2} + (c_{v,1} + c_{v,2v}) + c_{v,0} \\
&= - \sum_{k=v}^{2v-2} c_{k+1,2v-1} - c_{v,2v-1} = - d_{2v-1,2v-1}.
\end{aligned}$$

In the same way, we prove $d_{2v+1,2v+1} = -d_{2v,2v}$ and the proof is complete.

RESIDUES OF IDEALS

(after a mathematicians' party)

The room is filled with emptiness and smoke
The rug looks bored and sullen—nostalgic for laced oxfords, probably,
The large, brown, weighty kind that has impressed upon it for an evening
The necessary and sufficient condition of existence.

The chairs, which had been twelve at eight o'clock,
Twelve four-legged, seatable commodities, worth sixteen dollars each,
Are turned into a countable set of points, dense almost everywhere
And scattered random-fashion within party limits.

The tables, freed for five short hours from humdrum horizontalness
(Thanks to mathy minds) had taken to the air as intersecting, real,
Or tangential planes. But they came in now—with a crash (it's 1:AM)
The debris spreading wide. A lemma lost its sense
And lies dead on the floor, covered with paper napkins.
Toothpicks on window sills. Cherry stones. And salad leaves.
And fragmentary sandwich matter. Prime ideals, lower radicals
Smoldering among the ashes of cremated Chesterfields.

And at the bottom of a tall and empty glass a theorem lies quite dead (and wet)
The experts will identify the body.
Someone will bury it beside the lemma in a journal
And hide the grave under a corollary.

ISOTTA CESARI

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

OVALS WITH EQUICHORDAL POINTS

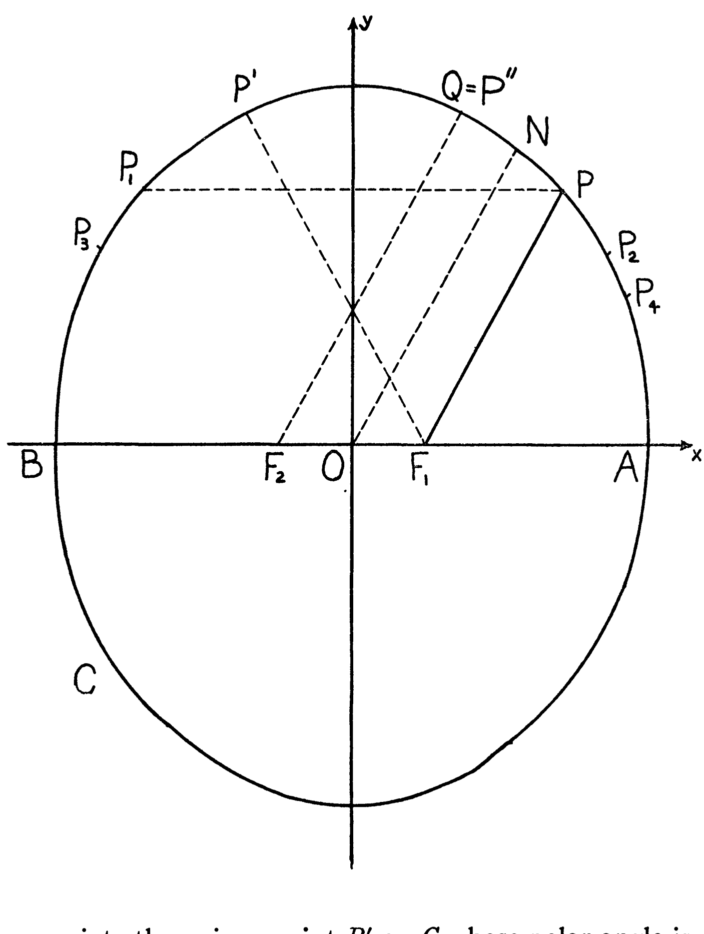
VIKTORS LINIS, University of Ottawa

A point F is called an equichordal point with respect to a curve C if all chords of C passing through F have equal length. Dirac* [1] has proved that if C is an oval (a bounded closed convex plane curve of class C^2) it cannot possess

* Dirac attributes this problem to Erdös, which is incorrect (cf. Math. Reviews, vol. 14, 1953, p. 309). The problem seems to be proposed by Blaschke, Rothe, and Weitzenböck (Aufgabe 552, Arch. Math. Phys., vol. 27, 1917, p. 82). It has been treated also by Süss [4], Dulmage [2] and Helfenstein [3].

more than two equichordal points, and if there are two such points F_1 and F_2, say, the oval must have the following properties: (i) C is symmetric about the line (F_1F_2); (ii) C is symmetric about the line orthogonal to (F_1F_2) passing through the point O, the midpoint of $[F_1F_2]$; (iii) if $r=r(\alpha)$ is the equation of C with respect to the pole F_1 (F_2 resp.) and (F_1F_2) as the polar axis (oriented as $[F_2F_1]$), then $r(\alpha)$ is a strictly increasing (resp. decreasing) function of α for $0\leqq\alpha\leqq\pi$. We will show that an oval possessing two equichordal points and the properties (i)–(iii) must be necessarily a circle. Consequently, no oval can have more than one equichordal point.

Proof. Let C be the oval (see the diagram), F_1 and F_2 the equichordal points, $AB=1$, $OF_1=OF_2=a<1/2$. Let (α, r) be the polar coordinates of the point P with respect to the pole F_1 and the ray $[F_1X)$; we choose $0\leqq\alpha\leqq\pi$. With every



point P we associate the unique point P' on C whose polar angle is $\pi-\alpha$. Then from the symmetry properties (i) and (ii) it follows $F_1P'=r(\pi-\alpha)=1-r(\alpha)$,

and also $F_1P' = F_2Q$ where F_2Q is parallel to F_1P and Q is a point on C. Thus, if ON is parallel to F_1P, $ON \geqq 1/2$ from the convexity of C, and the equality sign holds for $\alpha = 0$ or $\alpha = \pi$ only (unless $a = 0$), since we note the simple fact that no segment can belong to C.

Let $R = R(\beta)$ be the polar equation of C with respect to the pole O and the ray $[OX)$. Since $R(0) = 1/2$ it follows that $R(\beta)$ is strictly increasing in an interval $[0, \beta_0]$, $\beta_0 > 0$. For each α, $\beta_1(\alpha)$ and $\beta_2(\alpha)$ are the polar angles of P and P' respectively, where P is the unique point of C having angular coordinate α relative to F_1 as pole.

The relations

$$[R(\beta_1(\alpha))]^2 = r(\alpha)^2 + a^2 + 2ar(\alpha) \cos \alpha,$$

$$[R(\beta_2(\alpha))]^2 = (1 - r(\alpha))^2 + a^2 - 2a(1 - r(\alpha)) \cos \alpha,$$

define R as a differentiable function of α; also

$$\sin \beta_1(\alpha) = r(\alpha) \sin \alpha / R(\beta_1(\alpha)),$$

$$\sin \beta_2(\alpha) = (1 - r(\alpha)) \sin \alpha / R(\beta_2(\alpha)),$$

define β_1, β_2 as differentiable functions of α. Since $R \neq 0$, $\beta_1' \neq 0$, it follows that $R'(\beta_1(\alpha)) = 0$ if and only if $r'(\alpha)(r(\alpha) + a \cos \alpha) - ar(\alpha) \sin \alpha = 0$. Similarly $R'(\beta_2(\alpha)) = 0$ if and only if $r'(\alpha)(1 - r(\alpha) - a \cos \alpha) - a(1 - r(\alpha)) \sin \alpha = 0$. Since $r'(\alpha) = r'(\pi - \alpha)$ it follows that $R'(\beta_1(\alpha)) = 0$ if and only if $R'(\beta_2(\alpha)) = 0$.

Now we observe that there is on C a point $Y(r, \alpha)$ determined uniquely from the equation $r(\alpha) \cos \alpha + a = 0$, for which $R'(\beta_1(\alpha)) = 0$ and $\pi/2 < \alpha < \pi$. Denoting the associated points by dashes, we have at $Y'(1 - r(\alpha),\ \pi - \alpha) \neq Y$: $R'(\beta_1) = R'(\beta_2) = 0$. From the symmetry properties of the curve we have $R'(\beta) = 0$ also at the point P_1 obtained by reflecting P across OY. We construct the sequence $\{P_i\}_1^\infty$ where $P_1 = Y'$, and for $i \geqq 2$, $P_i = Q_{i-1}'$, where Q_{i-1} is the point obtained by reflecting P_{i-1} across OY. The sequence $\{P_i\}_1^\infty$ converges to A, at every point P_i we have $R'(\beta_1(\alpha)) = R'(\beta_2(\alpha)) = 0$, and thus there are such points in the interval $[0, \beta_0]$. This leads to contradiction unless $a = 0$, in which case all the above conditions are trivially satisfied. This completes the proof.

The condition that C belongs to C^2 can be relaxed to $C \subset C^1$, as can be seen from the proof and from Dirac's treatment.

The author thanks the referee for clarifying suggestions.

References

1. G. A. Dirac, Ovals with equichordal points, J. Lond. Math. Soc., vol. 27, 1952, pp. 429–437.

2. L. Dulmage, Tangents to ovals with two equichordal points, Trans. Roy. Soc. Canada. vol. 48, Sect. III, 1954, pp. 7–10.

3. H. Helfenstein, Ovals with equichordal points, J. Lond. Math. Soc. vol. 31, 1956, pp. 54–57.

4. W. Süss, Eibereiche mit ausgezeichneten Punkten, Tôhoku Math. J., vol. 25, 1925, pp. 86–98.

A NOTE ON THE SCHROEDER-BERNSTEIN THEOREM*

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Introduction. The proof of the Schroeder-Bernstein Theorem [1] is the critical step in the demonstration that the cardinal numbers are ordered. We may state the theorem as follows:

Let A and B be sets, and f, g be one-to-one mappings such that $f(A)\subset B$, $g(B)\subset A$; that is, f maps A onto a subset of B and g maps B onto a subset of A. Then there exists a one-to-one mapping F of A onto all of B.

To prove the theorem, we divide A into two parts, A_f and $A_{g^{-1}}$ and construct the mapping F of A onto B in such a way that $F=f$ on A_f and $F=g^{-1}$ on $A_{g^{-1}}$ [2]. The purpose of this note is to characterize all those sets A_f for which such a mapping F is one-to-one, and thus determine how inevitable is the choice of the set A_f used in the classical proof of the theorem.

1. Invariant subsets. We must begin with some preliminary observations about invariant sets. Let M be a set and let G be a one-to-one mapping of M onto itself. Then N is an invariant subset of M with respect to G whenever $G(N)=N$.

If $\{N_\alpha\}$ is an arbitrary collection of invariant subsets, then $\bigcup_\alpha N_\alpha$ and $\bigcap_\alpha N_\alpha$ are invariant subsets, since the mapping is one-to-one. Thus, there exists a minimal invariant subset (the intersection of all the invariant subsets), and a maximal invariant subset (the union of all the invariant subsets). Since the null set is invariant, it is the minimal invariant subset. The maximal invariant subset is $\bigcap_0^\infty G^k(M)$. To see this we observe that if N is any invariant subset, $G^k(N)=N$, so that $N=\bigcap_0^\infty G^k(N)$. Since $N\subset M$, $G^k(N)\subset G^k(M)$. Thus, $N=\bigcap_0^\infty G^k(N)\subset\bigcap_0^\infty G^k(M)$. Further, $\bigcap_0^\infty G^k(M)$ is itself invariant as is easily shown.

2. A Boolean equation. Now we shall show that the assertion that a certain Boolean equation has solutions implies that the Schroeder-Bernstein theorem is true.

If $F=f$ in A_f and $F=g^{-1}$ in $A_{g^{-1}}$, then for F to satisfy the conditions of the theorem, necessary and sufficient conditions are

(1) g^{-1} *must exist for all points of* $A_{g^{-1}}$, *so that* $g(B)\supset A_{g^{-1}}$,

(2) $$A = A_f + A_{g^{-1}},$$

(3) $$B = F(A_f) + F(A_{g^{-1}}) = fA_f + g^{-1}A_{g^{-1}}.$$

(By use of the $+$ instead of $\cup$ we imply that the sets involved are disjoint.)

Equation (3) shows that B is divided into two parts just as A is. That is, the mapping F^{-1} of B onto A is f^{-1} on fA_f and g on $g^{-1}A_{g^{-1}}$.

Let $g^{-1}A_{g^{-1}}=B_g$; thus (2), (3) become $A=A_f+gB_g$, $B=B_g+fA_f$, respectively. We can eliminate B_g and thus $A=A_f+g(B-f(A_f))=A_f+(gB-gfA_f)$, since g

* The authors wish to thank the referee for his valuable suggestions.

is a one-to-one mapping. Thus $A-A_f=gB-gfA_f$, since A_f and $gB-gfA_f$ are disjoint and $A_f\subset A$. Hence, $(A-A_f)+gfA_f=gB$, since gfA_f and $A-A_f$ are disjoint. Therefore,

$$gfA_f = gB - (A - A_f), \tag{4}$$

for the reason immediately above. Conversely, if A_f satisfies (4) and we set $B_g=B-fA_f$ then $A=A_f+gB_g$.

3. Solution of the equation. Inspection of (1), (2) and (4) shows that

$$A_f \supset A - g(B),\ A_f \supset S \textit{ implies that } A_f \supset gf(S).$$

From these two facts it follows by induction that $A_f\supset \underline{A}_f=\sum_0^\infty (gf)^k(A-g(B))$. (Here we have used $\sum$ instead of $\bigcup$ because the sets $(gf)^k(A-g(B))$ are all disjoint, a fact which is easily shown.)

It follows by direct substitution that $\underline{A}_f$ satisfies (1). If we let $A_f=\underline{A}_f+Q$, then, since gf is a one-to-one mapping, Q satisfies the equation $gf(Q)=Q$ so that Q is an invariant subset of A with respect to the mapping gf. Since we have used the $+$, the invariant subsets which we can use to construct solutions are, of necessity, disjoint from $\underline{A}_f$. However, this imposes no restriction on the class of invariant subsets because $\underline{A}_f$ is disjoint from $\bigcap_0^\infty (gf)^kA$, the maximal invariant subset, a fact which is easily shown.

We may summarize in the

THEOREM. *A_f satisfies equation* (1) *if and only if $A_f=\underline{A}_f+Q$ where $gf(Q)=Q$, and $\underline{A}_f=\sum_0^\infty (gf)^k(A-g(B))$.*

As an example, let A and B each be the positive integers and let f and g each be the successor function; *i.e.*, $f(N)=N+1$, $g(N)=N+1$. Then A_f is unique and consists of just the odd integers.

4. Inevitability of the choice of A_f. We see now that the set $\underline{A}_f$ to be used in the proof is unique up to the possible addition of a set invariant under the mapping gf. Since such an invariant set need not exist, the set $\underline{A}_f$ is the only set generally available for the proof.

References

1. J. E. Littlewood, A Mathematician's Miscellany, London, 1953, p. 10.
2. Stephen Cole Kleene, Introduction to Metamathematics, New York, 1952.

THE RANGE OF AN INTEGER-VALUED POLYNOMIAL

HAROLD S. SHAPIRO, New York University

It is a familiar and easily proven fact that a polynomial whose values at the integers are squares is the square of another polynomial. In this note we give a simple proof of the following generalization:

THEOREM. *Let $P(x)$ and $Q(x)$ be polynomials which are integer-valued at the integers, of degrees p and q respectively. If $P(n)$ is of the form $Q(m)$ for all n, or*

even for infinitely many blocks of consecutive integers of length $\geqq p/q+2$, *then there is a polynomial* $R(x)$ *such that* $P(x)=Q[R(x)]$.

Proof. Let $P(x)=a_px^p+\cdots+a_0$, $Q(x)=b_qx^q+\cdots+b_0$. Without loss of generality we may assume $a_p>0$ (for if $a_p<0$ we prove the theorem for the polynomials $-P$ and $-Q$). Also we may assume $P(n)=Q(m)$ for infinitely many blocks of *positive* integers n (in the contrary case, we would consider $P(-x)$ and $Q(-x)$). Finally, we can assume $b_q>0$. For, if q is even, this is necessarily the case, otherwise $Q(x)$ would take only finitely many positive values and so could not represent $P(n)$ for large positive n. If $Q(x)$ is odd, and $b_q<0$, we have only to replace $Q(x)$ by $Q(-x)$, which has the same range of values as $Q(x)$.

Now, let $x=\phi(w)$ denote a branch of the (algebraic) function inverse to the polynomial $w=Q(x)$, and indeed that branch for which $\phi(w)\sim(w/b_q)^{1/q}\to\infty$ as $w\to\infty$ along the positive real axis. $\phi(w)$ is positive and free of singularities if we restrict w to a suitable portion $w>w_0$ of the positive axis. Let now

$$f(x) = \phi[P(x)]. \tag{1}$$

Then $f(x)$ is a branch of an algebraic function, regular for $x>x_0$ and $\sim(a_p/b_q)x^{p/q}$ and we have $P(x)=Q[f(x)]$. Also, $f(n)$ is *integer-valued* at those values of n where $P(n)$ lies in the range of $Q(m)$, as we see from (1). Hence our theorem will be proved when we have proven the following

LEMMA. *Let* $f(x)$ *be a branch of an algebraic function real and regular for* $x_0<x<\infty$ *and satisfying* $|f(x)|<Cx^\alpha$ *where* $\alpha>0$, $C>0$. *If* $f(x)$ *is integer-valued for infinitely many blocks* $x=n, n+1, \cdots, n+r+1$, *where* r *is the least integer* $\geqq\alpha$, *then* f *is a polynomial.*

Proof of lemma. Suppose f is not a polynomial. Then $f^{(r+2)}(x)$ is a branch of an algebraic function, real for $x>x_0$, and $\not\equiv 0$. Hence $f^{(r+2)}(x)$ has only a finite number of zeros, and so $f^{(r+1)}(x)$ is monotonic (increasing or decreasing) for $x>x_1$ say. Therefore

$$\lim_{x\to\infty} f^{(r+1)}(x) = 0. \tag{2}$$

For, in the contrary case, we have either $f^{(r+1)}(x)\geqq a>0$ or $f^{(r+1)}(x)\leqq -a<0$ for $x\geqq x_2$. Integrating the first of these inequalities $r+1$ times from x_2 to x gives $f(x)>C_1x^{r+1}$ for large x, contradicting $f(x)<Cx^\alpha$. Similarly, the second alternative leads to a contradiction, and (2) is established.

Now, introducing the difference operator Δ, defined by $\Delta f(x)=f(x+1)-f(x)$, we know by the hypothesis that $\Delta^{r+1}f(n)$ is an integer for some arbitrarily large values of n. But by repeated application of the law of the mean, $\Delta^{r+1}f(x)=f^{(r+1)}(\xi)$ for some $x<\xi<x+r+1$. Hence for a sequence of $\xi_n\to\infty$, $f^{(r+1)}(\xi_n)$ is an integer, which must be zero for all large n because of (2). Thus $f^{(r+1)}(x)$, being an algebraic function vanishing at infinitely many points, is identically zero. Hence $f(x)$ is a polynomial, contrary to assumption. This completes the proof.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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INVARIANCE OF THE DISCRIMINANT OF A QUADRATIC FORM

L. A. RUBEL, The Institute for Advanced Study

The following elementary demonstration of the invariance of the discriminant of a quadratic form avoids laborious computations and may have some merit for presentation to elementary calculus classes. Suppose that the quadratic form is given as $Ax^2+Bxy+Cy^2+Dx+Ey+F$ with discriminant $\Delta=B^2-4AC$. Let the angle through which the axes are rotated be α, and let the form now have coefficients $A(\alpha)$, $B(\alpha)$, $\cdots$, $F(\alpha)$ and discriminant $\Delta(\alpha)=B^2(\alpha)-4A(\alpha)C(\alpha)$. A convenient way of writing the new coefficients is

$$
\begin{aligned}
2A(\alpha) &= (A-C)\cos 2\alpha + B\sin 2\alpha + A + C,\\
B(\alpha) &= B\cos 2\alpha + (C-A)\sin 2\alpha, \qquad (1)\\
2C(\alpha) &= (C-A)\cos 2\alpha - B\sin 2\alpha + A + C.
\end{aligned}
$$

To show that $\Delta(\alpha)$ is independent of α, we will show that $\Delta'(\alpha)\equiv 0$, primes denoting differentiation with respect to α. From (1) it is easily verified that

$$(2)\qquad A'(\alpha)=B(\alpha),\qquad B'(\alpha)=2[C(\alpha)-A(\alpha)],\qquad C'(\alpha)=-B(\alpha).$$

Now $\Delta'(\alpha)=2B(\alpha)B'(\alpha)-4A(\alpha)C'(\alpha)-4A'(\alpha)C(\alpha)$ becomes, by (2),

$$\Delta'(\alpha)B(\alpha)\{2\cdot 2[C(\alpha)-A(\alpha)]+4A(\alpha)-4C(\alpha)\}\cdot$$

Since this vanishes identically, the demonstration is complete.

BISECTOR OF AN INTERNAL ANGLE OF A TRIANGLE

P. SOMANADHAM, M. R. College, Vizianagram, India

If the equations of the sides BC, CA, AB of a triangle ABC are $a_ix+b_iy+c_i=0$, $i=1, 2, 3$, then the internal bisector of, say, angle A is, of course,

$$(1)\qquad \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \pm \frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}} = 0,$$

provided that the proper sign is chosen. This note establishes a simple criterion for making this choice.

The discussion will be simplified if the origin is shifted to the vertex A. Then the equations of the sides become $a_1x+b_1y+k=0$, $a_2x+b_2y=0$, $a_3x+b_3y=0$, respectively. The coordinates of B and C are

$$\left(\frac{-kb_3}{a_1b_3-a_3b_1}, \frac{ka_3}{a_1b_3-a_3b_1}\right), \quad \left(\frac{-kb_2}{a_1b_2-a_2b_1}, \frac{ka_2}{a_1b_2-a_2b_1}\right).$$

The ratio of the lengths of the sides AB and AC is

$$\left| \sqrt{a_3^2+b_3^2}(a_1b_2-a_2b_1)/\sqrt{a_2^2+b_2^2}(a_1b_3-a_3b_1) \right|.$$

It follows that the coordinates of the point P which divides BC internally in this ratio are

$$[-k(b_2\sqrt{a_3^2+b_3^2} \pm b_3\sqrt{a_2^2+b_2^2})/D,\ k(a_2\sqrt{a_3^2+b_3^2} \pm a_3\sqrt{a_2^2+b_2^2})/D],$$

where

$$D = \sqrt{a_3^2+b_3^2}(a_1b_2-a_2b_1) \pm \sqrt{a_2^2+b_2^2}(a_1b_3-a_3b_1),$$

according as

$$(a_1b_2-a_2b_1)/(a_1b_3-a_3b_1) \gtrless 0. \tag{2}$$

Since AP is the angle bisector, inequality (2) is the criterion sought; it is invariant under a translation and so holds for the triangle referred to the original axes. Thus (1) and (2) jointly give the equation of the internal bisector of angle A, a result that offers a simple method for finding the incenter of a triangle when the equations of its sides are given.

AN APPLICATION OF THE MEAN VALUE THEOREM

DAVID ZEITLIN, Remington Rand UNIVAC

Given $y=u^v$, where u and v are differentiable functions of x, the derivative D_xy is found by logarithmic differentiation. Thus,

$$D_x(u^v) = vu^{v-1}\cdot D_xu + u^v \log u \cdot D_xv \tag{1}$$

is derived in this manner. In most textbooks, no attempt is made to derive (1) using the definition of the derivative, *i.e.*, $\lim_{\Delta x\to 0} \Delta y/\Delta x$. For in doing so, one runs into difficulties involving a binomial expansion.

The mean value theorem is usually introduced in the study of indeterminate forms. At this point, the derivative of u^v may be easily found. Thus

$$\Delta y = \left\{(u+\Delta u)^{v+\Delta v} - u^{v+\Delta v}\right\} + \left\{u^{v+\Delta v} - u^v\right\}. \tag{2}$$

If we apply the mean value theorem to the first term in (2), treating $v+\Delta v$ as fixed, we find that

$$(u+\Delta u)^{v+\Delta v}-u^{v+\Delta v}=(\Delta u)(v+\Delta v)\cdot r^{v+\Delta v-1},$$

where $u+\Delta u<r<u$ or $u<r<u+\Delta u$. Thus

$$\frac{\Delta y}{\Delta x} = (v+\Delta v)\cdot r^{v+\Delta v-1}\cdot\left(\frac{\Delta u}{\Delta x}\right)+\left(\frac{u^{v+\Delta v}-u^v}{\Delta v}\right)\cdot\left(\frac{\Delta v}{\Delta x}\right) \tag{3}$$

As $\Delta x\to 0$, $\Delta u\to 0$, $\Delta v\to 0$, $r\to u$, and (1) is now established.

A VECTOR PROOF OF EULER'S THEOREM ON ROTATIONS OF E^3

M. K. FORT, Jr., University of Georgia

A well known theorem of Euler states that every orientation preserving isometry T of Euclidean 3-space that has a fixed point O is a rotation about a line through O. The most difficult part of the proof of Euler's theorem is in showing that there exists a nonzero vector $\mathbf{P}$ such that $T(\mathbf{P})=\mathbf{P}$. We give a proof of this last statement which does not use the theory of matrices and determinants, and which may be presented to a class that is studying elementary vector algebra.

Let O be the origin, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be a right hand orthogonal system of unit vectors. We define $\mathbf{i}'=T(\mathbf{i})$, $\mathbf{j}'=T(\mathbf{j})$, $\mathbf{k}'=T(\mathbf{k})$. Since T is an isometry, $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ is an orthogonal system of unit vectors. Since T is orientation preserving, $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ is also a right hand system.

The following computation shows that the triple scalar product $[\mathbf{i}-\mathbf{i}', \mathbf{j}-\mathbf{j}', \mathbf{k}-\mathbf{k}']$ vanishes.

$$
\begin{aligned}
&[\mathbf{i}-\mathbf{i}', \mathbf{j}-\mathbf{j}', \mathbf{k}-\mathbf{k}'] \\
&\quad = [\mathbf{i},\mathbf{j},\mathbf{k}] - [\mathbf{i},\mathbf{j},\mathbf{k}'] - [\mathbf{i},\mathbf{j}',\mathbf{k}] \\
&\qquad + [\mathbf{i},\mathbf{j}',\mathbf{k}'] - [\mathbf{i}',\mathbf{j},\mathbf{k}] + [\mathbf{i}',\mathbf{j},\mathbf{k}'] + [\mathbf{i}',\mathbf{j}',\mathbf{k}] - [\mathbf{i}',\mathbf{j}',\mathbf{k}'] \\
&\quad = 1 - \mathbf{i}\times\mathbf{j}\cdot\mathbf{k}' - \mathbf{j}'\cdot\mathbf{k}\times\mathbf{i} + \mathbf{i}\cdot\mathbf{j}'\times\mathbf{k}' - \mathbf{i}'\cdot\mathbf{j}\times\mathbf{k} + \mathbf{j}\cdot\mathbf{k}'\times\mathbf{i}' + \mathbf{i}'\times\mathbf{j}'\cdot\mathbf{k} - 1 \\
&\quad = 1 - \mathbf{k}\cdot\mathbf{k}' - \mathbf{j}'\cdot\mathbf{j} + \mathbf{i}\cdot\mathbf{i}' - \mathbf{i}'\cdot\mathbf{i} + \mathbf{j}\cdot\mathbf{j}' + \mathbf{k}'\cdot\mathbf{k} - 1 = 0.
\end{aligned}
$$

It follows that the vectors $\mathbf{i}-\mathbf{i}'$, $\mathbf{j}-\mathbf{j}'$, $\mathbf{k}-\mathbf{k}'$ are coplanar, and that there exists a non-zero vector $\mathbf{P}$ which is orthogonal to each of them. Thus, $\mathbf{P}\cdot\mathbf{i}=\mathbf{P}\cdot\mathbf{i}'$, $\mathbf{P}\cdot\mathbf{j}=\mathbf{P}\cdot\mathbf{j}'$, $\mathbf{P}\cdot\mathbf{k}=\mathbf{P}\cdot\mathbf{k}'$.

Since T is an isometry and $T(O)=O$, T is clearly linear. That is, if x, y and z are real numbers, then $T(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=x\mathbf{i}'+y\mathbf{j}'+z\mathbf{k}'$. Thus,

$$
\begin{aligned}
T(\mathbf{P}) &= T((\mathbf{P}\cdot\mathbf{i})\mathbf{i} + (\mathbf{P}\cdot\mathbf{j})\mathbf{j} + (\mathbf{P}\cdot\mathbf{k})\mathbf{k}) = (\mathbf{P}\cdot\mathbf{i})\mathbf{i}' + (\mathbf{P}\cdot\mathbf{j})\mathbf{j}' + (\mathbf{P}\cdot\mathbf{k})\mathbf{k}' \\
&= (\mathbf{P}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{P}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{P}\cdot\mathbf{k}')\mathbf{k}' = \mathbf{P}.
\end{aligned}
$$

ON A GRAPHICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS*

M. S. KLAMKIN, AVCO Research and Advanced Development Division

In a previous note [1] the author has given a method for a graphical solution of the first order linear differential equation. An alternative construction is given by Betz, Burcham, and Ewing [2]. In this paper higher order linear differential equations will be solved graphically by first transforming them into an equivalent set of simultaneous first order equations.

A linear differential equation with constant coefficients can be factored into the form

---

* Presented to the American Mathematical Society, August 31, 1955.

$$(1)\qquad [D - r_1][D - r_2] \cdots [D - r_n]y = F(x).$$

If we now let $[D-r_2] \cdots [D-r_n]y=S$, then $[D-r_1]S=F(x)$ and S then can be solved for graphically [1]. Consequently, (1) has been reduced in degree by one. Continuing in this fashion, we can determine y. In case some of the roots $\{r_n\}$ are complex, we can determine an alternative real factorization.

For example, let us consider

$$(2)\qquad [D^2 - 2aD + a^2 + b^2]y \equiv [D - a - bi][D - a + bi]y = F(x).$$

We can obtain a real factorization by letting $[D^2-2aD+a^2+b^2]\equiv[D-m]\times[D-n]$, where m and n are functions of x. Thus,

$$(3)\qquad m + n = 2a, \qquad mn - n' = a^2 + b^2.$$

Solving (3) simultaneously, we find that

$$(4)\qquad [D^2 - 2aD + a^2 + b^2]y \equiv [D - a - b\tan bx][D - a + b\tan bx]y.$$

Incidentally, by means of the exponential shift theorem, this factorization leads immediately to the closed form solution,

$$(5)\qquad y = e^{ax}\cos bx \int \frac{dx}{\cos^2 bx}\int e^{-ax}F(x)\cos bx\,dx,$$

of (2). By integrating (5) by parts, we can obtain the solution in the form one would obtain by the method of variation of parameters.

If, in (1), each r_n is a function of x, the solution is carried through in the same manner as indicated.

Let us now consider the general second order linear differential equation, $[D^2+2A(x)D+B(x)]z=G(x)$. As this equation cannot, in general, be factored, we have to resort to a different type of reduction. First, the equation is reduced to the canonical form,

$$(6)\qquad [D^2 + \phi(x)]y = F(x),$$

by means of the transformation $y=z\exp(\int A\,dx)$. We now factor the operator $[D^2+\phi(x)]$ into the form $[D+m][D+n]$. Consequently, m and n must satisfy the two equations

$$(7)\qquad m + n = 0, \qquad mn + n' = \phi(x).$$

Eliminating m, n must satisfy the Riccati equation

$$(8)\qquad dn/dx = n^2 + \phi(x).$$

Thus (8), taken simultaneously with

$$(9)\qquad [D - n][D + n]y = F(x),$$

is equivalent to (1). The transformation of (6) into (8) and (9), differs from the

usual transformation (*i.e.* by letting $dy/dx=p$) in that the dependent variables have been separated. This method of reduction may even be advantageous in some cases for a numerical solution.

To get a graphical solution of (8), we let $2n=p^2$. Then, $dp/dx=[p^2+4\phi(x)]/[4p]$, and we use the method [1] of the author for equations of the form $dy/dx=[\phi(x)-G(y)]/[\psi(x)-F(y)]$. Another method would be to plot the two curves C_1: $[x=n^2]$, and C_2: $[n=\phi(x)]$, as shown in Figure 1. The lines P_0A and P_0D

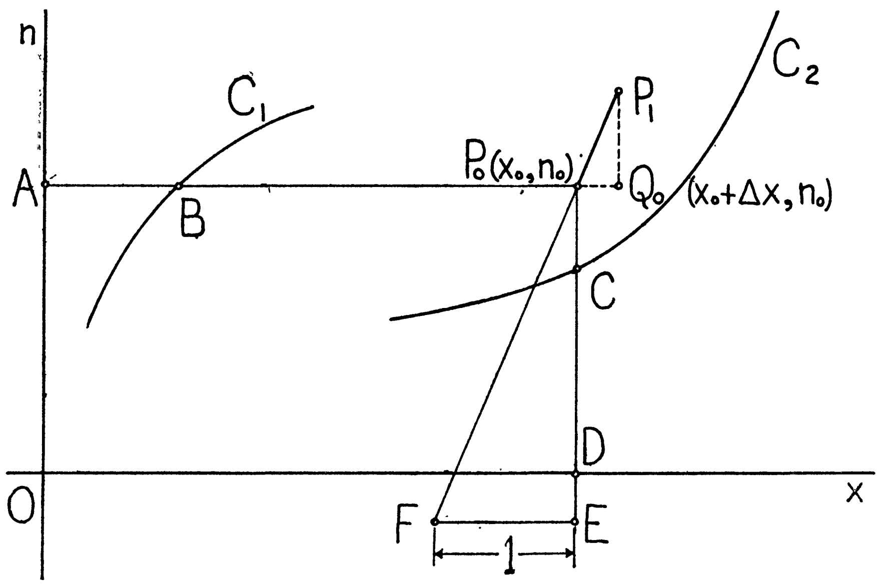

FIG. 1

are then drawn parallel to the axes. It then follows that $AB=n_0^2$ and $CD=\phi(x_0)$. Using a pair of dividers, we lay off P_0E which is the sum of AB and CD. The slope of line FP_0 is then $\phi(x_0)+n_0^2$. Thus, if P_0Q_0 represents Δx, P_1 is an approximation to a point on the integral curve of (8) through the point (x_0, n_0), n_0 being arbitrary. The procedure is then repeated from point P_1. The proximity of the successive points $P_0, P_1, \cdots$, *etc.*, will determine the accuracy of the construction. Equation (9) is now solved in a manner similar to that for (1); by reducing it to the two linear equations,

$$[D-n]S=F(x), \tag{10}$$

$$[D+n]y=S. \tag{11}$$

If the boundary conditions for (1) are $x=x_0$, $y=y_0$, $y'=y_1$, then the boundary condition for (10) is $x=x_0$, $S=y_1+n(x_0)y_0$.

If we have other boundary conditions, for example, the two point conditions

$x=x_0$, $y=y_0$; and $x=x_1$, $y=y_1$, we first determine two solutions $y=F_1(x)$, and $y=F_2(x)$ satisfying $F_1(x_0)=y_0$, $F_1'(x_0)=m_1$; $F_2(x_0)=y_0$, and $F_2'(x_0)=m_2$ (m_1 and m_2 are arbitrary). Then $y=aF_1(x)+bF_2(x)$, where $a+b=1$, $aF_1(x_1)+bF_2(x_1)=y_1$.

References

1. M. S. Klamkin, On a graphical solution of the first order linear differential equation, this MONTHLY, vol. 61, 1954, pp. 565–7.
2. H. Betz, P. B. Burcham, and G. M. Ewing, Differential Equations with Applications, New York, 1954.

NOTE ON INTEGRATION BY OPERATORS

ROGER OSBORN, University of Texas

Certain convenient devices for performing indefinite integrations may be found in the consideration of differential operators. The validity of the results, and of intermediate steps, may be established by differentiation procedures. Such devices as these can be presented to students of integral calculus, and they are especially useful to students of differential equations. The three indefinite integrations (particular integrals) which are shown here are illustrative of several which can be performed easily by operator methods.

The first is

$$\begin{aligned}\int e^{ax}\sin bx dx &= \frac{1}{D}e^{ax}\sin bx = e^{ax}\frac{1}{D+a}\sin bx\\ &= e^{ax}\frac{D-a}{D^2-a^2}\sin bx = \frac{e^{ax}}{-b^2-a^2}(D-a)\sin bx\\ &= \frac{e^{ax}}{-b^2-a^2}(b\cos bx - a\sin bx).\end{aligned}$$

Similarly,

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{-b^2-a^2}(-b\sin bx - a\cos bx).$$

A third is

$$\begin{aligned}\int x^n e^{ax}dx &= \frac{1}{D}e^{ax}x^n = e^{ax}\frac{1}{D+a}x^n = e^{ax}\frac{1}{a}\left(1+\frac{D}{a}\right)^{-1}x^n\\ &= \frac{e^{ax}}{a}\left(1-\frac{D}{a}+\frac{D^2}{a^2}-\frac{D^3}{a^3}+\cdots\right)x^n\\ &= \frac{e^{ax}}{a}\left[x^n-\frac{nx^{n-1}}{a}+\frac{n(n-1)x^{n-2}}{a^2}-\cdots+(-1)^{n-1}\frac{n!}{a^n}\right].\end{aligned}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1271. *Proposed by Ward Cheney and A. A. Goldstein, Convair-Astronautics, San Diego, California*

In E_3 let L denote a line not through the origin and not parallel to any coordinate plane. Does the point of contact of L with the smallest sphere having center at the origin necessarily lie in the same octant as the point of contact of L with the smallest cube having center at the origin and edges parallel to the coordinate axes?

E 1272. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2+\sin B/2+\sin C/2)^2 \leqq \cos^2 A/2+\cos^2 B/2+\cos^2 C/2.$$

E 1273. *Proposed by Alan Wayne, Baldwin, New York*

Prove or disprove the following proposition (which is true for $n=3$ and $n=4$): A plane n-gon with incircle of radius r and circumcircle of radius R is regular if and only if $r=R\cos(\pi/n)$.

E 1274. *Proposed by P. G. Kirmser, Kansas State College*

Given $p_i>0$, $q_i>0$, and $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i$, show that

$$\sum_{i=1}^n p_i \ln p_i \geqq \sum_{i=1}^n p_i \ln q_i.$$

E 1275. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development, Lawrence, Massachusetts*

Solve for x:

$$\int_0^x s^{8/3}(1-s)^{4/3}ds=\int_0^1 t^{8/3}/(1+t)^{-6}dt.$$

SOLUTIONS

A Holiday Greeting

E 1241 [1956, 723]. *Proposed by C. W. Trigg, Los Angeles City College*

The holiday greeting, *MERRY XMAS TO ALL*, is a cryptarithm in which

each of the letters is the unique representation of a digit, and each word is a square integer. Find all solutions.

Solution by L. A. Ringenberg, Eastern Illinois State College. We solve the problem using a table of squares as follows. Note that $ALL=100$, 144, 400, or 900. If $ALL=100$ or 144, then $XMAS=2916$ or 9216 and TO is not a square. If $ALL=400$, then $XMAS=1849$, 3249, 6241, or 8649. If $ALL=400$ and $XMAS=1849$, then $M=8$, $MERRY=81225$ and $E=X$. If $ALL=400$ and $XMAS=3249$, then $M=2$, $MERRY=27556$, and $TO=81$; hence the solution

(I) 27556 3249 81 400.

If $ALL=400$ and $XMAS=6241$, then TO is not a square. If $ALL=400$ and $XMAS=8649$, then $M=6$, $L=0$, and there is no solution for $MERRY$. If $ALL=900$, then $XMAS=1296$ or 7396. If $ALL=900$, $XMAS=1296$, then TO is not a square. If $ALL=900$, $XMAS=7396$, then $M=3$, $MERRY=34225$, and $TO=81$; hence the solution

(II) 34225 7396 81 900.

(I) and (II) are all of the solutions of the cryptarithm.

Also solved by A. W. Adler and F. C. Koch (jointly), Leon Bankoff, H. F. Bennett, Julian Braun, D. A. Breault, Dori Burke, Dorothy I. Carpenter, J. F. Daly, J. E. Darraugh, Monte Dernham, Underwood Dudley, Hazel E. Evans, H. M. Gehman, Basil Gordon and H. E. Sturgis (jointly), A. G. Grace, Jr., A. R. Hyde, Edgar Karst, C. D. Keim, H. R. Kingston, J. M. Kingston, J. D. E. Konhauser, Sidney Kravitz, A. M. Linn, Joe Lipman, R. L. London, D. C. B. Marsh, Erich Michalup, Herbert Nadler, C. S. Ogilvy, Marghrita L. Oneil, Bart Park, D. S. Passman, P. W. A. Raine, Azriel Rosenfeld, Frank Saunders, E. D. Schell, F. M. Stein, Alan Wayne, K. G. Whyburn, R. E. Wyllys, David Zeitlin, and the proposer. Late solutions by R. K. Guy and Tadao Shingu.

A number of the above solutions were incomplete.

Rosenfeld observed that if the further requirement is imposed that the *sum of the digits* of each word be a perfect square, then the solution is unique. Karst concocted the allied problem: The holiday greeting, $MERRY+XMAS=TOALL$, is a cryptarithm in which each of the letters is the unique representation of a digit and each word is divisible by 3. There is now a unique solution, $84771+5862=90633$.

A Circle Concentric with a Nine Point Circle

E 1242 [1956, 724]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the circle orthogonal to the circles inscribed in the squares of centers A', B', C' constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$.

Solution by the proposer. Denote the lengths of the sides BC, CA, AB of triangle ABC by a, b, c. Let O'' be the midpoint of $B'C'$; let N'' be the intersection with $B'C'$ of the radical axis $\Delta_{B'C'}$ of the two circles $(B', b/2)$, $(C', c/2)$; let A'' be the foot of the altitude $A'A''$ of triangle $A'B'C'$. Then we easily find

$$(1) \qquad (N''B')^2 - (N''C')^2 = (b^2 - c^2)/4 = 2(B'C')(O''N'')$$

and, by using the law of cosines on triangles $A'CB'$ and $A'BC'$,

$$(A'B')^2 = (a^2 + b^2)/2 + 2S, \qquad (A'C')^2 = (a^2 + c^2)/2 + 2S, \tag{2}$$

where S represents the area of triangle ABC. From (2) we find

$$(A'B')^2 - (A'C')^2 = (b^2 - c^2)/2 = 2(B'C')(O''A''),$$

whence, by comparison with (1), $O''N'' = (O''A'')/2$. It follows that $\Delta_{B'C'}$ passes through the nine point center N' of triangle $A'B'C'$. Similarly, $\Delta_{C'A'}$ and $\Delta_{A'B'}$ pass through N'. But the point of concurrence of the three radical axes of three circles taken in pairs is the center of the circle cutting the three circles orthogonally.

The proof can be modified to take care of the case of interiorly constructed squares.

Also solved by J. W. Clawson (essentially as above) and D. C. B. Marsh (analytically). Late solutions by R. K. Guy and Josef Langr.

Product of a Number and Its Reverse

E 1243 [1956, 724]. *Proposed by M. A. Rashid and M. A. Uppal, Lahore, West Pakistan*

Prove that the product of a number consisting of two digits and its reverse is never a square except when the two integers are equal.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Let the two-digit number be written as $10t+u$ (with t, u non-negative digits less than 10); its product with its reverse is

$$10t^2 + 101tu + 10u^2 = 10(t - u)^2 + 121tu \equiv 10(t - u)^2 \pmod{11}.$$

Since 10 is a quadratic nonresidue, modulo 11, it follows that $t-u \equiv 0 \pmod{11}$ and $t=u$.

II. *Solution by Alan Wayne, Baldwin, N. Y.* Let the number be $10t+u$, where t is the tens digit and u is the units digit. Consider $(10t+u)(10u+t)=k^2$, to be solved in positive integers. Obviously, when $t=u$, then $k=11t$ defines solutions. Suppose $t>u$ and let $x=t+u$, $y=t-u$. Then $(9y)^2+(2k)^2=(11x)^2$, an equation whose primitive solutions are

$$9y = p^2 - q^2, \qquad 2k = 2pq, \qquad 11x = p^2 + q^2,$$

where p and q are integers of opposite parity and relatively prime. It follows that $10t+u=p^2$, and $10u+t=q^2$. However, there are no two-digit squares ($t \neq u$) whose reversals are also squares. This completes the proof.

The property does not extend to three or four-digit numbers, as is shown by the examples $(961)(169)=(403)^2$ and $(9801)(1089)=(3267)^2$. There arises the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Also solved by W. A. Al-Salam, Leon Bankoff, H. F. Bennett, W. J. Blundon, D. A. Breault, J. E. Darraugh, J. E. D'Atri, Underwood Dudley, Hazel E. Evans, A. R. Hyde, I. M. Isaacs, J. H. Kaplan, M. S. Klamkin, J. J. Kohn, Joe Lipman, J. W. Mettler, Herbert Nadler, C. S. Ogilvy, D. S. Passman, Angelo Perlis, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, Gustavus Simmons, Art Steger, H. S. Valk, and David Zeitlin. Late solution by R. K. Guy.

Dudley solved the problem by actually enumerating and testing the 45 possible products. Some of the submitted solutions were open to grave objections.

A Series for Pi

E 1244 [1956, 724]. *Proposed by Leo Moser, University of Alberta*

Show that

$$\sum_{n=0}^{\infty} [5/(7n+1)(7n+6) + 3/(7n+2)(7n+5) - 1/(7n+3)(7n+4)] = \pi/\sqrt{7}.$$

Solution by A. E. Danese, Mt. Morris, N. Y. The given series can be written as

$$S = \sum_{n=0}^{\infty} [1/(7n+1) + 1/(7n+2) - 1/(7n+3) + 1/(7n+4) - 1/(7n+5) - 1/(7n+6)].$$

The expansion in partial fractions of the cotangent,

$$\pi t \cot \pi t = 1 + 2t^2 \sum_{k=1}^{\infty} 1/(t^2 - k^2), \qquad t \neq k,$$

with $t = x/7$ becomes

$$\phi(x) = (\pi/7) \cot (\pi x/7) = 1/x + \sum_{k=1}^{\infty} [1/(x-7k) + 1/(x+7k)].$$

Then $S = \phi(1) + \phi(2) - \phi(3) = \pi/\sqrt{7}$.

The series S occurs in Knopp, *Infinite Series*, p. 268, Ex. 108 (d).

Also solved by Leonard Carlitz, N. J. Fine, Emil Grosswald, F. W. Herlihy, A. R. Hyde, M. S. Klamkin, Erich Michalup, Chih-yi Wang, and David Zeitlin. Late solution by R. K. Guy.

This problem was published in *Newsletter of the Canadian Mathematical Congress*, 2nd issue, Autumn, 1954, by the same proposer. For an allied problem proposed by Chih-yi Wang, see Problem 287 in the 1956 Nov.–Dec. issue of *Mathematics Magazine*.

Convergence of Two Associated Sequences

E 1245 [1956, 724]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$b_{n+1} = \int_0^1 \min (x, a_n) dx, \qquad a_{n+1} = \int_0^1 \max (x, b_n) dx,$$

prove that the sequences $\{a_n\}$ and $\{b_n\}$ both converge and find their limits.

Solution by N. J. Fine, University of Pennsylvania. For any a_0, b_0, it is easy to see that a_n and b_n both lie between 0 and 1 for all $n \geqq 2$. The recurrence formulas then become (for $n \geqq 2$)

$$a_{n+1} = (1 + b_n^2)/2, \qquad b_{n+1} = a_n - a_n^2/2.$$

If we assume that $\lim a_n = a$, $\lim b_n = b$, they must satisfy

$$a = (1 + b^2)/2, \qquad b = a - a^2/2,$$

from which we get

$$a + b - 1 = (a + b - 1)(b - a + 1)/2.$$

Since the factor $(b - a + 1)/2 \neq 1$, we have $a + b = 1$, and this yields $a = 2 - \sqrt{2}$, $b = \sqrt{2} - 1$. To show that $a_n \to a$, $b_n \to b$, we write $a_n = a + \delta_n$, $b_n = b + \epsilon_n$. The recurrence formulas become, after an easy reduction,

$$\delta_{n+1} = (b + \epsilon_n/2)\epsilon_n, \qquad \epsilon_{n+1} = (b - \delta_n/2)\delta_n.$$

Now $|\epsilon_n| = |b_n - b| \leqq \max(b, 1 - b) = a$, and $|\delta_n| = |a_n - a| \leqq \max(a, 1 - a) = a$. Hence $|b + \epsilon_n/2| \leqq b + a/2 = 1/\sqrt{2}$, $|b - \delta_n/2| \leqq b + a/2 = 1/\sqrt{2}$. Therefore $|\delta_{n+1}| \leqq |\epsilon_n|/\sqrt{2}$ and $|\epsilon_{n+1}| \leqq |\delta_n|/\sqrt{2}$. This shows that $\delta_n \to 0$, $\epsilon_n \to 0$, and the proof is complete.

Also solved by Nathaniel Grossman, Emil Grosswald, A. R. Hyde, P. G. Kirmser, D. C. B. Marsh, Mary Payne, Lawrence Shepp, L. K. Williams, and the proposer. Late solution by R. K. Guy.

A number of the submitted solutions failed to show convergence.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4743. *Proposed by Aaron Herschfeld, Canisius College, Buffalo, N. Y.*

Given an arbitrary positive number u_0, consider the sequence $\{u_n\}$, where $u_n = u_{n-1} + 1/u_{n-1}$. Prove the relation $u_n = \sqrt{2n} + (\log n)/\sqrt{32n} + O(1/\sqrt{n})$.

4744. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

Three congruent ellipses are mutually tangent. Determine the maximum and the minimum of the area bounded by the three ellipses.

4745. *Proposed by Marvin Marcus, National Bureau of Standards and the University of British Columbia*

Let $A(t)$ be a complex n-square matrix function, continuous on $[0, \infty)$. Let A^* be the conjugate transpose of A and let $X(t)$ be the real part of the trace of $A(t)$. Assume that

$$\text{(i)} \qquad A(t) + A^*(t) \geqq 0, \quad 0 \leqq t < \infty; \qquad \text{(ii)} \quad \limsup_{t\to\infty} \int_0^t X(s)ds < \infty.$$

Show that all solutions of the linear vector-matrix differential equation $\dot{y} = A(t)y$ are bounded on $[0, \infty)$.

4746. *Proposed by Leonard Carlitz, Duke University*

Let p be an odd prime. Show that the number of solutions of the system

$$x + y + z \equiv 3, \qquad xyz \equiv 1 \pmod{p} \tag{1}$$

is equal to $p-2-(-3|p)$, where $(-3|p)$ denotes the quadratic character of -3. If $p>3$, show also that the number of solutions is the same for the system

$$x + y + z \equiv 3, \qquad x^3 + y^3 + z^3 \equiv 3 \pmod{p}. \tag{2}$$

4747. *Proposed by B. J. Boyer, Lafayette, Ind.*

Write $S_n = \sum_{k=0}^{\infty} B_k^2$, where the B_k are Fibonacci numbers, $B_0=1$, $B_1=1$, $B_{n+2}=B_{n+1}+B_n$. Find the value of $\sum_{n=0}^{\infty}(-1)^n/S_n$.

SOLUTIONS

Simson Line and Euler Line

4695 [1956, 426]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. Using complex coordinates and taking the circle to be the unit circle, let the vertices of the quadrangle be t_1, t_2, t_3, t_4, $|t_i|=1$. The Simson line of t_1, with respect to the triangle formed by t_2, t_3, t_4, has the equation

$$2t_1^2 z - 2t_1 s_3 \bar{z} + s_3 + t_1 s_2 - t_1^3 - t_1^2 s_1 = 0,$$

where s_1, s_2 and s_3 are the elementary symmetric functions of t_2, t_3 and t_4. The Euler line of triangle t_2, t_3, t_4 has the equation $s_2 z - s_1 s_3 \bar{z} = 0$. If the two lines are perpendicular, one clinant is the negative of the other, whence $s_3/t_1 = -s_1 s_3/s_2$, or upon simplifying, $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0$. The symmetry of this result guarantees that the property holds equally for any vertex.

Also solved by J. W. Clawson, G. W. Courter, R. Deaux, Beckham Martin, O. J. Ramler, Robert Sibson, Chih-yi Wang, and the proposer.

Mangoldt's Function

4696 [1956, 426]. *Proposed by T. M. Apostol, California Institute of Technology*

Let $\Lambda(n)$ denote Mangoldt's function, defined as follows:

$$\Lambda(n) = \begin{cases} \log p, \text{ if } n \text{ is a power of the prime } p, \\ 0, \text{ otherwise.} \end{cases}$$

For $n > 1$, let $f_m(n)$ denote the number of representations of n as a product of m factors, where each factor is greater than 1 and the order of the factors is taken into consideration (*e.g.*, $f_2(24) = 6$ and $f_3(16) = 3$). Prove the identity

$$\Lambda(n) = (\log n) \sum_{m=1}^{\nu(n)} \frac{(-1)^{m+1}}{m} f_m(n) \qquad (n > 1),$$

where $\nu(n)$ is the total number of prime factors of n, *i.e.*, $\nu(n) = a_1 + a_2 + \cdots$, if $n = p_1^{a_1} p_2^{a_2} \cdots$.

Solution by J. V. Whittaker, University of California, Los Angeles. The generating functions for $\Lambda(n)$ and $f_m(n)$ are

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \qquad (\zeta(s) - 1)^m = \sum_{n=1}^{\infty} \frac{f_m(n)}{n^s}.$$

The first of these is well known, and the second can easily be obtained if we agree that $f_m(n) = 0$ when $m > \nu(n)$. Differentiating the second equation and rearranging factors, we have

$$\sum_{n=1}^{\infty} \frac{(\log n) f_m(n)}{m n^s} = -(\zeta(s) - 1)^{m-1} \zeta'(s).$$

Multiplying both sides by $(-1)^{m+1}$ and summing from $m = 1$ to ∞, we find that

$$\sum_{n=1}^{\infty} \frac{\log n}{n^s} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} f_m(n) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

The convergence of the series is assured, say, for $s \geqq 2$. The desired result now

follows from the uniqueness of the Dirichlet series representation.

Also solved by Leonard Carlitz, Nathaniel Grossman, J. B. Johnston, C. D. Olds, Abe Sklar, and the proposer.

Generalization of De Moivre's Quintic

4697 [1956, 427]. *Proposed by Leonard Carlitz, Duke University*

Find the roots of the equation

$$y^n - ny^{n-2} + \frac{n(n-3)}{2!}y^{n-4} + \cdots + (-1)^r \frac{n(n-r-1)\cdots(n-2r+1)}{r!}y^{n-2r} + \cdots = 2a,$$

in particular, when $a=1$.

Solution by D. C. B. Marsh, Colorado School of Mines. Let $y=2\cos\theta$, and the given equation reduces to $\cos n\theta=a$ by a familiar result (See, *e.g.*, Chrystal, *Textbook of Algebra*, II, p. 276, (7), where the result is credited to James Bernoulli.) The roots are therefore $y=2\cos\{(2\pi k+\text{Arc}\cos a)/n\}$, $k=1,\cdots,n$.

Also solved by H. W. Gould, C. D. Olds, Chih-yi Wang, and the proposer.

Editorial Note. In Dickson, *First Course in the Theory of Equations*, p. 139, a solution is indicated for the equation

$$y^n - nqy^{n-2} + \frac{n(n-3)}{1\cdot 2}q^2y^{n-4} - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}q^3y^{n-6} + \cdots = c,$$

of which the present problem is a particular case. The values of y are given by $A+B$ and $A\epsilon^j+B\epsilon^{n-j}$, $j=1,\cdots,n-1$, where A is one value of $\{(c+\sqrt{c^2-4q^{2k}})/2\}^{1/k}$, $B=q/A$, and ϵ is a primitive nth root of unity. See also problems E 223 [1937, 109] and 4568 [1955, 189].

An Extension of a Result of Hilbert

4698 [1956, 497]. *Proposed by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland*

Prove that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m a_n}{m+n+1/6} \leq 2\pi\sum_{r=0}^{\infty}a_r^2,$$

provided the right-hand side converges.

Editorial Note. As pointed out by several correspondents, this problem is a special case of III, 169, Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, pp. 117, 290, with slight change in notation.

Solved by Leonard Carlitz, N. J. Fine, Basil Gordon, Peter Henrici, J. Horváth, Marvin Rosenblum, Chih-yi Wang, and the proposer.

4-, 5-, 6-, and 7-Lines

4700 [1956, 498]. *Proposed by J. W. Clawson, Ursinus College*

It is well known that the midpoints of the diagonals of a 4-line lie on a straight line which has been called the *Newtonian* of the 4-line. It is also known that the Newtonians of the five 4-lines obtained by omitting in turn each of the sides of a 5-line concur in a point which we may call the *Newtonian point* of the 5-line.

Prove (1) that the Newtonian points of the six 5-lines obtained by omitting in turn each of the sides of a 6-line lie on a conic which may be called the *Newtonian conic* of the 6-line; (2) that the Newtonian conics of the seven 6-lines obtained by omitting in turn each of the sides of a 7-line concur in three points, two of which may be imaginary.

Solution by Roland Deaux, Faculté Polytechnique, Mons, Belgium. We suppose that no two of the sides 1, 2, 3, 4, 5, 6, 7 of a 7-line are parallel, that no three of them are concurrent, and that no six are tangent to a conic. By omitting for instance 6, 7, we obtain the 5-line 12345 and the five 4-lines 2345, 1345, 1245, 1235, 1234 whose Newtonians $\overset{1}{n}_{67}$, $\overset{2}{n}_{67}$, $\overset{3}{n}_{67}$, $\overset{4}{n}_{67}$, $\overset{5}{n}_{67}$ concur in the Newtonian point N_{67} of the 5-line.

Such notations as $\overset{1}{n}_{67} \equiv \overset{1}{n}_{76}$, $\overset{6}{n}_{17}$, $\overset{7}{n}_{16}$ clearly show the identity of those three lines. The Newtonian points N_{67}, N_{17}, N_{16} are thus collinear.

We have proved [*Bulletin de l'Institute Polytechnique de Jassy*, vol. 7, 1948, p. 28] that the points at infinity of the pairs of lines

$$(1, \overset{1}{n}_{67}), \qquad (2, \overset{2}{n}_{67}), \qquad (3, \overset{3}{n}_{67}), \qquad (4, \overset{4}{n}_{67}), \qquad (5, \overset{5}{n}_{67})$$

are five pairs of a quadratic involution $\dot{I}$ (See also Remark 1, below.) A similar property holds for the pairs

$$(1, \overset{1}{n}_{57}), \qquad (2, \overset{2}{n}_{57}), \qquad (3, \overset{3}{n}_{57}), \qquad (4, \overset{4}{n}_{57}), \qquad (6, \overset{6}{n}_{57}).$$

The pencils $(\overset{1}{n}_{67}\overset{2}{n}_{67}\overset{3}{n}_{67}\overset{4}{n}_{67})$, $(\overset{1}{n}_{57}\overset{2}{n}_{57}\overset{3}{n}_{57}\overset{4}{n}_{57})$ being homographic, their centers N_{67}' N_{57} and the common points N_{17}, N_{27}, N_{37}, N_{47} of any two corresponding rays lie on the Newtonian conic ν_7 of the 6-line 123456. Hence the first part of the problem.

We now consider the Newtonian conic $\nu_6 \equiv N_{16}N_{26}N_{36}N_{46}N_{56}N_{76}$ of the 6-line 123457 which cuts ν_7 in N_{67} and in three points A, B, C, two of which may be imaginary. The lines $\overset{1}{n}_{67}$, $\overset{2}{n}_{67}$, $\overset{3}{n}_{67}$, $\overset{4}{n}_{67}$, $\overset{5}{n}_{67}$ pass through N_{67}, meet ν_6 in N_{16}, N_{26}, N_{36}, N_{46}, N_{56} and meet ν_7 in N_{17}, N_{27}, N_{37}, N_{47}, N_{57}.

Since the ranges

$$(N_{16}, N_{26}, N_{36}, N_{46}, N_{56}, A, B, C), \qquad (N_{17}, N_{27}, N_{37}, N_{47}, N_{57}, A, B, C)$$

are projectively related, so are such pencils as

$$N_{16}(N_{26}, N_{36}, N_{46}, N_{56}, A, B, C), \qquad N_{17}(N_{27}, N_{37}, N_{47}, N_{57}, A, B, C).$$

Their centers N_{16}, N_{17} and the common points N_{12}, N_{13}, N_{14}, N_{15}, A, B, C of any two corresponding rays lie therefore on the Newtonian conic ν_1 of the 6-line 234567. Hence the second part of the problem.

Remarks. 1. The existence of involution $\dot{I}$ may also be established as follows. In the 5-line 12345 let M_1, M_2, M_3 be the midpoints of the diagonals joining the vertex 45 with the vertices 23, 31, 12. We have $N_{67}M_1=\overset{1}{n}_{67}$, $N_{67}M_2=\overset{2}{n}_{67}$, $N_{67}M_3=\overset{3}{n}_{67}$ and the lines M_2M_3, M_3M_1, M_1M_2 are respectively parallel to the sides 1, 2, 3. The three pairs of opposite sides of the quadrangle $N_{67}M_1M_2M_3$ are cut by the line at infinity in three pairs of points belonging to $\dot{I}$. The same reasoning applied to the vertices 35, 25 and the remaining triangles 124, 134 will prove the theorem.

2. In the 5-line 12345 let P be the symmetric of the vertex 45 with respect to the point N_{67}. The quadrangle whose vertices are P and those of triangle 123 may replace the quadrangle $N_{67}M_1M_2M_3$ for the determination of $\dot{I}$. The unit points of $\dot{I}$ are thus cyclic or on two rectangular directions according as P is the orthocenter of triangle 123 or lies on the circumcircle. Hence the theorem:

Consider a 5-line circumscribed to an ellipse or a hyperbola with center 0. *If the symmetric with respect to* 0 *of a vertex is the orthocenter of the triangle formed by the sides which do not contain that vertex or lies on the circumcircle of that triangle, the same property holds for the other* 9 *vertices.*

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. III: *Astronomy and Physics*. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. ix+252 pp. \$6.25.

The portion of the Third Berkeley Symposium on Mathematical Statistics and Probability reported in this volume of the *Proceedings* was held in December, 1954, in conjunction with the meeting in Berkeley of the AAAS. The papers here reported were given in sessions jointly sponsored by mathematicians, astronomers, and physicists. There are eight papers dealing with astronomy, and

five with physics. They are of uniformly high quality; but it is natural to expect that the papers of greatest interest to the readers of this MONTHLY will be the papers with considerable mathematical interest, including the five on mathematical physics.

Statistical Mechanics and Probability Theory by André Blanc-Lapierre and Albert Tortrat is the first of this quintet. It deals with those problems of statistical mechanics that stem from the fact that the systems considered have many degrees of freedom. The authors point out that the reduction of the problems of statistical mechanics to probability theory results more from the similarity of the mathematical expressions than from the nature of the problems. The results are interpreted for Bose-Einstein, for Fermi-Dirac, and for classical statistics. Both the method of probability density and the method of characteristic functions are used. *Foundations of Kinetic Theory* by M. Kac has a superb quality of exposition. It re-examines critically the derivation of the reduced Boltzmann equation and the conclusions drawn from it. There are clear indications of the limitations in the validity of results and of theorems needing proof. As the author says, more questions are raised than answered. In *Random Solutions of Partial Differential Equations* by J. Kampé de Fériet, the author considers boundary value problems for linear partial differential equations of elliptic and parabolic type, but replaces the function $f(A)$ defining the boundary value by a random function $f_\omega(A)$ where ω represents a point in a suitable probability space. This point of view is motivated by physical considerations. The limitation of the discussion to one elliptic equation in a finite domain, and one parabolic equation in an infinite domain is accepted because of the insurmountable difficulties in applying the "random" point of view to the Navier-Stokes equations corresponding to a given random velocity field at time $t=0$.

The *Theory of the Vibration of Simple Cubic Lattices with Nearest Neighbor Interactions* by Elliott W. Montroll is a powerful analysis of two of the problems discussed and a more brief handling of the third and fourth. There are two appendices dealing with extensive proofs of mathematical theorems needed in the body of the paper. Some of the mathematical problems have application in the theory of random walks on discrete lattices.

In *Nonlinear Prediction and Dynamics*, Norbert Wiener gives a most interesting and suggestive treatment of the theory of prediction in the nonlinear case, and related matters. There are quotable passages about the nature and role of hypothesis, about the nature of the errors of an instrument or a computational system, and about the nature of stability. The paper is very short, and repays reading.

Of the papers on astronomy, the one with the most mathematical interest is by Jerzy Neyman, Elizabeth L. Scott and C. D. Shane. But all the papers are of considerable astronomical interest. There are five dealing with the Hertzsprung-Russell diagram and three dealing with Spatial Distribution of Galaxies. Since the names of the authors were reported in an earlier number of this MONTHLY

(vol. 63, 1956, p. 441), they are omitted here. It should be remarked, however, that the authors are among the ablest working in these fields, and that the collection of the papers in these groupings presents an illuminating view of the subjects.

MINA REES
Hunter College, New York

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. V: *Econometrics, Industrial Research, and Psychometry.* Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956, viii+184 pp. $5.75.

This book contains ten papers presented either in December 1954 or July-August 1955 at the symposium of the title. Of the four papers in the section entitled Contributions to Econometrics, three actually are concerned with the foundations of probability and decision theory. It is to be hoped that potential readers interested in this field will discover these articles under the guise of econometrics.

The one paper in the econometrics section which does not deal with foundations is *Reduction of Constrained Maxima to Saddle-point Problems* by K. J. Arrow and L. Hurwicz. The main result here lies in two theorems which imply the existence of a local nonnegative saddle-point for a modified Lagrangian expression under conditions which are somewhat less restrictive than those assumed by Kuhn and Tucker in a similar existence theorem.

The second paper is E. W. Barankin's *Toward an Objectivistic Theory of Probability*, which sets forth "the initial ideas and implications of a . . . theory of behavior." Arguing that utility should be a set function rather than a point function, that *irrational* is to a theory of behavior as *magical* is to a theory of physics and that distinctions between conscious and unconscious motivations are vain, the author follows human free will as a *guiding light* to a mathematical characterization of the concept *personality process*. The theory yields the following: (a) *Probability* and *utility* are one and the same thing—in fact, "the single key to the full resolution of the nature of reality." (b) an individual (or any *thing*) is a complex of personality (stochastic) processes, (c) all behavior is discrete, (d) "the three concepts of *eventuality*, *probability*, and *act* express the entire substance of reality." *Probability* for the theory is given by a modified relative frequency approach which seems to avoid some of the pitfalls of von Mises' theory. It is hoped that this exciting paper will be followed by further amplification by Barankin and investigation by others.

In the third paper, *Problems of Value Measurement for Theory of Induction and Decisions*, C. W. Churchman argues that "the experimental approach to the problem of decision-making lacks a theory of data collection (*i.e.* a theory of stability of information)," and "until such a theory is at least tentatively formed, we lack any foundations for decision theory." While Barankin's theory in the

previous paper suggests possible answers to some of the questions raised by this paper, it is not clear that the theory gets at this main one.

In *The Role of Subjective Probability and Utility in Decision-Making*, Patrick Suppes takes an entirely different tack at foundations and gives an axiomatization of decision theory which is very similar to Savage's but differs from it in that (i) "the number of states of nature is arbitrary rather than infinite" and (ii) "a 50–50 randomization of two pure decisions is permitted."

In the section on industrial research A. H. Bowker surveys continuous sampling plans and suggests directions for research in this field, C. Daniel discusses fractional replication and completes the published lists of fractional replicates for the useful ranges of the number of factors and number of runs, and M. Sobel extends Birnbaum's results on a sequential procedure for selecting that particular one of two or more exponential populations with the largest expected life.

In the section on psychometry T. W. Anderson and H. Rubin discuss some methods of factor analysis. They restrict their attention to one general probability model, use it to "point up features of model-building and statistical inference that occur in other areas," and state and prove some new results for the given model. Then F. Mosteller discusses stochastic learning models for simple psychological experiments, and H. Solomon brings forth some unsolved or unclear issues for problems of item analysis and classification techniques. Each of the three papers in this section could well serve as an introduction to the topic being discussed.

In the opinion of the reviewer almost every one of the papers in this volume should serve as a stimulant to further research. The book is to be recommended to all those interested in econometrics, industrial research, psychometry, or the foundations of probability.

FRANK L. WOLF
Carleton College

Random processes in automatic control. By J. Halcombe Laning, Jr. and Richard H. Battin. McGraw-Hill, New York, 1956. ix+434 pp. $10.00.

This book is a presentation of the techniques useful in analyzing and synthesizing control systems whose inputs are random functions of time. It contains sufficient background in statistical methods to allow the control systems engineer, starting with only a speaking acquaintance with statistics, to acquire a working knowledge of these methods.

The general aim of the book is to give the engineer the tools to enable him to fashion some sort of figure of merit for a control system subject to random inputs.

Two preliminary chapters devoted to an exposition of *Basic Concepts of Probability Theory* and *Descriptions of Random Processes* prepare the student (or engineer) for the analyses of actual systems. The properties of correlation functions, stationary and ergodic processes, and the energy spectral density are all

treated. These chapters are interlarded with enough specific examples, illustrative of the ideas presented in the text, that the reader can gain some facility in giving analytic expression to physical statistical problems. These preliminary chapters are sufficiently comprehensive that the reader need not, in the subsequent discussions, be limited by only an intuitive grasp of probability and random processes. On the other hand, a complete assimilation of this background material is not required in order to make use of the techniques presented in the principal chapters concerned with systems engineering.

In these subsequent chapters the extensive use of the mean-squared error as a performance index and its limitations are discussed candidly. In fact, the careful delineation of the state of the techniques discussed is one of the principal features of the book. For those areas for which inadequate analytical theory exists, the situation is outlined and the analogue computation methods, which take the place of the undeveloped mathematical theory, are explained. Since this is the situation for systems with variable coefficients, or with non-stationary inputs, these analogue computing methods are finding much use in analyzing actual control system problems such as occur in fire control system design. The time-saving method of adjoint simulation for obtaining mean-squared errors is spelled out in some detail. For systems with stationary inputs, and constant coefficient control systems, mathematical means are developed for obtaining the mean-squared error in terms of the system weighting function, or transfer function, and the input spectral density.

The final two chapters in the book deal with system synthesis rather than analysis. The Wiener theory for synthesizing the optimum system weighting function is developed. In the last chapter it is extended to cases for which the input signal is not stationary, and for which the past history is known over only a limited time.

A well annotated bibliography is provided, and a short but complete explanation of the analogue computer is given in an appendix.

For either the student or the practising control systems engineer, this book offers an understandable statement of the current techniques (and their limitations) available for the analysis and synthesis of control systems subject to random input signals.

W. F. Cartwright
U. S. Naval Ordnance Test Station
China Lake, California

Topics in Number Theory. By William J. LeVeque. Addison-Wesley, Reading, 1956. vol. I, x+198 pp. \$5.50; vol. II, viii+270 pp. \$6.50.

These two volumes constitute an outstanding contribution to the literature of number theory, because there is much material in volume II, and also in the later chapters of volume I, which is not readily accessible elsewhere. The author develops the subject from the beginning, presuming no previous knowledge of

number theory on the part of the reader. Although a wide variety of topics is presented in the two volumes, the writer has placed much more emphasis on analytic number theory, and less emphasis on Diophantine equations and quadratic forms, than most American books on the subject.

Volume I is suitable for a first course for advanced undergraduate and beginning graduate students. Although calculus is used in two of the nine chapters, no other technical knowledge is required. The writing, while very clear and in some places remarkably illuminating, is at a fairly sophisticated level and will not be easy going for the mediocre student. Similarly, while there is a good collection of problems at the ends of the sections, very few of them are routine computational exercises. Clearly the author has had in mind an audience of serious, competent students of mathematics.

Volume I opens with a chapter on the nature of the subject and the types of proof used in mathematics. This is followed by several chapters comprising the basis of number theory: the Euclidean algorithm, congruences, linear Diophantine equations, primitive roots and quadratic residues. The writer then turns to number-theoretic functions and applications to elementary results in the theory of prime numbers, for example the Erdös proof of the Bertrand conjecture. The representation of a number as a sum of two squares, and the fact that every positive integer is a sum of four squares, are given. In Chapter 8 there is a treatment of the Pell equation with various applications, and also a proof, based on Farey series, of the Hurwitz theorem that to any irrational number ξ there correspond infinitely many rational numbers x/y such that $|\xi-x/y| < 1/(\sqrt{5}y^2)$. This proof is not based on continued fractions. In fact, continued fractions are introduced in the last chapter of volume I in a rather novel way: as a device to get the "good" rational approximations to a real number.

One of the author's main reasons for writing the books was to provide the advanced topics of volume II. Here the level of mathematical maturity required for understanding is much higher, a working knowledge of the theory of analytic functions being a prerequisite in two of the seven chapters. The first chapter treats binary quadratic forms from the geometric viewpoint based on the modular group. This formulation provides an interesting contrast with the arithmetic treatment commonly given. The next two chapters are concerned with algebraic numbers and applications to rational number theory, principally Kummer's theorem that for a regular prime p the equation $x^p+y^p+z^p=0$ has no solution in rational integers x, y, z for which $(p, xyz)=1$, and the Delaunay-Nagell theorem that $x^3+dy^3=1$ has at most one non-trivial solution for any given integer d.

Chapter 4 is titled *The Thue-Siegel-Roth Theorem*, and it could have been labeled *The Liouville-Thue-Siegel-Dyson-Schneider-Roth-Le Veque Theorem*. Without citing all the results in this chain, let us recall that Liouville proved that if α is any algebraic number of degree $n \geqq 2$, then there is only a finite number of rationals h/k satisfying the inequality $|\alpha-h/k| < 1/(k^t)$ for fixed real $t>n$. (By use of this result, Liouville was the first to establish the existence of transcendental numbers.) Roth proved in 1955 that there is only a finite number of

rationals h/k satisfying the inequality for fixed $t>2$. LeVeque extends this to an algebraic formulation (for the first time in print, apparently) as follows. For any algebraic number β define $H(\beta)$ as the maximum of the absolute values of the coefficients of the unique monic minimal polynomial of β. Let K be an algebraic number field, and let α be algebraic of degree $n \geqq 2$ over K. Then for fixed real $t>2$ there is only a finite number of solutions β in K satisfying $|\alpha-\beta| < 1/(H(\beta))^t$.

Irrationality and Transcendence is the title of Chapter 5, which includes recent work of Mahler on arithmetic properties of the exponential function, of which the generalized Lindemann theorem and the transcendence of π are corollaries. Also given is a general formulation by Schneider of the Gelfond result that a^b is transcendental (apart from obvious special cases) for algebraic a and b.

The last two chapters deal with the distribution of prime numbers. Chapter 6 includes two proofs of the Dirichlet theorem on the infinitude of primes in an arithmetic progression, one non-elementary and one elementary, *i.e.* with and without the use of analytic function theory. The prime number theorem is proved (by non-elementary methods) in the final chapter, and an extension is made to the asymptotic estimate of the number of primes in an arithmetic progression.

There are notes at the ends of the chapters disclosing much of the author's source material, which will be most helpful to every student of number theory. Furthermore, the books are enlivened by frequent discussions of the author's reasons for choosing one approach rather than another, one proof rather than another. These handsomely printed volumes provide a welcome addition to the advanced expository literature.

IVAN NIVEN
University of Oregon

Applied Analysis. By C. Lanczos. Prentice-Hall, Englewood Cliffs, N. J., 1957. xx+539 pp. $9.00.

The book under review is designed to serve as an introduction to one of the most vital and significant fields of current mathematical research, the field of computational analysis. It serves its purpose admirably.

The style is pleasantly informal and leisurely, yet an enormous amount of material is included. In each chapter there is an interesting historical introduction to the principal problems to be discussed, followed by clear-cut formulations of these problems, a description of the methods to be employed in their treatment, and a number of completely worked-through numerical examples.

Much of the material is of current vintage, developed by the author himself in the course of many years spent in the company of digital computers. Occasionally the author cannot resist the temptation to pluck a plum from higher mathematical branches than those with which the remainder of the book deals. However, this is not necessarily a fault and should serve to whet the appetite of the reader for further reading in the field of analysis.

Let us now briefly survey the contents of the various chapters.

I. *Algebraic equations.* Since determination of the roots of a polynomial equation plays an essential part in stability analysis, a number of the most frequently used methods are presented in this opening chapter.

II, III. *Matrices and eigenvalue problems.* Matrices are introduced in conjunction with the eigenvalue problem and its geometric interpretation. A number of iterative techniques are discussed in connection with the problem of solving systems of linear equations of large degree.

IV. *Harmonic analysis.* The author motivates the treatment of finite Fourier series by means of an interpolation problem for data collected at equidistant time intervals. From there he goes on to the subject of Fourier series and Fourier integrals, and continues with a cross-section of the theory of orthogonal expansions.

Particularly useful is his discussion of the numerical inversion problem for the Laplace transform which escapes the tables. This is seldom discussed.

V. *Data processing.* This chapter contains a discussion of a number of important topics arising in the use of empirical functions defined by means of numerical tables. There is a discussion of least squares, determination of derivatives of empirical functions, and other aspects of the general smoothing problem.

VI. *Quadrature methods.* Here the problem is that of evaluating definite integrals of functions which may be empirical, or which may not possess simple analytic integrals. In addition to the classical methods of Simpson and Gauss, a number of other techniques are discussed as well.

VII. *Power expansions.* Although the method of power series expansions is one of the most powerful in analysis, it cannot be applied efficiently in the majority of cases without careful planning. The author discusses a number of ways in which the convergence of these expansions may be greatly increased with particular emphasis upon the use of Chebyshev polynomials.

An important application of these techniques is to the solution of differential equations with rational coefficients.

Finally let us note that the book is printed in a very readable and attractive type.

RICHARD BELLMAN
The RAND Corporation

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor W. L. Williams, University of South Carolina, was the representative of the Association at the inauguration of President F. R. Veal of Allen University on March 30, 1957.

Institute of Mathematical Sciences, New York University: Professor William Eberlein, University of Wisconsin, Dr. Joe Foote, University of Oklahoma, Dr. Frank Karal, Jr., Magnolia Petroleum Company, Dr. Martin Kneser, University of Heidelberg, Dr. Paul Koosis, University of Michigan, Dr. Walter Kyner, Northwestern University, Mr. Seymour Parter, Los Alamos Scientific Laboratory, Professor H. E. Rauch, University of Pennsylvania, Dr. Gian-Carlo Rota, Yale University, Professor J. T. Schwartz, Yale University, Dr. Soloman Schwartzman, Johns Hopkins University, and Dr. Shlomo Sternberg, Johns Hopkins University are in residence under the Temporary Membership Program.

Northwestern University: Visiting Associate Professor H. C. Wang, Columbia University, has been appointed Associate Professor; Dr. Teruhisa Matsusaka, University of Chicago, has been appointed Visiting Associate Professor.

University of Kentucky: Associate Professor Sallie E. Pence has been promoted to Professor; Mr. Richard Sprague, part-time instructor, has been appointed Instructor; Professor V. F. Cowling has been made Research Professor under a grant from the National Science Foundation; Mr. J. B. Cornelison and Mr. J. B. Wells have been made research instructors and are working on the National Science Foundation Contract project.

Vassar College: Associate Professor Abba V. Newton has been promoted to Professor; Associate Professor Winifred Asprey has been awarded an IBM Post-Doctoral Industrial-Research Fellowship for 1957–58 and will be on leave of absence.

Associate Professor J. W. Andrushkiw, Seton Hall University, has been promoted to Professor.

Dr. J. H. Bell, Head, Radar Development Department, Research Laboratory Division, Bendix Aviation Corporation, Detroit, Michigan, has a position as a scientific staff consultant at AC Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

Dr. H. F. Bright, Deputy Director, Technical Services, Human Resources Research Office, George Washington University, is now an operations research specialist at General Electric Company, Plainville, Connecticut.

Mr. C. B. Brown is a mathematician at Sohio Petroleum Company, Oklahoma City, Oklahoma.

Assistant Professor P. L. Butzer, McGill University, is on leave of absence and has been appointed Visiting Professor at the University of Mainz, Germany.

Mr. D. A. Cope, Research Engineer, Consolidated-Vultee Aircraft Corporation, San Diego, California, is a project engineer for Technical Research Group, New York, New York.

Mr. H. L. Dachslager, has been appointed a mathematician-economist at Remington Rand Univac, St. Paul, Minnesota.

Dr. Gus DiAntonio, Carnegie Institute of Technology, has accepted a position as aerodynamicist with Bell Aircraft Corporation, Buffalo, New York.

Mr. E. L. Dubowsky, Teacher, Colby Community High School, Kansas, has been appointed Instructor at the University of Wichita.

Associate Professor Charles Fox, McGill University, has been promoted to Professor.

Mr. C. B. Germain, Graduate Student, Iowa State College, has been appointed Assistant Professor of Statistics at the University of Manitoba.

Mr. T. L. Glahn, Technical Engineer, General Electric Company, Evendale, Ohio, has a position as a design specialist at Glenn L. Martin Company, Denver, Colorado.

Assistant Professor Simon Green, University of Tulsa, has been promoted to Associate Professor.

Assistant Professor Gerald Harrison, Wayne State University, has accepted a position as a systems engineer with Teleregister Corporation, Stamford, Connecticut.

Dr. J. F. Heyda, General Motors Corporation, has accepted a position as a mathematician with General Electric Company, Cincinnati, Ohio.

Mr. J. F. Jakobsen, Instructor, University of Missouri, has been appointed Instructor at the University of New Hampshire.

Mr. J. B. Lackey, University of Kentucky, has been appointed a mathematical analyst at Ballistics Reduction Group, Ordnance Test Activity, Yuma Test Station, Arizona.

Mr. G. M. Leibowitz has a position as an assistant engineer at the Ford Instrument Company, Missiles Development Division, Long Island City, New York.

Professor C. B. Lindquist, University of Minnesota, Duluth, has been appointed Chief for Natural Sciences and Mathematics, Department of Health, Education, and Welfare, Office of Education, Washington, D. C.

Assistant Professor D. S. McManus, Norwich University, has been appointed Instructor at the University of Wisconsin.

Dr. G. W. Medlin, Wake Forest College, is on leave of absence and is a mathematician at Oak Ridge National Laboratory, Tennessee.

Assistant Professor W. K. Moore, Albion College, has been promoted to Associate Professor.

Mr. J. E. Mullins, Mathematician, North American Aviation, Propulsion Field Laboratory, Santa Susana, California, is now an assistant mathematician at the Rand Corporation, Lincoln Laboratory, Lexington, Massachusetts.

Mr. F. E. Nemmers has a position as an analyst with the AC Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

Mr. E. F. Ormsby has been appointed Director of EDPM, International Business Machines World Trade Corporation, New York, New York.

Assistant Professor W. A. Rutledge, University of Tulsa, has been promoted to Associate Professor.

Miss Inez I. Sausen, Instructor, Garden City Junior College, has been appointed Instructor at the University of Wichita.

Dr. Paul Slepian, Ramo-Wooldridge Corporation, has a position as a research physicist, Hughes Research and Development Laboratories, Culver City, California.

Dr. M. D. Springer, Naval Ordnance Plant, Indianapolis, has accepted a position as a senior operations analyst with Technical Operations, Fort Monroe, Virginia.

Associate Professor D. D. Strebe, University of South Carolina, has been appointed Professor at the Teachers College at Oswego, State University of New York, effective September 1957.

Mr. Ralph Surasky, Graduate Assistant, University of South Carolina, has been appointed Assistant Professor at North Georgia College.

Dr. F. B. Thompson, Project Engineer, General Analysis Corporation, Santa Monica,

California, is now Manager, Operations Analysis, Computer Department, General Electric Company, Phoenix, Arizona.

Mr. A. W. Wallace, Instructor, University of Massachusetts, has a position as a mathematician at Instrumentation Laboratory, Massachusetts Institute of Technology.

Miss Marjorie Watson, Chattanooga, Tennessee, has accepted a position as engineer with Westinghouse Electric Corporation, Baltimore, Maryland.

Dr. G. L. Weiss has been appointed to an instructorship at DePaul University.

Mr. Nathan Wetrogan, Mathematical Engineer, Lockheed Aircraft Corporation, Marietta, Georgia, has a position as a research engineer at Grumman Aircraft Engineering Corporation, New York, New York.

Dr. W. D. Wood, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as Vice-President of the Dikewood Corporation, Albuquerque.

Professor H. E. Arnold, Wesleyan University, died on March 2, 1957. He was a member of the Association for thirty-three years.

Mr. J. H. Fishel, Research Assistant, University of Illinois, died on January 20, 1957.

Professor N. A. Goldsmith, Chicago Teachers College, died on January 20, 1957. He was a member of the Association for seven years.

Professor Emeritus J. M. Howie, Nebraska Wesleyan University, died on December 7, 1956. He was a charter member of the Association.

Rev. A. J. O'Leary, Chairman, Department of Mathematics, St. Anselm's College, died on February 3, 1957. He was a member of the Association for twenty-three years.

Professor J. M. Rankin, College of Idaho, died on June 8, 1956. He was a member of the Association for thirty-nine years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 117 persons have been elected to membership by the Board of Governors on applications duly certified.

WALTER L. ANDERSON, M.S. (West Virginia) Instr., West Virginia University.

MILTON T. AUSTIN, Student, Rutgers University.

RAY AUTHEMENT, Ph.D. (Louisiana S.U.) Asso. Professor, McNeese State College.

MRS. ANNE B. BARNES, M.A. (Texas) Instr., University of Texas.

BETTY L. BERNHARDT, M.A. (Ohio S.U.) Instr., Ohio University.

ROBERT T. BLACKBURN, Ph.D. (Chicago) Asst. Professor, San Francisco State College.

JOAN B. BOENITSCH, A.B. (Mary Washington) Math., Army Map Service, Washington, D. C.

JOSEPH R. BRASHEAR, Student, Carnegie Institute of Technology.

MRS. HELEN H. BROSS, Ph.D. (Yale) Asst. Professor, Talladega College.

WALTER C. BUTLER, M.A. (Colorado S.C.) Asst. Professor, Colorado Agricultural and Mechanical College.

WILLIE B. CAMPBELL, M.A. (Columbia) Instr., Texas Southern University.

MR. CHU C. CHANG, M.A. (Washington) Instr., Seattle University.

BENJAMIN CHEREEK, M.S. (Georgia) Analyst,

Air Force Armament Center, Eglin Air Force Base, Fla.

WILLIAM H. COLBERT, JR., B.S. (Nevada) Grad. Asst., University of Nevada.

CAROL A. CONN, M.A. (Cornell) Instr., Hobart College.

DONALD CONWAY, A.B. (San Jose S.C.) Instr., San Jose Junior College.

GEORGE T. CROCKER, B.S. (Union U.) Teaching Fellow, Alabama Polytechnic Institute.

MORTON L. CURTIS, Ph.D. (Michigan) Asso. Professor, University of Georgia.

MERLE L. DECKARD, Student, Lawrence Institute of Technology; Product Designer, Ford Motor Co., Detroit, Mich.

GUS DIANTONIO, Ph.D. (Pittsburgh) Aerophysicist, Bell Aircraft Corp., Niagara Falls, N. Y.

MORTON M. DICKSTEIN, B.A. (Hunter) Actuarial Trainee, George B. Buck & Co., New York, N. Y.

WILLIAM DONAHUE, U. S. Navy, Retired, Long Beach, California.

C. TOLAND DRAPER, B.S. (Lane) Grad. Student, University of California, Berkeley.

DONALD S. EVANS, B.S. (Stanford) Project Engr., Electro Instruments, San Diego, Calif.

LOUIS J. EVANS, B.S. (Detroit) Stress Analyst, Long Manufacturing Co., Detroit, Mich.

RUDOLF N. FESTA, Ph.D. (Vienna) Visiting Professor, University of Alabama.

FLOYD G. FISHER, Ph.D. (California) Math., Bureau of Ordnance, Dept. of Navy.

HAROLD FLEISHER, Ph.D. (Case I.T.) Manager, Research Communications, I.B.M., Poughkeepsie, N. Y.

EDWARD H. FLETCHER, B.A. (California, Berkeley) Programming Math., National Advisory Committee on Aeronautics, Cleveland, Ohio.

EUGENE A. FRANCIS, M.A. (Columbia) Asst. Professor, University of Puerto Rico.

MR. AUBYN FREED, Ph.D. (Illinois) Instr., Smith College.

MARION R. FRIEL, A.M. (Loyola) Teacher, Wright Junior College.

JOHN H. GLOVER, A.B. (Michigan); B.S. (M.I.T.) Product Design Engr., Ford Motor Co., Dearborn, Michigan.

HERMAN R. GLUCK, Student, New York University.

JOHN L. GORDON, B.S. (Texas Tech.C.) Math., Holloman Air Development Center, N. Mex.

FEDERICO GRABIEL, M.S. (Chicago) Res. Physicist, Hughes Research and Development Labs., Culver City, Calif.

JOHN W. GREINER, M.A. (Colgate) Asst. Professor, University of Florida.

ROLLIE J. HARP, M.S. (Florida S.U.) Instr., University of Chattanooga.

JAMES L. HATFIELD, M.A. (Virginia) Asst., Professor, College of William and Mary, Norfolk.

MICHAEL HERSCHORN, M.A. (McGill) Lecturer, McGill University.

ABRAHAM P. HILLMAN, Ph.D. (Princeton) Asst. Professor, State College of Washington.

J. EUGENE HOGAN, M.Ed. (Rochester) Head, Department of Mathematics, Madison High School, Rochester, N. Y.

ROBERT B. JACKSON, JR., B.S. (Davidson) Asst. Professor, Davidson College.

BRUCE A. JENSEN, M.S. (Wisconsin) Instr., Dana College.

NEIL W. JOHNSON, B.A. (Concordia) Grad. Asst., University of Nebraska.

JOHN W. JONES, Student, Gannon College.

RICHARD C. KAO, Ph.D. (Illinois) Asso. Math., Rand Corp., Santa Monica, Calif.

EVERETT H. KLINZING, B.S. (Wisc.S.C., Platteville) Teacher, Washington High School, New London, Wisc.

PAUL J. KNOPP, S.J., Student, Spring Hill College.

EUGENE E. KOHLBECKER, Ph.D. (Illinois) Asst. Professor, University of Utah.

JOHN T. KONZACK, M.A. (Nebraska) Chm., Division of Math. & Science, Hastings College.

BENOIT V. LACHAPELLE, B.S. (Montreal) Attaché de Recherches, University of Montreal.

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ALAN G. LAW, Student, University of British Columbia.

FRANK A. LEE, JR., M.A. (Virginia) Instr., Marion Institute.

EARL A. LEONHARDT, M.Ed. (Nebraska) Asst. Professor, Union College.

JAMES V. LEWIS, Ph.D. (California, Berkeley) Asso. Professor, University of New Mexico.

CALVIN T. LONG, Ph.D. (Oregon) Asst. Professor, State College of Washington.

RICHARD LORD, B.A., B.S. (Idaho S.C.) Grad. Student, Idaho State College.

STANLEY M. LUKAWECKI, M.S. (Alabama P.I.) Grad. Student, Alabama Polytechnic Institute.

JENS L. LUND, A.B. (California, Berkeley) Instr., Miramonte High School, Orinda, Calif.

ROBERT J. LUNDEGARD, Ph.D. (Purdue) Asst. Professor, Syracuse University.

ERNEST J. LYTLE, JR., Ph.D. (Florida) Asso. Math., I.B.M., Poughkeepsie, N. Y.

ANGELO MARGARIS, Ph.D. (Cornell) Instr., Oberlin College.

NORMAN L. MENZIE, B.S. (York) Grad. Asst., University of Nebraska.

JOSEPH A. MINAHAN, Student, College of St. Thomas.

NORMAN R. MINOR, Student, College of the Holy Cross.

ANDREV MOISEENKO, Mathematic (Smolensk) Draftsman, International Projector Corp., Bloomfield, N. J.

DONALD L. MUELLER, B.A. (Minnesota) Minneapolis, Minnesota.

DONALD E. MUIR, B.S. (Idaho) Acting Instr., University of Idaho.

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RONALD A. MYERS, Student, Gonzaga University.

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ROBERT B. OSBORN, M.A. (Missouri) Asso. Professor, Colorado School of Mines.

HUBERT V. PARK, Ph.D. (North Carolina) Professor, North Carolina State College.

THOMAS K. PENNINGTON, Student, Fordham University.

GEORGE A. POLCHIN, Student, Seton Hall University.

CLARA Y. POTTER, Student, Gonzaga University.

CARL PRINCE, M.S. (George Peabody) Math., Army Ballistics Missile Agency, Huntsville, Ala.

MRS. UPSHUR S. PUCKETTE, Sewanee, Tennessee.

WILLIAM J. PURCELL, M.A. (Columbia) Teacher, Chicago Teachers College.

MURIEL RALPH, M.A. (Columbia U., Teachers C.) Teacher, Butte High School, Mont.

LLOYD H. RHODES, II, B.A. (Eastern New Mexico) Math., Holloman Air Force Base, N. Mex.

DONALD E. ROOT, Student, University of Washington.

JAMES A. ROSSAS, A.B. (California) Instr. in Physics, Oroville Union High School, Calif.

JOHN W. ROYAL, B.A. (Maine) Instr., University of Maine.

LÉOPOLD SAUVÉ, Student, University of Ottawa.

LOUIS A. SCHMITTROTH, Ph.D. (Stanford) Asst. Professor, Montana State University.

ROBERT A. SEBASTIAN, M.A. (Missouri) Math., Ballistic Research Lab., Aberdeen Proving Ground, Md.

ALEXANDER J. SEIDLER, B.A. (N.Y.U.) Asst. Math., Rand Corp., Lexington, Mass.

GERALD M. SIEGEL, B.A. (Wayne S.U.) Detroit, Michigan.

SISTER M. JUSTA SMITH, O.S.F., M.S. (St. Bonaventure) Head, Mathematics, Rosary Hill College.

SISTER M. LORETTA ANN COLBERT, M.A. (Gonzaga) Instr., Marylhurst College.

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WILLIAM S. WINN, M.A. (North Carolina) Instr., Armstrong Junior College.

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TRAVEL GRANTS FOR THE 1958 INTERNATIONAL CONGRESS OF MATHEMATICIANS

Travel grants will be made to a limited number of mathematicians who wish to attend the International Congress of Mathematicians in Edinburgh, August 14–21, 1958. It is expected that funds available may provide travel assistance for about twenty-five mathematicians. Grants will be made on the basis of recommendations by the Committee on Travel Grants, which has been appointed by the Division of Mathematics of the National Academy of Sciences—National Research Council. In order to work out a fair distribution, the Committee has asked for the cooperation of various mathematical societies. A committee of the Mathematical Association of America will submit to the Committee on Travel Grants an ordered list of those members of the Association who desire grants.

Members of the Association who desire travel grants should (1) notify the Secretary of the Association, Professor H. M. Gehman, University of Buffalo, Buffalo 14, N. Y., and supply the information described below, (2) send a written request to the National Science Foundation, Washington 25, D. C., for an application form for foreign travel grants, and (3) fill out this form and return it to the National Science Foundation. The deadline for receiving applications is October 15.

The additional information referred to above, to be sent to the Secretary of the Association, is as follows: Each applicant should (1) send his complete address, (2) state whether he intends to present a paper and, if so, whether this is by invitation, and (3) describe any other travel funds that are available to him. Note that each application to the National Science Foundation should be matched by an application through the Association as described above.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-fourth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi on February 15–16, 1957. The Friday afternoon meeting was held in two concurrent sessions. Professor Elsie T. Church, Louisiana Vice-Chairman and Professor Roy D. Sheffield, Mississippi Vice-Chairman presided. Dean W. H. Bradford, McNeese State College, Chairman of the Section, presided at the Friday evening and Saturday morning sessions.

There were 117 persons registered including 65 members of the Association.

The following officers were elected for the coming year: Chairman, Professor A. C. Grimes, Mississippi State College; Vice-Chairman for Louisiana, Professor S. M. Spencer, Louisiana College; Vice-Chairman for Mississippi, Professor N. A. Childress, University of Mississippi; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

At the business meeting the section approved the sponsoring of a high school mathematics contest jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics. A 15 member committee was appointed by the chairman with a 5 man steering committee, Professor H. T. Karnes of Louisiana State University as chairman. A full report of this committee is to be brought before the section at the next meeting. A membership committee for the section was also appointed.

The invited speaker for the meeting was Dr. William T. Guy, Jr. of the University of Texas. His lecture on Friday evening was entitled "What Do Mathematicians Do?" He discussed some weaknesses of mathematicians and gave some inspirational challenges on how to make mathematics alive. The Saturday morning address was on "Cybernetics." The relation of cybernetics to statistical mechanics, communication and control was discussed.

The following papers were presented:

1. *The new look in mathematics*, by Professor Lester M. Garrison, Louisiana Polytechnic Institute.

This talk summarized some of the recent developments in high school and college mathematics and the new emphasis on modern mathematics. The need for special programs for the gifted students and the recommendations of the Committee on Study of Admission with Advanced Standing and the M.A.A. Committee on the Undergraduate Program were discussed.

2. *Making better use of the etymology of mathematical terms as an aid in teaching*, by Professor T. F. Mulcrone, S. J., Loyola University.

The speaker pointed out that the following objectives are better attained by an interest in the etymology of mathematical terms: (1) a better understanding of the mathematics taught, (2) the remembering of definitions, (3) the identification of concepts in their historical setting, and (4) the generalization and unification of mathematical types. Some methods were suggested for introducing in the class room references to the root meaning of words.

3. *Certain constructions completed*, by Professor Benjamin Ernest Mitchell, University of Mississippi.

If we are to have free and unrestricted use of the terms "continuity," "conversely," "in general" and "without exception" in our geometry, we must come to grips with the imaginary. It cannot be dumped into the discard, nor shunted to the side track, nor carried as excess baggage. It must be part and parcel of the paraphernalia of geometry. The explication at such a place for the imaginary is the objective of this paper as it relates to certain constructions. In particular the constructions of triangles in the cases $s-s-s$ and $s-s-a$, and the common harmonic conjugate of two given point-pairs on a line or two line-pairs on a point.

4. *Squares and diamonds*, by Professor V. B. Temple, Mississippi Southern College.

Draw circles C_1: $\rho_1=2a_1 \cos\theta$ and C_2: $\rho_2=2a_2 \sin\theta$. From the origin draw any line cutting C_1 in H_1 and C_2 in H_2. A horizontal through H_2 and a vertical through H_1 meet in P_1; a vertical through H_2 and a horizontal through H_1 meet in P_2. P_1 and P_2 define the loci of two continuous and closed line segments perpendicular to each other and whose equations are:

$$P_1: \begin{cases} x = 2a_1 \cos^2\theta, \\ y = 2a_2 \sin^2\theta. \end{cases} \qquad P_2: \begin{cases} x = 2a_2 \sin\theta\cos\theta, \\ y = 2a_1 \sin\theta\cos\theta. \end{cases}$$

If $a_1=a_2$ we have the T square; if $a_1\neq a_2$ the L squares. If double signs are used in the equations they are those of a square when $a_1=a_2$, and of a diamond when $a_1\neq a_2$.

5. *Curve tracing*, by Professor Elsie T. Church, Northwestern State College.

The paper presented a brief description of the use of the analytical triangle and the analytical polygon to find the approximate form of a curve at certain points, to determine all of the asymptotes and to determine the intercepts on the X-axis, Y-axis and the line at infinity. Several examples were given.

6. *The STIP program*, by Professor Houston T. Karnes, Louisiana State University.

A survey of the Science Teaching Improvement Program was given. This included the conditions which gave rise to the program and an outline of the work to date. The paper ended with a discussion of the work of the Regional Consultants which is the most recent phase of the program. The author is one of these consultants.

7. *The graduate program in mathematics at Louisiana State University*, by Professor R. D. Anderson, Louisiana State University.

A survey is given of the various facets of the graduate program in mathematics at Louisiana State University. Topics covered include the mathematical fields of interests of the members of the faculty; the available means of support for graduate students; course offerings, seminars and colloquia; and plans for future activity.

8. *A demonstration on the use of the electric analog computer*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

The differential equation $(D^2+2.4D+144)y=250$ is solved. The equation is first transformed into the machine equation by proper scale factors. The machine was brought along and several demonstrations were given during the meeting.

9. *A general theory for linear systems*, by Professor Roy D. Sheffield, University of Mississippi and Convair, Inc.

For a singular matrix L, a pseudo-inverse M of L is defined by the relation $LML=L$. If a matrix equation $Lx=b$ is consistent, then it follows that Mb is a particular solution; moreover, the column vectors of $I-ML$ span the linear space of solutions of $Lx=0$. If L is an $n\times n$ matrix of rank r, then a calculation for an M is given whereby $I-ML$ has exactly $n-r$ non-zero columns. In particular, this means that a system of linear equations can be solved by machine methods regardless of the nature of the matrix.

10. *Another note on quasi-idempotent matrices*, by Professor B. E. Mitchell, Louisiana State University.

The canonical form of a quasi-idempotent matrix is determined. Certain properties of quasi-idempotent matrices are proved by means of this canonical form.

11. *Centrifugal multiplication*, by Professor Haskell Cohen, Louisiana State University, introduced by the Secretary.

R. J. Koch has asked whether there exists a topological semigroup S, defined on the two cell with the property that $S^2 = \{xy: x \text{ and } y \in S\}$ contains the boundary of S but $S^2 \neq S$. The example given below not only meets the desired conditions but actually has S^2 equal to the boundary of S. Let S be the unit disk in the complex plane with center at the origin. Using polar coordinates, define the product of (r_1, θ_1) and (r_2, θ_2) to be the point $(1, 2\pi \max\{r_1, r_2\})$. It is easy to check the required properties.

T. L. REYNOLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 15–16, 1957, at Emory University, Emory University, Georgia. Professor G. B. Huff, Chairman of the Section, and Professor Trevor Evans presided over the general sessions; Professors R. G. Blake, C. G. Latimer, J. D. Novak, W. L. Strother and M. C. Wicht presided over subsections.

There were about 300 in attendance including 185 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Trevor Evans, Emory University; Vice-Chairman, Professor D. E. South, University of Florida; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College. The following were named as the Section's Committee on High School Contests, and will report at the next annual meeting: Professors J. B. Banks (Chairman), A. H. Herrington, E. Myrtice Lynch, J. D. Mancill, E. D. Nichols, J. W. Sawyer, C. Eucebia Shuler and F. L. Wren.

The following program was presented:

1. *Numbers that are irrational*, by Professor C. G. Phipps, University of Florida.

An extension of the note in this MONTHLY, vol. 63, 1956, p. 247, to other bases of notation leads easily to conclusions about the irrationality of the various roots of 2 and other positive integers.

2. *Some test questions in mathematics: A comparison between the Union of Soviet Socialist Republic and the United States*, by Professor Robert Kalin, Florida State University.

To enroll as freshmen in the Mechanics-Mathematics Department at the University of Moscow, U.S.S.R., applicants must pass an examination consisting of three algebra and three geometry problems. Thirty-five sample problems from the 1950–53 examinations indicate that greater skill in following a maze of algebraic technique and geometric information is required than is needed to handle the *typical* U. S. college entrance test. However, the University of Moscow may be *atypical*, and some U. S. examinations require a higher level of mathematical attainment and perhaps greater understanding of basic mathematical principles.

3. *Mathematics for social scientists*, by Professor E. B. Shanks, Vanderbilt University.

An experimental course at Vanderbilt for graduate students in the social sciences, based upon recommendations of the Madow Committee, was discussed. The point of view that many essential concepts in mathematics could be "freed" from mathematical contexts and thereby more easily applied to other fields of knowledge was expressed and illustrated; also, that constructive definitions should replace descriptive, where feasible. As an illustration, groups were discussed as certain types of square arrays. An important feature of the latter discussion was the consideration of the associative law stated in the following manner: $ab = c$ implies $a(bx) = cx$.

4. *Āryabhaṭa, first among the mathematicians of India*, by Mr. W. J. Mays, Actuary Imperial Life Insurance Company.

Āryabhaṭa (476 A.D.—c. 550 A.D.) was the most celebrated among a very early group of Indian astronomer-mathematicians. The Sanskrit treatise known as the *Āryabhaṭīya* (edited by Dr. H. Kern, Leiden, 1874) is attributed to him. It contains 118 aphorisms, or maxims, on astronomy and mathematics, composed in verse. Of particular interest are Āryabhaṭa's treatments of quadratic equations and linear indeterminate equations, as well as his famous approximation of *pi*, 3.1416. Although no proof or derivation of the results is given, a critical study of the text leads to certain plausible conjectures about some of his reasoning.

5. *New amicable pairs*, by Professor Mariano Garcia, Jr., University of Puerto Rico.

This paper lists over one hundred and fifty new amicable pairs obtained by methods related to those used for finding multiply perfect numbers. Several of the new pairs are of the "miscellaneous" type, of which relatively few examples are known.

6. *Properties of quasi-idempotent matrices*, by Professor G. B. Huff, University of Georgia.

A square matrix $A=(a_{ij})$ over the field of complex numbers is said to be *quasi-idempotent* if there is a polynomial matrix $(f_{ij}(x))$ such that for each positive integer r, $A^r=(f_{ij}(r))$. A matrix A is quasi-idempotent if there is a positive integer k such that $(A-E)^kA=0$, where E is the identity matrix; and, in this case, $f_{ij}(x)$ is of virtual degree $(k-1)$ for all i, j. (This MONTHLY, Vol. 62, 1955, pp. 334–339). Professor Huff pointed out the role of these matrices in elementary matrix theory and indicated some applications.

7. *On the universal sums of nine values of a cubic function*, by Professor R. L. Yates, University of Florida.

Necessary conditions are found for the coefficients ϵ, σ, ρ in order that the function $F(x)=(\epsilon/6)(x^3-x)+\sigma x+\rho$ may have nonnegative integral values for all the integers $x\geqq t$ and represent 0 and 1 for two such values. This gives rise to formulae for determining all such possible functions for which all integers are represented as the sum of nine values.

8. *A note on the Laguerre quadratic formula for the solution of algebraic equations*, by Professor Stephen Kulik, University of South Carolina.

Let $f(x)=0$ be an algebraic equation of degree N with real roots. Let x be a real number between any two roots of the equation, u an arbitrary number not equal to x or a root of the equation, and n an even number. Then the two real values of x_1, which satisfy the equation $[Q_n(u, x)+Nf^n(x)](x-x_1)^n=(u-x_1)^nf^n(x)$, where $Q_n(u, x)$ depends on $f(x)$ and its first n derivatives and is calculated recursively, approximate both roots adjacent to x. The accuracy increases with the increase of n.

9. *On a universal ternary quadratic form*, by Professor E. H. Hadlock, University of Florida.

Let C_1, C_2 be any given positive, odd and relatively prime integers. Let C_3 satisfy the congruences $C_3\equiv -C_2(\text{mod } C_1)$, $C_3\equiv -C_1(\text{mod } C_2)$ and its prime factors satisfy certain congruences mod $8C_1C_2$. Let $C_4=C_1+C_2-C_3$ and C_3 be greater than C_1+C_2 if $C_3>0$. If the prime factors of C_4 also satisfy certain congruences mod $8C_1C_2$, then the form $f=2X^2+C_1C_2Y^2+C_3C_4Z^2$ is universal. In particular if C_1, C_2 and C_3 are restricted to classes modulo 8, then there exist infinitely many primes $|C_4|=C_1+C_2-C_3$ for which f is universal.

10. *Examples of maximal non-prime ideals*, by Professor M. L. Curtis, University of Georgia.

In teaching modern algebra one proves that in a commutative ring with unit every maximal ideal is prime. There are very simple examples to show that both commutativity and a unit are necessary. This note gives a presentation which yields a class of such examples and has side results which are interesting from a pedagogical point of view.

11. *On the solutions of a binary quadratic congruence*, by Professor C. G. Latimer, Emory University.

In this paper one considers the binary congruence $x^2+y^2+1\equiv 0 \pmod{m}$, where m is a positive odd integer. The function Ψ is defined by

$$\Psi(m) = m[1 - p_1^{-1}(-1 \mid p_1)][1 - p_2^{-1}(-1 \mid p_2)] \cdots [1 - p_n^{-1}(-1 \mid p_n)]$$

where the p's are the distinct prime factors of m and $(-1 \mid p)$ is Legendre's Symbol. It is shown that the number of distinct solutions is $\Psi(m)$ where the solutions (x_1, y_1), (x_2, y_2) are considered as distinct only if $x_1 \not\equiv x_2$ or $y_1 \not\equiv y_2 \pmod{m}$. Note that x^2+y^2 is the norm of a Gaussian complex integer. The proof uses properties of such integers and may be extended to the integers in any quadratic field.

12. *Veblen-Wedderburn systems*, by Professor J. R. Wesson, Birmingham Southern College.

In 1907, Veblen and Wedderburn showed that certain finite algebras could be used to coordinatize some finite projective planes. There are redundancies in the postulates, some more apparent than others. A unit may be introduced by modifying the system slightly, and the new systems are usually employed in the construction of "Veblen-Wedderburn" planes. The Veblen-Wedderburn system with eight elements is a field.

13. *A vector proof of Euler's theorem on rotations*, by Professor M. K. Fort, Jr., University of Georgia.

Using elementary properties of vectors (especially properties of the triple scalar product), it is shown that if T is an orientation-preserving isometry of 3-space and 0 is a point such that $T(0)=0$, then there exists $P\neq 0$ such that $T(P)=P$. Euler's theorem on rotations follows easily from this fact. The proof given differs from the usual one in that no use is made of the theory of determinants and linear equations.

14. *Planar families of lines*, by Professor Andrew Sobczyk, University of Florida.

A family of lines $F=\{-x \sin \alpha+y \cos \alpha=d(\alpha)\}$ in the Euclidean plane E_2 is *representative* if it contains exactly one line in each direction α, $0\leqq\alpha<\pi$, *continuous* if the function $d(\alpha)$, extended by $d(\pi)=-d(0)$, is continuous on $0\leqq\alpha\leqq\pi$. A point P is *simply covered* if there is exactly one line L in F which passes through P. The following theorems are proved. Each representative and continuous planar family F fills the plane. Each planar family of lines such that the points exterior to a circle are simply covered, which is not a sheaf of parallel lines, must be representative and continuous.

15. *The Apollonius contact problem of three circles*, by Professor C. N. Mills, Florida State University.

Historically the circle-contact problem was first considered by Archimedes, but it is generally known as the problem of Apollonius. This paper considers the ten cases of geometrical construction from three null circles, to the general problem of three circles tangent externally mutually two by two. The constructions make use of many modern geometry theorems. When the three circles are mutually external, but not tangent, no general algebraic solution has been given which will give the radius of each of the eight circles. Fermat was first to give a solution of the problem to determine the radii of the two spheres tangent to four spheres tangent externally three by three.

16. *Generalizations of curvature properties of curves on a surface*, by Professor J. D. Novak, University of South Carolina.

At a point P of a curve on a surface, the T-normal curvature, the T-geodesic curvature, and the T-geodesic torsion of the curve relative to a variable tangent vector of the surface are defined. Certain generalizations are obtained which reduce to the classical theorems when the variable vector

is tangent to the curve. Analogues of Euler's equation and the Dupin indicatrix are proved for the principal T-normal curvatures at P.

17. *Remarks on points of uniform convergence*, by Professor C. W. McArthur, Florida State University.

If the sequence of functions (f_n) and the function f are defined on a topological space X to a metric space Y let $B=[x_0\in X:\epsilon>0$ implies that there is an integer N and an open set G about x_0, such that $d(f_n(x), f(x))<\epsilon$ if $n>N$ and $x\in G]$. Let A be the set of x_0 satisfying the above with $f(x)$ replaced by $f(x_0)$. In general, neither set is necessarily a subset of the other. If $\lim_{n\to\infty} f_n(x)=f(x)$ on X then $A\subset B$ and f is continuous at each $x\in A$. If, in addition, f_n is continuous for infinitely many n, $A=B$.

18. *On a problem of Henriksen and Isbell*, by Professor J. D. McKnight, Jr., University of South Carolina.

Let X and Y be completely regular spaces. THEOREM. *If X is first countable, then $X\times Y$ is pseudo-compact if and only if X and Y are both pseudo-compact.* COROLLARY (*using a result of Glicksberg*). *If X is first countable, then $\beta(X\times Y)=\beta X\times\beta Y$ if and only if both X and Y are pseudo-compact.*

19. *Some remarks concerning hyperconvex metrics*, by Professor R. L. Plunkett, Florida State University.

Let X be a compact topological space. It is known that (1) if X has a convex metric with unique segments, then X is an absolute retract (Abstract, *Bull. Amer. Math. Soc.*, vol. 61, 1955, p. 70) and (2) if X has a hyperconvex metric, then X is an absolute retract. (Aronszajn and Panitchpakdi, *Pacific J. Math.*, vol. 6, 1956, pp. 405–439.) Whether (1) implies (2) or vice versa is unknown. In this paper, if d is a metric for X, it is proved that any two of the following statements imply the remaining one: "d is a convex metric with unique segments," "d is a hyperconvex metric," and "X is a dendrite."

20. *Convergence in the overlap topology*, by Professor R. A. Lytle, University of South Carolina.

Suppose (A, T) and (B, T') are topological spaces and f_m is a sequence of evenly continuous functions on A into B. Then f_m converges to the continuous function f in the overlap topology if, and only if, f_m converges to f in the compact open topology.

21. *Retracts from neighborhood retracts*, by Professor W. L. Strother, University of Miami.

In an article to appear in *Proc. Amer. Math. Soc.*, Linda Falcao answers in the affirmative Wojdyslawski's question (*Fund. Math.* 1939) of whether the space of subsets of an absolute retract is again an absolute retract. In this note it is established further that if X is a connected metric absolute neighborhood retract then the space of the subsets of X is an absolute retract.

22. *One+One=One*, by Professor P. R. Halmos, University of Chicago.

An exposition of some of the algebraic and set-theoretic ideas involved in the Banach-Tarski paradox.

23. *Mathematics curriculum survey*, by Professor H. A. Robinson, Agnes Scott College.

From 48 replies to curriculum study suggested at Section Officers Seattle meeting, 21 reported recent changes; 12 appeared to be due to initial sectioning. Two institutions placed lower three-fourths in traditional courses while others studied Griffin's *Introduction to Mathematical Analysis*; 17 offered only algebra and trigonometry to freshmen, 21 analytics and 6 a combination with

calculus; 7 gave integrated courses and 2 universal mathematics; 12 offered intermediate algebra and 2 plane geometry. Sophomore courses were mainly traditional.

24. *Criteria for a logarithmic solution of a certain type of linear differential equation of third order with a regular singular point*, by Professor R. W. Cowan, University of Florida.

A compact expression for the coefficient of a general term in the series solution of the differential equation is obtained by the method of Frobenius. From this expression certain criteria required for a logarithmic solution are developed. These criteria depend upon whether the difference of the roots of the indicial equation are integral multiples of the difference in successive powers of the series solution and the nature of the roots of a certain cubic equation.

25. *On a revised tabulation of fiducial limits for point biserial correlation*, by Professor N. C. Perry, Alabama Polytechnic Institute.

An extensive tabulation of 5% fiducial limits for point biserial correlation (r_{pb}) is incorrect because the supporting non-central t theory of Lev (1949) was partially invalidated in a paper by Tate (1954). The speaker presents the results of a preliminary investigation of the numerical difference between the two theories, obtained by using Tate's normalizing $tanh^{-1}$ transformation. It was found that fiducial limits agree to two decimals for samples larger than 22, and differ by .01 for samples ranging in size from $n=22$ to $n=11$.

26. *On non-linear behavior of shallow ellipsoidal shells*, by Professor W. A. Nash, University of Florida.

Equilibrium equations are written for an element corresponding to a thin elastic open shell having small curvatures. These equations are specialized to the case of a shell in the form of an elliptic paraboloid. The case of a completely restrained-edge shell subject to uniform normal loading is considered. Infinite series for each of the three orthogonal displacement components are employed and all boundary conditions are satisfied identically. Coefficients in these series are determined and in this manner displacements and stresses throughout the shell are known.

27. *Order relations in the ring of polynomials*, by Professor Trevor Evans, Emory University.

The non-archimedean orderings of $J[x]$, the integral domain of polynomials over the integers, are well-known. It is pointed out in this note that there are uncountably many different ways of introducing an archimedean order into $J[x]$. These are obtained from the isomorphisms of $J[x]$ and sub-domains of the real numbers, x corresponding to different transcendentals.

28. *On a certain limit*, by Mr. Neal Bradley, student, Georgia Institute of Technology.

While trying to find an elementary proof that, if $u(x)$ has a limit as $x\to\infty$, then $\lim_{x\to\infty} [1+u(x)/x]^x = \exp [\lim_{x\to\infty} u(x)]$, it was noticed that the following theorem must hold: *If* $u(x)$ *is smooth for all sufficiently large* x, *and* $\lim_{x\to\infty} u(x)$ *exists, then:*

(1) *Either* $u'(x)$ *has no limit at all, or* $\lim_{x\to\infty} u'(x)=0$.

(2) *The same is true for* $xu'(x)$. (*The case* $u'(x)\to\pm\infty$ *is also excluded.*)

29. *On the summability of series by Nørlund means*, by Visiting Lecturer C. N. Moore, University of South Carolina (Professor Emeritus, University of Cincinnati).

In the case of the well known Cesàro means, the weights used in forming the means are the coefficients in the binomial expansion. A natural extension of this method is to replace the binomial coefficients by the coefficients of an arbitrary power series. Some restriction must be made in order to make the method "regular," that is to make sure that it will sum a convergent series to the proper value. If we represent the coefficients as $c_0, c_1, \cdots$, and set $C_n = \sum_{k=0}^{n} c_k$, the necessary

and sufficient conditions for regularity are $\lim_{n\to\infty} c_{n-m}/C_n=0$ $(0\leqq m\leqq n)$, $\sum_{k=0}^{m}|c_k|<H|C_m|$, when H is a positive constant and $m\geqq 0$.

30. *A development of logarithms using the function concept*, by Professor C. L. Seebeck, Jr., University of Alabama, and Professor John Jewett, University of Georgia, read by Professor Jewett.

The purpose of this paper is to present a definition of the logarithm as a function by means of properties which are simple enough to be understood readily by freshman students. The authors define the logarithm to be a function L which assigns to each positive real number a real number such that (1) $L(a\cdot b)=L(a)+L(b)$, (2) $L(10)=1$, and (3) if $a>1$, then $L(a)>0$. The usual elementary properties of logarithms follow easily from this definition and it can be shown that the function L agrees with the logarithm function defined by other methods.

31. *The mathematics in forward looking high school mathematics programs*, by Professor E. D. Nichols, Florida State University.

Contents of programs which most likely will exert influence on future courses are examined. The program of the University of Illinois Committee on School Mathematics has many interesting features. This group has developed materials which are being used in public schools under typical conditions. These materials represent drastic changes in content. Among aspects examined in this paper are the following: emphasis on distinction between a name of a thing and a thing itself; concept of variable; elementary aspects of set theory; linear and quadratic equations in terms of sets of points, intersection and union of sets.

32. *Mathematics in an industrial computing center*, by Dr. R. A. Willoughby, The Babcock & Wilcox Company, Atomic Energy Division, read by Mr. G. A. Duncan, Lockheed Aircraft Company, Marietta, Georgia.

The application of digital computers to the solution of scientific and technical problems in industry and the numerical analysis associated with the solution of these problems are placing a fresh emphasis on the basic concepts of undergraduate mathematics and their relation to physical problems. A broad horizon of job opportunities are opening to mathematics majors. It may well be that industrial experience will greatly benefit even those mathematics graduates who intend to become high school or college teachers.

33. *The stress distribution in a rotating disk of orthotropic material*, by Professor C. B. Smith, University of Florida.

A circular disk of orthotropic material is considered to be rotating about its center at a constant angular velocity. Since the thickness of the disk is taken to be small as compared with its radius, variation of stresses over thickness can be neglected and the problem is assumed to be a two-dimensional one. Usual equilibrium equations are modified to include body forces given by a potential function. Solution is obtained by aid of complex variables, and shows that stress distribution is much more complicated than that of the corresponding problem for an isotropic disk.

34. *Note on a system of differential equations in reactor statics*, by Professor J. A Nohel, Georgia Institute of Technology.

The differential equations for the fast and slow flux in a two region nuclear reactor form a system of two linear second order differential equations with variable coefficients; moreover, one of these coefficients is unknown. In treatments previously given this coefficient is assumed to be a constant. In this paper the above system is solved in general without this assumption, but subject to the condition that the unknown coefficient is analytic in some region containing the origin. Special cases of practical interest are then considered.

35. *A recurrence matrix for the response of an airplane in landing*, by Professor R. E. Wheeler, Howard College.

The problem of finding the time during landing at which there is a maximum stress at different positions on an airplane wing introduces interesting applications of elementary matrix theory and difference equations. By considering solutions of a difference equation at various positions on the wing, matrix relationships can be obtained which when simplified reduce to a recurrence matrix that gives the displacements at any particular time in terms of three preceding displacements. These can be used to find the time and the amount of the maximum stress.

36. *Nonlinear creep in pin-jointed structures*, by Professor R. C. Meacham, University of Florida.

Determination of the strains and stresses in a pin-jointed truss structure, the material of which obeys an mth power creep rate law, is shown to depend upon solving a system of non-linear, ordinary differential equations. The solution can be effected for any truss structure which contains only one redundant member; for higher degrees of redundancy the problem becomes formidable.

37. *Recursion and interrelations for Miles-Williams biharmonics*, by Professor M. C. Wicht, North Georgia College.

After a brief discussion of basic sets of polyharmonic polynomials, the author develops for the Miles-Williams biharmonics (this MONTHLY, vol. 63, 1956, p. 528) the recursion formula:

$$B_\beta^n = \cos^2(\beta\pi/2)\left[xB_\beta^{n-1} - (n-\beta-1)yB_{\beta+1}^{n-1}\right] + \sin^2(\beta\pi/2)\left[y/nB_{\beta-1}^{n-1} + (n-\beta)xB_\beta^{n-1}\right],$$

where $\beta = 0, 1, 2, 3$. Further, the author develops a set of eight relations between the Miles-Williams biharmonics and the biharmonics $P_m(x, y)$ of Zweiling.

38. *A nomogram for reducing the cubic equation*, by Professor R. G. Blake, University of Florida.

A design determinant is developed for a nomogram to determine p and q in the transformation $x^3+ax^2+bx+c=0$ to $X^3+pX+q=0$ by the substitution $x=X-a/3$.

39. *Some remarks on functional completeness*, by Professor F. L. Hardy, Emory University.

Consider the finite set of natural numbers $1, \cdots, m$. It is shown that all possible functions from this set to itself can be generated by a single function of two variables, the operation being substitution. This means in m-valued logics that all functionally complete logics may be given by a single truth-table in the same way as the 2-valued logic may be given by the Sheffer stroke function.

40. *On simultaneous reduction of sets of normal real matrices to canonical form*, by Mr. R. D. Boswell, Jr., University of Georgia.

An orthogonal matrix T is said to reduce a normal real matrix A if $T^{-1}AT$ is a block diagonal matrix with entries which are either 1×1 blocks or 2×2 blocks of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is proved that if F_t, $-\infty < t < \infty$, is a one-parameter group of $n\times n$ normal real matrices, then there exists a single orthogonal matrix S which reduces F_t for each real t.

41. *A method of reducing a positive ternary quadratic form*, by Professor P. B. Patterson, University of Florida.

It is shown by means of the minimum values of the coefficients that the positive ternary

quadratic form can be transformed into a reduced form by a certain set of three transformations.

42. *A careful study of the equation*, $x=f(x)$, by Professor W. R. Mann, University of North Carolina, and Doctor R. A. Willoughby, The Babcock and Wilcox Company, Atomic Energy Division, read by Professor Mann.

The problem of solving $x=1+\sin x$ accurate to a specified number of significant digits is used as a motivation for a careful investigation of a number of important topics in advanced calculus and numerical analysis (e.g. mean value theorem, convergence of sequences, Taylor series, numerical uncertainty and error analysis). In studying the equation $x=f(x)$ strong intuitive appeal can be given through the use of analytic geometry. Properly guided, the student can have the thrill of discovering many interesting and important properties.

43. *A lemma in matrices*, by Mr. R. T. Stubbs, graduate student, Georgia Institute of Technology.

Let A be any $n\times n$ complex constant matrix with characteristic roots λ_i, $i=1, \cdots, n$, such that $\Re(\lambda_i)\leqq -\mu<0$, $i=1, \cdots, n$. If σ is a constant such that $0<\sigma<\mu$, then there exists a constant K greater than zero such that $\|e^{tA}\|\leqq Ke^{-\sigma t}$, $t\geqq 0$ where K depends only on μ, σ, and A; and $\|e^{tA}\|$ is defined to be the sum of the absolute values of the elements of the matrix e^{tA}. Although the above result is well known, it is not readily found in the literature.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, State Teachers College, Mankato, October 5, 1957.
MISSOURI
NEBRASKA
NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.
NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.
NORTHERN CALIFORNIA, San Francisco State College, January, 1958.
OHIO
OKLAHOMA, Oklahoma City University, October 25, 1957.
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA, November 30, 1957.
ROCKY MOUNTAIN
SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.
SOUTHERN CALIFORNIA
SOUTHWESTERN
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GAME-THEORETIC SOLUTION OF BACCARAT

JOHN G. KEMENY AND J. LAURIE SNELL, Dartmouth College

The famous gambling game of baccarat has been the subject of many mathematical studies. There is, however, considerable variation in the solutions offered and it is now clear that these variations have come from trying to solve a game of strategy without a precise meaning for the solution of such a game. The theory of games has made this precise; it is now interesting to carry out the solution of baccarat in terms of the modern concepts of game theory and to compare the solution obtained with the earlier attempts based upon the more vague ideas previously available.

Baccarat is usually played by three men, one banker playing against a pair of players. But there is a popular variant of the game, known as *chemin de fer*, in which the banker plays against a single player. This is the version commonly discussed in the literature, and we too will take this two-person version of baccarat as our subject.

1. Description of the game. The game to be considered is a card game with the special feature that hands are evaluated modulo 10. A hand will consist of two or three cards. Each card from ace through 9 is worth its face value, while each card from 10 through king is worth 10 points (and hence 0 modulo 10). Thus a hand consisting of an 8 and a 3 gives a count of 1. A count of 9 is the best possible hand.

The banker serves as dealer, dealing his opponent and himself two cards each, face down. If either man has a count of 8 or 9, he announces this fact, and hands are compared at once.

If neither player has a count of 8 or 9, then the player has the option of taking an additional card. If he elects to do so, this card is dealt him *face up*. The banker may then decide to take an additional card, if he wishes one. The hands are then compared.

When hands are compared, the higher count (modulo 10) wins. If the two men have equal counts, the game is declared a draw.

Since nonplayers may bet on the player's hand, the player's strategy is restricted by the rules of the game. He must draw if he has a count of 4 or less, and is not allowed to draw on a count of 6 or more. His only free choice arises when he has a count 5. The banker is free to make his own decisions.

2. Strategies for the player and the banker. The player has just two pure strategies: One in which he draws if dealt a 5, and one in which he does not draw in this case. The banker, on the other hand, has a wealth of strategies. He is free to decide whether to draw or not (assuming that neither player has an 8 or 9 dealt to him), knowing what cards were dealt to him, whether the player drew, and knowing what he drew. His hand may have a count from 0 to 7. The player may not have drawn, or may have drawn a card of value 0 through 9. Thus the banker has 88 possible situations in which he must decide whether to

draw. He has 2^{88} pure strategies. We are thus confronted with a 2 by 2^{88} game.

3. Reduction to a 2 by 16 game. As could be guessed from game theory, most of the strategies of the banker are dominated. Instead of pure strategies, we will discuss the 88 decisions. (A pure strategy P is said to be dominated by the strategy Q if Q is at least as good as P against all strategies of the opponent, and is sometimes better. Clearly, if Q dominates P then we may omit P from our considerations, since any time that P is used we would do as well or better to use Q.)

PLAYER'S DRAW

	None	0	1	2	3	4	5	6	7	8	9
Player does not draw on 5	5	3	3	4	4	4	5	6	6	2	2
Player does draw on 5	6	3	4	4	4	5	5	6	6	2	3

Knowing the player's strategy, the banker should draw if his count is less than or equal to the entry in this table.

FIG. 1

If, in a situation calling for a decision on the part of the banker, it is better to draw (or not to draw) against both strategies of the player, this alternative need not be considered further. Thus our first step is to compute the best choice for the banker, in each situation he might face, assuming that he knows the strategy for the player. This involves some rather complicated conditional expectation computations. The results are shown in Figure 1.

It is seen from this figure that all but four of the alternatives have the property that the best choice for the banker is the same against either strategy of the player. Thus we need only consider the banker's strategy as it relates to these four alternatives. They are:

A. The player does not draw and the banker is dealt a count of 6.
B. The player draws a 1 and the banker is dealt a count of 4.
C. The player draws a 4 and the banker is dealt a count of 5.
D. The player draws a 9 and the banker is dealt a count of 3.

There are then $2^4 = 16$ strategies which need to be considered for the banker, corresponding to possible choices of drawing or not drawing in situations A, B, C, and D.

The game has thus been reduced to a 2 by 16 game and all that remains is to compute the payoff matrix and solve the game. The payoff matrix is shown in Figure 2. The entries represent payments from the banker to the player.

4. Solution of the 2 by 16 game. We denote the banker's strategy by *NNNN*, *NNND*, *NNDN*, *etc.* We first look for dominances among his 16 strategies. We find that the strategy *NNDD* dominates *NDNN*, *NDND*, and *NDDN*. The

GAME MATRIX

Banker on A	N								D							
Banker on B	N				D				N				D			
Banker on C	N		D		N		D		N		D		N		D	
Banker on D	N	D	N	D	N	D	N	D	N	D	N	D	N	D	N	D
Player does not draw on 5	−4636	−4565	−4635	−4564	−4529	−4458	−4528	−4457	−2764	−2693	−2763	−2692	−2657	−2586	−2656	−2585
Player does draw on 5	−3585	−3690	−3600	−3705	−3590	−3695	−3605	−3710	−4001	−4106	−4016	−4121	−4006	−4111	−4021	−4126

All entries must be multiplied by 16/4826809. N means "does not draw," D means "draws."

FIG. 2

strategy *DNDD* dominates the strategies *DDNN*, *DDND*, *DDDN*. A mixture of *NNDD* and *DNDD* dominates *NDDD*, *DNNN*, *DNND*, and *DNDN*. Finally a mixture of *NNNN* and *NNDD* dominates *NNND*. This reduces the game to a 2 by 5 game. By a theorem of game theory, there is at least one 2 by 2 subgame (called a kernel) such that the solution of this 2 by 2 game gives the solution of the 2 by 5 game. Trying the 2 by 2 subgames, we find that the unique kernel is the game given by the strategies *NNDD* and *DNDD* for the banker.

The optimal mixed strategy for the player is (2/11, 9/11). The banker's mixed strategy is (1429/2288, 859/2288). The value of the game is $-679568/53094899$ or approximately -0.0128.

The banker's optimal strategy means that (1) in position A the banker should draw with probability 859/2288, (2) in position B he should not draw, (3) in C and D he should always draw.

5. Comparison with previous solutions. There have been many different "mathematically best" solutions to the game we have considered. Besides the obvious difficulty of not having a precise definition of solution, the early attempts often suffered from using heuristic ideas of expectations and conditional expectations.

It is interesting to observe that the results of modern game theory tell us the form the answer is likely to take without carrying out any computations. Our game is a 2 by N game. As such, its kernel must be 1 by 1 or at most 2 by 2. In the former case the game is strictly determined, and neither participant has a choice. But if the player is to use a mixture, as most writers correctly guessed, then the banker must mix exactly two strategies. The only alternative to this analysis would be having more than one kernel—but with the size of the numbers appearing, this is *a priori* most implausible.

As an example of a previous solution, we can find in the *Encyclopaedia Britannica* that the player (1) should not draw in A or B, (2) he should draw in C, (3) he has an "option" in D. This gives a mixture of *NNDN* and *NNDD*. While the proportions are not specified, no mixture of this kind is optimal. For example, a mixture of (1/2, 1/2) will result in an expected decrease of more than 5 per cent in the value for the banker. We also find the amazing statement: "The banker's advantage cannot be exactly calculated, but has been estimated to be 7 per cent of all the money bet against it." This is surprising, both in its negative attitude and in the much too high value attributed to the game for the banker.

A second "best" solution is given by M. Boll (*La Chance et Lés Jeux de Hasard*, Paris, 1936). In this work there is a very complete and correct treatment of the conditional expectations needed for solving the game. However, having found these, the author argues as follows: The player must disguise his intention to draw or not to draw on a 5. The banker wishes to guess the player's intention. There are four possibilities corresponding to the guess of the banker and whether or not he is right. If the players are equally matched, these four will be equally likely. This leads him to the strategy (1/2, 1/2) for the player

and the pure strategy *NNDD* for the banker. This strategy is optimal against the mixed strategy (1/2/, 1/2) for the player. He arrives at a value of -0.0137 for the game. The nonoptimal mixed strategy thus increases the player's expected loss by about 7 per cent.

A more detailed presentation of these results as well as an indication of the methods used will appear in a forthcoming publication of the Committee on the Undergraduate Program of the Mathematical Association of America.

ON THE CAUCHY CRITERION FOR THE CONVERGENCE OF AN INFINITE SERIES*

ALBERT WILANSKY, Lehigh University

1. The general principle of convergence for a sequence $x=\{x_n\}$, $n=1, 2, \cdots$, of real numbers is equivalent to the following: *x is convergent if and only if for each pair of increasing sequences $\{p_n\}$, $\{q_n\}$ of positive integers with $p_n \neq q_n$ for each n, we have $x_{p_n}-x_{q_n} \to 0$ as $n \to \infty$.*

Let A^{pq} be a matrix whose nth row, for each $n=1, 2, \cdots$, consists entirely of 0's except that in the p_n place there is 1 and in the q_n place, -1. Then if a sequence x is written as a column vector, the matrix product $A^{pq}x$ is the column vector with $x_{p_n}-x_{q_n}$ as its nth entry, for each n.

The general principle of convergence then asserts that x is convergent if and only if $A^{pq}x$ is null (*i.e.* converges to 0) for each A^{pq}. The number of such A^{pq} is uncountable and the result to be proved here is that no countable such set could be sufficient for convergence. To phrase this accurately and more generally: let a *Cauchy matrix* be a matrix A with the property that Ax is null whenever x is a convergent sequence (=column vector). Each A^{pq} is a Cauchy matrix. Let a sequence x be said to *satisfy a Cauchy condition* if, for some Cauchy matrix A, Ax is null.

THEOREM. *No countable collection of Cauchy conditions is sufficient for convergence; more precisely, given a countable collection of Cauchy conditions, there exists an unbounded sequence which satisfies them all.*

2. This theorem was given by S. Mazur and W. Orlicz [1] as a paraphrase of a special result in the theory of summability; it occurs as 4.3 on page 158 of [1]. We shall give an indication of its proof. The theorem was originally stated by Mazur and Orlicz but the first published proofs are due to Karl Zeller [2, 3]; one by classical methods, inspired by ideas of R. P. Agnew; the other by means of functional analysis. Many proofs are now known.

3. The first proof of the theorem is computational and fairly long. One first proves that given a Cauchy matrix A, there exists an increasing sequence

* Invited address at the meeting of the Philadelphia Section of the Mathematical Association of America, November 24, 1956.

$r=\{r_n\}$ of positive integers such that a sequence x must obey the Cauchy condition corresponding to A if it satisfies

$$(*) \qquad \lim_{n} \max_{r_n \leq u,v \leq r_{n+1}} |x_u - x_v| = 0.$$

Corresponding to a sequence of Cauchy matrices we get a sequence $\{r^k\}$ of sequences. There exists a single sequence r such that any x satisfying $(*)$ must satisfy each condition obtained from $(*)$ by replacing r by r^k. Since there is clearly an unbounded sequence satisfying $(*)$, this concludes the proof. (For details see [3], pp. 140, 141, and §12, p. 150.)

4. The second proof depends on deeper ideas but has the advantages of brevity and intrinsic interest.

5. Given a Cauchy matrix A, let $\mathfrak{A}$ be the set of all sequences obeying the corresponding Cauchy condition *i.e.* such that Ax is null. It is possible—the exact details can be omitted—to introduce a definition of distance between two elements of $\mathfrak{A}$ such that the various following statements are correct. See [2] for details. Given any linear and continuous real function f defined on $\mathfrak{A}$, there exists a sequence $\{b_k\}$ of real numbers such that for each bounded sequence $x\in\mathfrak{A}$ we have $f(x)=\sum b_n x_n$. (See [2], pp. 476, 479.) In the following lemma, c_0 is the subset of $\mathfrak{A}$ consisting of the null sequences.

LEMMA 1. *If such a function f vanishes on c_0, then $f(x)=0$ for each bounded $x\in\mathfrak{A}$*

Proof. As above, $f(x)=\sum b_n x_n$. Now suppose that f vanishes on c_0. Let, for $k=1, 2, \cdots$, δ^k be the sequence of 0's save for 1 in the kth place. Then $0=f(\delta^k)=b_k$ for each k and so $f(x)=0$.

LEMMA 2. *Given a Cauchy condition, at least one unbounded sequence must satisfy it.*

This means that if A is a Cauchy matrix, Ax is null for at least one unbounded x. Assume the contrary. Then $\mathfrak{A}$ contains only bounded sequences. We shall deduce from this the false conclusion that $\mathfrak{A}$ contains only null sequences, contradicting the definition of a Cauchy matrix. Let $L_n(x)=x_n$, $n=1, 2, \cdots$; this defines L_n for each n, a linear and continuous real function on $\mathfrak{A}$. Our hypothesis is that $\{L_n(x)\}$ is bounded for each x. By the uniform boundedness principle $\{L_n\}$ is an equicontinuous family, *i.e.* given $\epsilon>0$, there exists $\delta>0$ such that $|x_n-y_n|<\epsilon$ for each n if the distance between x and y is less than δ.

For arbitrary $x\in\mathfrak{A}$, let $\epsilon>0$ be given, and choose $y\in c_0$ with $|x_n-y_n|<\epsilon$ for each n. This can be done since, by Lemma 1, c_0 is dense in $\mathfrak{A}$. Then $|x_n|\leq|y_n|+|x_n-y_n|<|y_n|+\epsilon$. Hence $\lim\sup|x_n|\leq\epsilon$ and so x is null.

This computation is simply the verification that an equicontinuous family which is convergent on a dense set is convergent everywhere.

6. We now turn to the proof of the main theorem. Given a sequence of Cauchy conditions, the set of sequences satisfying all these conditions is the intersection of the sets satisfying each one, hence the intersection of a sequence of metric spaces of the type we are considering. Such a space can again be given a metric of the same type: specifically, each metric is given by a sequence of seminorms and we get the metric for the intersection by using all the seminorms. (For details see [2], 4.7, p. 472.)

On the intersection space we again have the same representation of a linear and continuous real function as is mentioned above before Lemma 1. Hence the proofs of Lemmas 1 and 2 proceed unchanged and we obtain our result.

References*

1. S. Mazur and W. Orlicz, On linear methods of summability, Studia Math., vol. 14, 1954, pp. 129–160 [16, 814].

2. Karl Zeller, Allgemeine Eigenschaften von Limitierungsverfahren, Math. Z., vol. 53, 1951, pp. 463–487 [12, 604].

3. Karl Zeller, Faktorfolgen bei Limitierungsverfahren, Math. Z., vol. 56, 1952, pp. 134–151 [14, 158].

* The numbers in square brackets refer to *Mathematical Reviews*, volume and page.

MÖBIUS TETRADS*

SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India

Introduction. An incomplete model of a pair of Möbius tetrads served as a hanger until the following chain of observations made its construction simple and led to the very interesting results enumerated below. Excepting the existence of the Möbius tetrads explained in Baker's *Principles of Geometry* (vol. 3, pp. 67–68, 138), all the results established here are original.

(i) A quadric Q_1 is discovered interlocked with a pair of Möbius tetrads T_0 and T_1 that leads to the construction of infinitely many mutually interlocked tetrads all further interlocked with Q_1. Sets of four pairs of Möbius tetrads are observed, each set having the same eight vertices.

(ii) T_0 and T_1 are harmonic inverses of each other with respect to a pair of skew generators of the quadric Q for which T_0, T_1, and Q_1 are self-polar. These generators form a pair of polar lines for Q_1, and Q and Q_1 have a pair of skew generators in common.

(iii) There are two pairs of polar lines for Q_1 upon which lie the vertices, one on each, of a third tetrad T_2 that is interlocked with T_0 and T_1 independently of the infinity of tetrads in (i). Given one vertex, T_2 is uniquely determined.

* Read at the 42nd session of I.S.C.A., Baroda, January, 1955.

We proceed to establish these results in the following sections.

1. Existence.

1.1 Discovery of a quadric. Let $T_0=ABCD$ and $T_1=A_1B_1C_1D_1$ be a pair of Möbius tetrads such that $ABCD_1$, $BCDA_1$, $CDAB_1$, $DABC_1$; $A_1B_1C_1D$, $B_1C_1D_1A$, $C_1D_1A_1B$, $D_1A_1B_1C$ are the eight coplanar tetrads that constitute the faces of T_0 and T_1. This arrangement suggests that AB_1, A_1B, CD_1, C_1D are a set of four skew lines that intersect another set, *viz.*, AB, CD, A_1B_1, C_1D_1, thus forming the two systems of generators of a quadric Q_1 evidently inscribed as well as circumscribed to both T_0 and T_1, *i.e.*, T_0 and T_1 are m.i.* For, any face of either tetrad contains a pair of generators of the quadric and hence is a tangent plane, as is evident from Figure 1. Evidently, there are three quadrics of the type Q_1, each

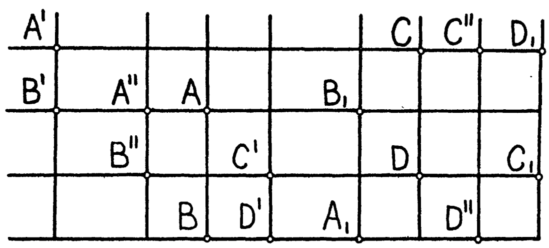

FIG. 1

having a pair of opposite edges, of each tetrad, as generators, thus proving that the vertices of the two Möbius tetrads form a set of eight associated points.

1.2 Infinity of tetrads. If we take any two generators of the second system of the preceding paragraph, one intersecting any two lines of the first set in A', B' and the other intersecting the other two in C', D', we find that $A'B'C'D'$ is a tetrad T' that is interlocked with Q_1, and in an order with T_0 and T_1 (Fig. 1.) Again, if we construct another tetrad T'' in the same manner as T', we easily see that Q_1, T', T'', T_0, and T_1 are all m.i., suggesting thus the construction of infinitely many m.i. tetrads all interlocked with Q_1.

Incidentally, we may now observe the three pairs of Möbius tetrads ABC_1D_1, A_1B_1CD; AB_1CD_1, A_1BC_1D; AB_1C_1D, A_1BCD_1 that, together with T_0 and T_1 form a set of four pairs of Möbius tetrads having the same eight vertices. Any pair gives rise to such a set.

1.3 Equation of the quadric. If the coordinates of the vertices of T_1 referred to T_0 be written† [**B-3**, p. 138] as $(0,\ -n_1,\ m_1,\ -l_1)$, $(n_1,\ 0,\ -l_1,\ -m_1)$, $(-m_1,\ l_1,\ 0,\ -n_1)$, $(l_1,\ m_1,\ n_1,\ 0)$, the equation of Q_1 is found to be $l_1(yz+tx)+m_1(yt-zx)=0$, whose generators are

---

* m.i. denotes mutually interlocked.

† **B-i** denotes H. F. Baker, Principles of Geometry, vol. i (i = 1, 2, 3, 4).

$$x - ky = 0 = l_1z + m_1t - k(m_1z - l_1t),$$
$$x - k'(l_1z + m_1t) = 0 = y - k'(m_1z - l_1t),$$

k and k' being the parameters. The lines of the first set mentioned in Section 1.1 are given by $k = \infty$, 0, l_1/m_1, $-m_1/l_1$, and those of the second set by $k' = \infty$, 0, $-n_1/(l_1^2 + m_1^2)$, $1/n_1$.

2. Self-polar quadric.

2.1 Transversals. Let U, U' and V, V' be pairs of conjugate points for Q_1 of Section 1.1 on AA_1 and BB_1, respectively. Since AA_1 and BB_1 form a pair of polar lines for Q_1, UV and $U'V'$ will also be a pair of polar lines for Q_1 (Fig. 2). Again,

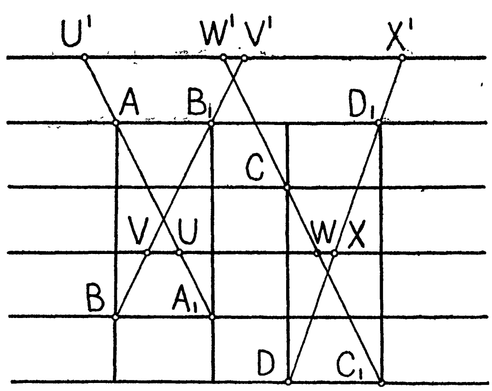

FIG. 2

if UV meets CC_1 in W, and W' be on CC_1 conjugate to W for Q_1, $U'V'W'$ will be the polar plane of W for Q_1 that must pass through the polar line DD_1 [B-3, p. 67] of CC_1 for Q_1. Further, if UVW meets DD_1 also, say in X, and X' be on DD_1 conjugate to X for Q_1, then U', V', W', X' become collinear in the polar line of $UVWX$ for Q_1. This proves that the two transversals of the four lines, AA_1, BB_1, CC_1, DD_1, joining the corresponding vertices of a pair of Möbius tetrads, T_0 and T_1, constitute a pair of polar lines for Q_1.

2.2 Harmonic inversion with respect to a pair of skew lines. Two points, B, B_1, are harmonic inverses of each other with respect to skew lines x, y, if BB_1 meets the lines in points B', B'', that separate B, B_1 harmonically. If C, C_1 are also harmonic inverses with respect to x, y, the lines BC and B_1C_1 are said to be harmonic inverses of each other with respect to x, y, for any point on BC is the inverse of a point on B_1C_1 and conversely, and the cross-ratio is unaltered by this inversion [B-3, p. 3]. If D, D_1 is a third pair of points that are harmonic inverses with respect to x, y and neither collinear with BC nor with B_1C_1, the planes BCD, $B_1C_1D_1$ are said to be harmonic inverses with respect to x, y. The line l joining the meets, L', L'', of x, y with either plane, say BCD, is its own

inverse and the other plane, $B_1C_1D_1$, is none other than the fourth harmonic of the first plane with respect to the planes lx, ly (Fig. 3). If x, y happen to be a pair of generators of the same system of a quadric Q, it is easy to see that any point has for its inverse with respect to x, y, its conjugate with respect to Q. A pair of conjugate points or planes for Q has for its inverse a pair of conjugate points or planes for Q (cross-ratio property). If x, y is a pair of polar lines for a quadric, the quadric is its own harmonic inverse.

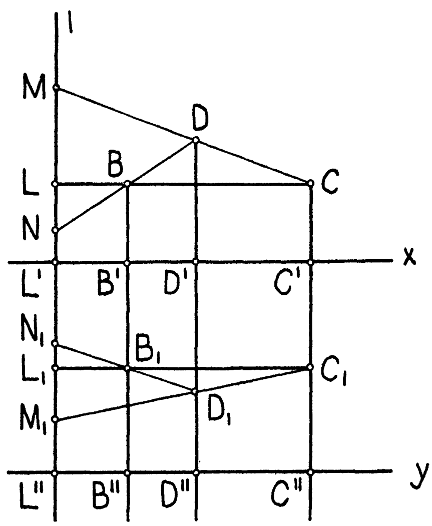

FIG. 3

We may now observe that a pair of Möbius tetrads, T_0, T_1, are harmonic inverses of each other (Fig. 2) with respect to the transversals (Sec. 2.1) $x \equiv UVWX$, $y \equiv U'V'W'X'$, of the four joins of the corresponding vertices of the tetrads [**B-3**, p. 144], and Q_1 is its own inverse.

2.3 Construction. (a) Let us construct a quadric Q having $x \equiv UVWX$, $y \equiv U'V'W'X'$ as defined in Section 2.1 as a pair of skew generators and the triangle BCD self-conjugate for Q. Since AA_1, BB_1, CC_1, DD_1, are pairs of harmonic inverses with respect to x, y (Fig. 2), they are evidently pairs of conjugate points for Q. The polar planes of B, C, D are then CDB_1, BDC_1, BCD_1, respectively, all containing A (Sec. 1.1), proving that BCD is the polar plane of A, *i.e.*, T_0 is self-polar for Q.

(b) Since T_1 is the harmonic inverse of T_0 with respect to a pair of generators of Q, it must also be self-polar for Q (Sec. 2.2; [**B-3**, p. 68, Ex. 22]). Hence, the pair of transversals to the four lines joining the corresponding vertices of a pair of Möbius tetrads are generators of the quadric for which the given tetrads are self-polar. It is interesting to observe that the polar reciprocal of Q_1 (Sec. **1.1**)

with respect to Q is Q_1 itself. This follows since the polar reciprocal, being defined as the locus of the poles of tangent planes of Q_1 as well as the envelope of the polar planes of points of Q_1 with respect to Q, must be Q_1 itself, since it passes through seven independent vertices and touches seven independent faces of the tetrads, T_0, T_1, which are poles and polar planes, respectively, of seven tangent planes and seven points of Q_1 with respect to Q. Hence, Q_1 is also self-reciprocal or self-polar for Q.

2.4 Generators common to two quadrics. Let us now construct the pair of generators u, v, of the quadric Q_1 (Sec. 1.1) that separate harmonically the two pairs of its generators of the first system, AB_1, A_1B and CD_1, C_1D, which we

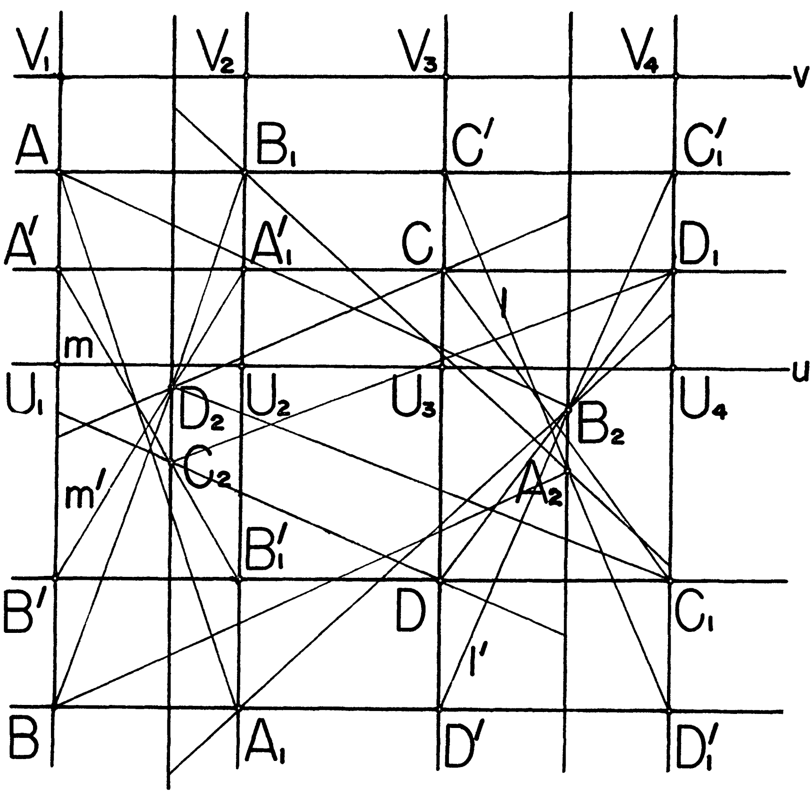

FIG. 4

notice to be the pairs of polar lines for Q. This is true since, by the preceding section, the polar planes A_1BCD and $A_1BC_1D_1$ of A, B_1 with respect to Q have the line A_1B in common, and the polar planes C_1DAB, $C_1DA_1B_1$ of C,D_1 with respect to Q have the line C_1D in common. From the nature of the construction of u, v we see that U_i, V_i ($i=1, 2, 3, 4$) on u, v, respectively separate harmonically the two pairs of conjugate points for Q on each of the generators AB, A_1B_1, CD,

C_1D_1 (Fig. 4), of the second system of Q_1 where they meet the two pairs of polar lines, AB_1, A_1B and CD_1, C_1D for Q; that is U_i, V_i are the united elements of the involution set up by the two pairs of conjugate points for Q on the respective lines and hence are none other than the pairs of intersections of these lines with Q. In other words, we prove that they lie on Q, showing that u, v are the common generators of Q and Q_1.

2.5 Equations. The equation of Q referred to T_0 is found to be $x^2+y^2+z^2+t^2=0$, and u, v are given by (Sec. 1.3) $k=\pm i$, that is, $x\pm iy=0=z\mp it$. To find the polar reciprocal of Q_1 with respect to Q, let $lx+my+nz+pt=0$ be a tangent plane of Q_1. The condition of tangency implies [**B-3**, p. 24]

$$\begin{vmatrix} 0 & 0 & -m_1 & l_1 & l \\ 0 & 0 & l_1 & m_1 & m \\ -m_1 & l_1 & 0 & 0 & n \\ l_1 & m_1 & 0 & 0 & p \\ l & m & n & p & 0 \end{vmatrix} = 0.$$

Its pole referred to Q is (l, m, n, p) and hence the locus of this point, defined to be the polar reciprocal of Q_1, is given by

$$\begin{vmatrix} 0 & 0 & -m_1 & l_1 & x \\ 0 & 0 & l_1 & m_1 & y \\ -m_1 & l_1 & 0 & 0 & z \\ l_1 & m_1 & 0 & 0 & t \\ x & y & z & t & 0 \end{vmatrix} = 0,$$

which on simplification is found to be Q_1. It is easy to verify that T_1 is self-polar for Q.

3. A third tetrad.

3.1 Location. Let $A_2B_2C_2D_2$ be a tetrad T_2 interlocked with both T_0 and T_1 such that A_2 lies in BCD and $B_1C_1D_1$, B_2 in ACD and $A_1C_1D_1$, C_2 in ABD and $A_1B_1D_1$, D_2 in ABC and $A_1B_1C_1$. Evidently, the line l common to BCD and $B_1C_1D_1$ is polar to the line l' common to ACD and $A_1C_1D_1$, and the line m common to ABD and $A_1B_1D_1$ is polar to the line m' common to ABC and $A_1B_1C_1$ for Q_1 (Fig. 4). Hence, the vertices of T_2, independently of the infinity of m.i. tetrads (Sec. 1), lie on two pairs of polar lines for Q_1.

3.2 Three quadrics. If T_2 is interlocked with T_0 and T_1 as described in the preceding paragraph, then, following Section 1, AB, A_2B_2, CD, C_2D_2, with AB_2, A_2B, CD_2, C_2D, will generate a quadric Q_2, and A_1B_1 A_2B_2, C_1D_1, C_2D_2, with A_1B_2, A_2B_1, C_1D_2, C_2D_1, generate a quadric Q_{12}. The three quadrics Q_1, Q_2, Q_{12}, have a pair of skew generators, say u', v', in common. This follows since Q_1 and

Q_2, having a pair of skew generators AB, CD in common, will have another pair of generators of the other system in common, which will therefore intersect all the generators of the first system and, in particular, A_1B_1, C_1D_1, of Q_1 and A_2B_2, C_2D_2, of Q_2. Hence, they will be generators of Q_{12} also.

To locate these generators, we notice that A_2, B_2 and C_2, D_2, being pairs of conjugate points for Q_1 (Sec. 3.1), A_2B_2, C_2D_2 will meet Q_1 in pairs of points, conjugate for Q_2 and Q_{12}, that must lie on u', v', and which will therefore separate harmonically (Fig. 4) the pairs of generators of the same system of both Q_2 and Q_{12} through A_2, B_2 and C_2, D_2, respectively. Then A, B; C, D; A_1, B_1; C_1, D_1 will be pairs of harmonically inverse points with respect to u', v' (following the property of related ranges, [**B-3**, p. 3]), thus identifying u', v' with u, v of Section 2.4.

3.3 Self-polar for Q. Since u, v are generators of Q (Sec. 2.3), A_2, B_2 and C_2, D_2 form pairs of conjugate points for Q also. Furthermore, A_2 is conjugate to A as well as A_1 for Q, since it lies in their polar planes BCD and $B_1C_1D_1$ with respect to Q, and the plane $B_2C_2D_2$ contains A as well as A_1, since T_2 is assumed to be interlocked with both T_0 and T_1. We can say, therefore, that the polar plane of A_2 with respect to Q, passing through three of its conjugate points, B_2, A, A_1, is none other than the plane $B_2C_2D_2$. Similarly, it can be proved that the polar planes of B_2, C_2, and D_2 for Q will be $A_2C_2D_2$, $A_2B_2D_2$, and $A_2B_2C_2$, respectively, showing that T_2 is also self-polar for Q.

3.4 Construction. We are now in a position to construct a tetrad $T_2 \equiv A_2B_2C_2D_2$,interlocked with both T_0 and T_1 (Sec. 1.1), as follows:

Through any point A_2 on the line l (Sec. 3.1) draw a transversal to u, v that will meet l' in B_2. Then, l, l', u, v, CC_1, DD_1 (Fig. 4) generate a quadric following the property of related ranges on CD and C_1D_1 [**B-3**, p. 3]. The quadric generated by m, m', u, v, AA_1, BB_1, and that by AB, CD, A_2B_2, having the generators AB, u, v in common, must have another generator in common that meets m, m' in C_2, D_2, respectively. Then $A_2B_2C_2D_2$ is the required tetrad T_2 that is interlocked with T_0. This follows, since A_2 lying in BCD means that A_2B intersects CD, so that A_2B intersects A_2B_2, AB, and hence C_2D_2 also. The lines AB_2, CD_2, C_2D have a similar property, thus determining the quadric Q_2 of Section 3.2. To show that T_2 is also embedded with T_1, we observe that C_2D_2 intersects AA_1 as well as AB_2, showing that A_1B_2 intersects C_2D_2; but A_1B_2 intersects A_1B_1 and C_1D_1 also. Similarly, A_2B_1, C_1D_2, C_2D_1 intersect the four lines A_1B_1, A_2B_2, C_1D_1, C_2D_2, proving finally what is required. Hence, given one vertex, T_2 is uniquely determined.

4. Further results. In addition to the results (i), (ii), (iii) stated in the introduction, it is possible to prove the following:

(iv) The tetrads T_0, T_1, T_2, give rise to another tetrad T_3, forming a set of four mutually interlocked tetrads that lead to another set of four such tetrads, having the same sixteen vertices (faces) that lie (touch) by sixes on sixteen

conics (cones), one in (through) each face (vertex), having in common with one another a pair of vertices (faces as tangent planes). The eight tetrads of the two sets are all self-polar with respect to Q. There exist thirty-six quadrics out-polar (in-polar) to Q, each one passing through (enveloping) four conics (cones) and twelve vertices (faces) that form two pairs of Möbius tetrads, one pair from each set. Nine quadrics pass through (envelop) each conic (cone).

(v) There are eighteen quadrics Q_i for the set of tetrads T_i ($i=0, 1, 2, 3$), three for each pair, divided into three sets of six each, each set having a fixed pair of skew generators in common with Q. The same is the case with the second set of tetrads. The relations of these thirty-six quadrics with those of (iv) are quite remarkable.

Proofs of these results will appear later.

COMPLEX NUMBERS AND TRIGONOMETRY

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1. Introduction. The present paper is concerned with a method of presenting trigonometry which greatly reduces the time which needs to be devoted to the subject and also contributes significantly to the mathematical maturity of the student. This method is to treat trigonometry as a branch of the theory of complex numbers. The present treatment is a development of an approach used at Williams College in order to handle in a single freshman course students who have had no preparation in trigonometry along with those who have had such a preparation. The treatment is sufficiently novel to interest those already familiar with the subject without being too difficult for beginners. The material prior to Section 6 could very well be taught in the fourth year of high school. I regard this as entirely practicable. It has the advantage of familiarizing students with complex numbers and vectors and opening the way to significant connections with other branches of mathematics.

2. Complex numbers. Our treatment of complex numbers is along conventional lines with one exception. Specifically:

(1) Two complex numbers, $\alpha=a+ib$ and $\beta=c+id$, are defined to be equal if and only if $a=c$ and $b=d$.

(2) Sums, differences and products are defined by

$$\alpha \pm \beta = (a \pm c) + i(b \pm d), \alpha\beta = (ac - bd) + i(ad + bc).$$

(3) Complex numbers are represented by vectors which are added and subtracted in the familiar way.

(4) The absolute value $|\alpha|$ of α is defined by $|\alpha|^2=a^2+b^2=\alpha\bar{\alpha}$ where $\bar{\alpha}=a-ib$.

The angle θ of α is to be measured (in degrees or radians) counterclockwise from the positive real axis.

(5) However, instead of justifying the usual geometric representation of the product of α and β by introducing the polar form and appealing to the addition formulas of trigonometry, we adopt an idea from Dresden's *An Invitation to Mathematics* (pp. 89, 90) to prove directly that when two factors are multiplied, their magnitudes are multiplied and their angles are added. The proof, which differs in detail from his, follows in Section 3.

3. Multiplication of complex numbers.

a) It is easy to verify that $\overline{\alpha\beta}=\bar{\alpha}\bar{\beta}$, that is, that the conjugate of the product is the product of the conjugates. Hence

$$|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = \alpha\beta\bar{\alpha}\bar{\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = |\alpha|^2|\beta|^2.$$

Since magnitudes are nonnegative, $|\alpha\beta| = |\alpha|\cdot|\beta|$.

b) In discussing the angle of the product, it is sufficient to assume that $|\alpha|=|\beta|=1$, since an alteration in the magnitudes of α and β would not change the direction of their product. Let θ and ϕ be the angles of α and β. (See Figs. 1a, 1b.) Then chord $BC=|\alpha\beta-\beta|=|(\alpha-1)\beta|=|\alpha-1||\beta|=|\alpha-1|=$chord AR. Hence $\angle BOC=\theta$ and $\angle ROC=\phi+\theta$ or $\phi-\theta$.

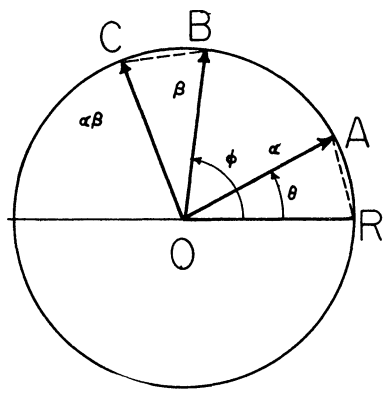

FIG. 1a

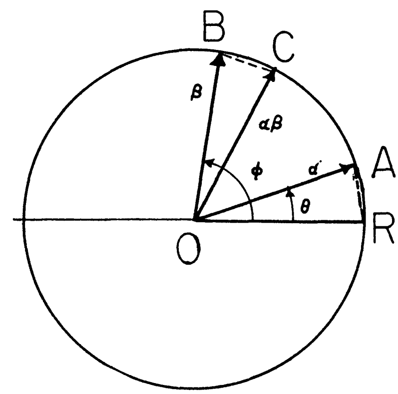

FIG. 1b

Interchanging the role of α and β in the proof gives $\angle ROC=\theta+\phi$ or $\theta-\phi$. The desired conclusion follows unless $\phi-\theta=\theta-\phi$, that is, unless $\phi-\theta=0°$ or $180°$. For the first alternative, $\alpha=\beta$, and we are asked to believe that $\alpha^2=1$. This is possible only if $\alpha=\beta=1$ or $\alpha=\beta=-1$, when addition and subtraction of angles give equivalent answers (identifying angles which differ by 360°). The alternative $\phi=\theta+180°$ is handled similarly.

4. The addition formulas of trigonometry. A complex number α of unit magnitude ($|\alpha|=1$) is completely specified by the angle θ between it and the posi-

tive real axis. (We shall use degree measure.) It will be appropriate to write $\alpha=\alpha(\theta)$. We define $\cos\theta$ to be the real part of $\alpha(\theta)$ and $\sin\theta$ to be its imaginary part. (See Fig. 2.)

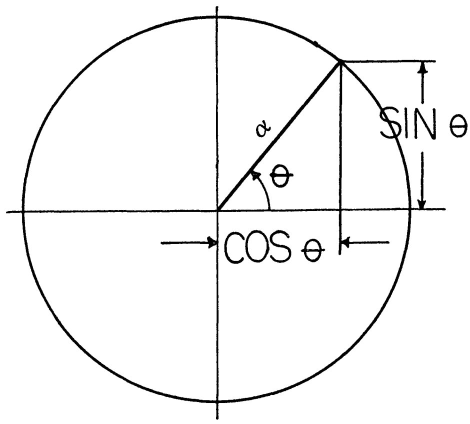

FIG. 2

Let $\alpha(\theta)$ and $\alpha(\phi)$ have unit magnitude. By definition

$$\alpha(\theta) = \cos\theta + i\sin\theta,\ \alpha(\phi) = \cos\phi + i\sin\phi.$$

Multiply $\alpha(\theta)$ by $\alpha(\phi)$, first by using the definition, secondly by using the theorem of Section 3 which gives

$$\alpha(\theta)\alpha(\phi) = \alpha(\theta+\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi).$$

Equating the real and imaginary parts of these two results gives the addition formulas for the cosine and sine. This proof is completely general, involving no separate consideration of quadrants. It has the additional merit of being extremely simple.

To prove the subtraction formulas, multiply the equation

$$\alpha(\phi-\theta)\alpha(\theta) = \alpha(\phi)$$

by $\bar{\alpha}(\theta)$, using $\alpha(\theta)\bar{\alpha}(\theta)=1$, to obtain

$$\alpha(\phi-\theta) = \alpha(\phi)\bar{\alpha}(\theta).$$

Multiply out on the right and equate real and imaginary parts. The special case $\phi=0$ gives $\alpha(-\theta)=\bar{\alpha}(\theta)$ and hence $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta$.

5. Multiple angles. Calculation of tables. From the results of Section 4, it follows immediately that $[\alpha(\theta)]^n=\alpha(n\theta)$ which yields the well-known formula of de Moivre

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

and thence formulas for $\cos n\theta$ and $\sin n\theta$, in particular, for $\cos 2\theta$ and $\sin 2\theta$.

The sines and cosines of the quadrantal angles and of 30°, 45°, and 60°, with their relatives in other quadrants, are easily obtained by simple geometrical arguments. It is of interest to note how easily one may construct a complete table of sines and cosines for all angles which are integral multiples of 1°.

We may begin by calculating $\alpha=\alpha(72°)$. Clearly $\alpha^5=1$ or $\alpha^5-1=0$. Removing the uninteresting root $\alpha=1$, we have $\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$. Dividing by α^2 and noting that $1/\alpha=\bar{\alpha}$,

$$\alpha^2+\alpha+1+\bar{\alpha}+\bar{\alpha}^2=0. \tag{1}$$

Now $\alpha+\bar{\alpha}=2\cos 72°=x$, say, and squaring and simplifying, $\alpha^2+\bar{\alpha}^2=x^2-2$. Then (1) becomes $x^2+x-1=0$ whose positive root is $(\sqrt{5}-1)/2$. From $\cos 72° =(\sqrt{5}-1)/4$, we obtain successively cos 36°, cos 18°, cos 9° and sin 9°.

Similarly, $\alpha(40°)$ satisfies $\alpha^9-1=0$. The same procedure, using easily derived expressions for $\alpha^3+\bar{\alpha}^3$ and $\alpha^4+\bar{\alpha}^4$, gives $x^4+x^3-3x^2-2x+1=0$ for $x=2\cos 40°$. Removing the obvious root $x=-1$, we must solve

$$x^3-3x+1=0 \tag{2}$$

for its largest positive root (1.53208). From cos 40°, one readily calculates cos 20°, cos 10° and sin 10°. From cos 9° and sin 9°, previously calculated, and the subtraction formulas we have sin 1°, cos 1°, and thence the sines and cosines of all integral multiples of 1°.

It is to be noted that cos 40° is not constructible by ruler and compass while cos 72° and cos 60° are so constructible. It follows that the only angles with an integral number of degrees which are constructible are multiples of 3°.

The method described furnishes a natural bridge to problems in the theory of equations. In this connection it is pertinent to show the solution of (2) by *iteration*, a method which deserves to be included in our elementary courses. We write (2) in the form $x=\sqrt[3]{3x-1}$ and substitute under the radical a guess (x_0) at the required solution, obtaining $x_1=\sqrt[3]{3x_0-1}$. Feed back x_1 under the radical to obtain $x_2=\sqrt[3]{3x_1-1}$ and continue until agreement is reached to the requisite accuracy. Concretely, if this process is applied to the present case, starting with $x_0=1.5$, one obtains from a table of cube roots, the sequence $x_1=1.5183$, $x_2=1.526$, $x_3=1.529$, $x_4=1.531^-$, $x_5=1.531^+$. Agreement to three decimal places is reached at $x=1.532$.

We may well ask for assurance that this process converges and that it yields the desired solution. Clearly

$$x_{n+1}^3=3x_n-1. \tag{3}$$

Then

$$x_{n+2}^3=3x_{n+1}-1, \qquad x_{n+2}^3-x_{n+1}^3=3(x_{n+1}-x_n).$$

If therefore $x_{n+1}>x_n$, then $x_{n+2}>x_{n+1}$.

Since in fact $x_1>x_0$, the solutions $x_0, x_1, x_2, \cdots$ form an increasing sequence.

This sequence has 2 as an upper bound. For, if $x_{n+1}>2$, by (3) $x_n>3>2$ and it would follow that $x_0>2$, contradicting $x_0=1.5$. The sequence $\{x_n\}$ therefore converges to a limit $L\leqq 2$. It is easy to reduce this upper bound.

Since $x_n\to L$, $x_{n+1}\to L$ and from (3), $L^3=3L-1$, so that L is indeed a solution of the required equation.

6. Calculus of trigonometric functions. We now turn to those properties of the sine and cosine functions which require calculus ideas for their expression and proof. For this purpose we use radian or arc measure. Let x be the arc length measured counterclockwise from A and $\alpha(x)$ the complex number represented by the vector $\overrightarrow{OP}$ (see Fig. 3). We proceed to determine the derivative of $\alpha(x)$.

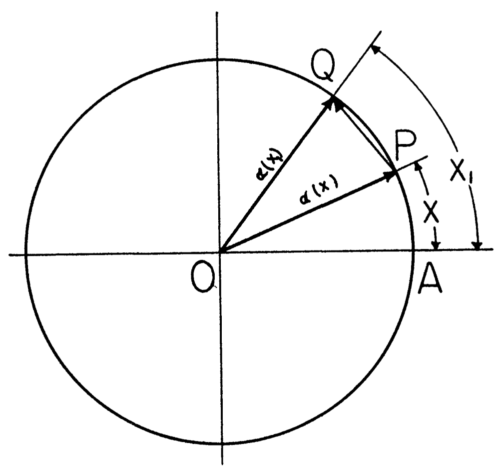

FIG. 3

Let x_1 be a second value of the arc length and $\alpha(x_1)$ the corresponding complex number. The derivative $D\alpha$ is defined to be the limit as $x_1\to x$ of the ratio $[\alpha(x_1)-\alpha(x)]/[x_1-x]$, provided that this limit exists.

Now $\alpha(x_1)-\alpha(x)$ is represented by the vector $\overrightarrow{PQ}$ along the chord of the unit circle while x_1-x is the length of the arc PQ, and $[\alpha(x_1)-\alpha(x)]/[x_1-x]$ is directed along $\overrightarrow{PQ}$. As $x_1\to x$, the ratio of the length of chord PQ to arc PQ approaches 1 and the direction approaches that of the tangent to the circle at P. Hence

$$\frac{\alpha(x_1)-\alpha(x)}{x_1-x}\to\tau(x),$$

where $\tau(x)$ is the complex number represented by the unit tangent vector PT (Fig. 4). But $\tau(x)=i\alpha(x)$, since multiplication by i rotates $\alpha(x)$ through a right

angle. Hence

$$D\alpha = i\alpha. \tag{4}$$

This result gives almost immediately the derivatives of sin x and cos x, since

$$D(\cos x + i \sin x) = i(\cos x + i \sin x)$$

or

$$D \cos x + iD \sin x = i \cos x - \sin x.$$

Hence

$$D \cos x = - \sin x, \qquad D \sin x = \cos x.$$

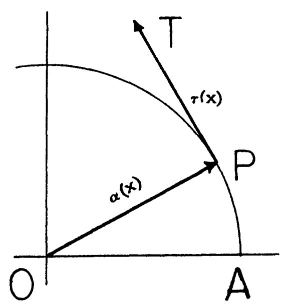

FIG. 4

Equation (4) suggests the notation e^{ix} for $\alpha(x)$ since (4) is then the analogue of $De^{ax} = ae^{ax}$. It is of course open to us to define e^{ix} to be the function previously called $\alpha(x)$, so that in this new notation

$$e^{ix} = \cos x + i \sin x,$$

the famous Euler formula. We shall not, however, adopt this notation since we wish our treatment to be independent of any prior knowledge of the exponential function.

We proceed to solve (4) for $\alpha(x)$ subject to the condition, $\alpha(0) = 1$. Integrating from 0 to x, we have the equivalent equation

$$\alpha(x) - 1 = i \int_0^x \alpha(x) dx. \tag{5}$$

We solve (5) by iteration in a form suitable for computation, starting on the right with $\alpha_0(x) = 1$ and finding in succession $\alpha_1(x)$, $\alpha_2(x)$, $\cdots$, where

$$\alpha_{n+1}(x) = 1 + i \int_0^x \alpha_n(x) dx. \tag{6}$$

We obtain

$$\alpha_1(x) = 1 + ix,$$

$$\alpha_2(x) = 1 + i\left(x + \frac{ix^2}{2}\right) = 1 + ix + \frac{(ix)^2}{2},$$

$$\alpha_3(x) = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{2.3}.$$

By induction, one easily proves that

$$\alpha_n(x) = 1 + ix + \frac{(ix)^2}{2!} + \cdots + \frac{(ix)^n}{n!}.$$

It is of interest to construct $\alpha_n(x)$ graphically. For simplicity we assume that $x<1$. This is no handicap in practice, since we require $\sin x$ and $\cos x$ (and hence $\alpha(x)$) only for $x \leqq \pi/4$.

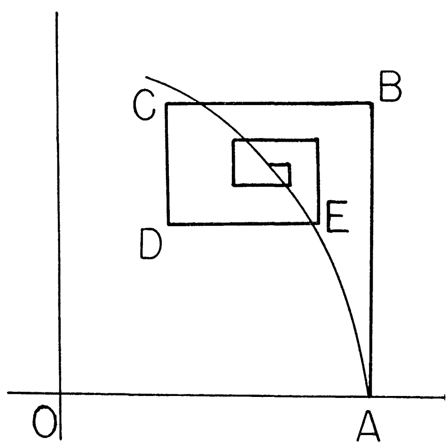

FIG. 5

We plot as points in the complex plane in succession (Fig. 5)

$$A: \quad \alpha_0 = 1,$$

$$B: \quad \alpha_1 = 1 + ix,$$

$$C: \quad \alpha_2 = 1 + ix - \frac{x^2}{2!},$$

$$D: \quad \alpha_3 = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!},$$

$$E: \quad \alpha_4 = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!}.$$

Joining O to A, to B, to C, *etc.*, by straight-line segments gives a spiral. Each term, $(ix)^n/n!$, which is added, differs from its predecessor by the factor ix/n. The factor i corresponds to a counterclockwise rotation of 90°. With $x<1$, as assumed, $x/n<1$, so that the successive segments of the spiral become shorter. Moreover, $x^n/n!<1/n!<1/n$, so that the length of the nth segment approaches zero. The α_n's converge to a limit $\alpha(x)$. In fact, if $r_n(x)=\alpha(x)-\alpha_n(x)$, it is geometrically clear that $|r_n(x)|<2/n$. It remains to prove that $\alpha(x)$ is the required solution so that

$$\alpha(x) = 1 + i\int_0^x \alpha(x)dx.$$

Proof. We have from (6)

$$\begin{aligned}\alpha_{n+1}(x) &= 1 + i\int_0^x \alpha_n(x)dx \\ &= 1 + i\int_0^x \alpha(x)dx - i\int_0^x r_n(x)dx.\end{aligned}$$

Since $\alpha_{n+1}(x)\to\alpha(x)$, (5) follows if $\int_0^x r_n(x)dx\to 0$. Now

$$\left|\int_0^x r_n(x)dx\right| \leqq \int_0^x |r_n(x)|\, dx \leqq \int_0^x \frac{2}{n}\, dx = \frac{2x}{n} \leqq \frac{2}{n},$$

so that $\int_0^x r_n(x)dx\to 0$ as $n\to\infty$.

From

$$\alpha(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

one obtains the well-known series for cos x and sin x by taking the real and imaginary parts, respectively. Thus

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

References

1. R. Courant and H. Robbins. What is Mathematics?, 3rd ed., Oxford, 1945, pp. 88–100, 120–140.

2. L. E. Dickson, New First Course in the Theory of Equations, New York, 1939, pp. 163–175.

3. A. Dresden, An Invitation to Mathematics, New York, 1936, pp. 89–90.

4. H. Fehr, Secondary Mathematics, Boston, 1951, pp. 254–272, 382–405.

5. I. Niven, The Concept of Number, Yearbook N.C.T.M., 1956, Ch. II.

6. W. F. Osgood, First Course in the Differential and Integral Calculus, Rev. ed., New York, 1940, p. 403.

7. D. E. Richmond, Fundamentals of the Calculus, New York. 1950, pp. 156–199.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the seventeenth William Lowell Putnam Mathematical Competition held on March 2, 1957, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Everett C. Dade, David B. Mumford, and Rohit J. Parikh; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Columbia College, New York, New York. The members of the team were David Bloom, Jonathan Lubin, and Jerrold Rubin; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Frans M. Djorup, Stanley Kaplan, and Robert F. Riley; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Luis Baez-Duarte, H. Jerome Keisler, and Charles J. Stone; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are David Bloom, Columbia College; Richard Bumby, Massachusetts Institute of Technology; Everett C. Dade, Harvard University; Rohit J. Parikh, Harvard University; and Ian Richards, University of Minnesota. Each of these will receive a prize of fifty dollars.

The six succeeding persons (tenth and eleventh tied) ranking highest in the examination, named in alphabetical order, are Armand Brumer, Brandeis University; H. Jerome Keisler, California Institute of Technology; David B. Mumford, Harvard University; Robert F. Riley, Cornell University; Paul A. Schweitzer, College of the Holy Cross; and Lawrence Shepp, Polytechnic Institute of Brooklyn. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: Oberlin College, Oberlin, Ohio, the members of the team being George Hannauer, Daniel Kleinman, and John Miller; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being James Ax, Donald Passman, and Lawrence Shepp; University of California, Los Angeles, California, the members of the team being Richard S. Grote, Louis A. Jaeckel, and Michael Raugh; and University of Michigan, Ann Arbor, Michigan, the members of the

team being John S. Denton, Jr., James Penquite, and Charles Sims.

Twelve individuals were given honorable mention. The names, in alphabetical order, are: David M. Dahm, Rice Institute; John S. Denton, Jr., University of Michigan; John D. Ferguson, Yale University; Stanley Kaplan, Cornell University; Jacob Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; Jonathan Lubin, Columbia College; John P. McCabe, Manhattan College; Michael Raugh, University of California, Los Angeles; Robert Solovay, Harvard University; Harold Nathaniel Ward, Swarthmore College; and Peter Wolk, Massachusetts Institute of Technology.

A total of 505 individuals from 102 institutions entered the competition this year. Of this number 128 individuals and 9 institutions were unable to compete, due to various reasons. Therefore, a total of 377 undergraduates from 93 institutions actually took part in the competition.

The following is a list, in alphabetical order, of all colleges and univerities which entered teams in the Competition: Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, Alabama Polytechnic Institute, Arizona State College (Flagstaff), Arizona State College (Tempe), Bethune-Cookman College, Brandeis University, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Davidson College, DePaul University, Georgia Institute of Technology, Harvard University, Haverford College, Kent State University, Kenyon College, Knox College, Lebanon Valley College, Manhattan College, Massachusetts Institute of Technology, McGill University, Muskingum College, New Mexico College of Agriculture and Mechanical Arts, New York University, Oberlin College, Phillips University, Polytechnic Institute of Brooklyn, Princeton University, Purdue University, Queen's University (Ontario), Rice Institute, Rose Polytechnic Institute, Royal Military College of Canada, Rutgers College of South Jersey, Saint Francis Xavier University, Saint Martin's College, Saint Mary's College (Indiana), Saint Olaf College, San Jose State College, Siena College, Spring Hill College, Stanford University, State College of Washington, Swarthmore College, The Cardinal Stritch College, The College of Saint Catherine, The College of the City of New York, The Ohio State University, The University of British Columbia, Union College, United States Naval Academy, University of Arizona, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Detroit, University of Michigan, University of Minnesota, University of Nebraska, University of Notre Dame, University of Oregon, University of Rochester, University of Tennessee, University of Toronto, University of Washington, Ursinus College, Vanderbilt University, Vassar College, Wayne State University, Wesleyan University (Connecticut), and Yale University.

The following colleges and universities, alphabetically arranged, entered individual contestants only: Blackburn College, Colorado School of Mines, Fordham University, Friends University, Hamilton College (of McMaster University), Iowa State College, Long Beach State College, Manchester College, Miami University, Northwestern University, Seattle University, Syracuse University, Talladega College, The College of Saint Francis, The College of Saint Thomas, The Cooper Union, The University of Texas, The University of Western Ontario, University of Alaska, University of Buffalo, University of Chicago, University of Cincinnati, Washington University (Missouri), and West Virginia Wesleyan.

The individual rankings of contestants (except for the relative ranks of the

first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Material for this department should be sent to Ivan Niven, Department of Mathematics, University of Oregon, Eugene, Oregon.

THE NUMBER OF ELEMENTS OF GIVEN PERIOD IN FINITE SYMMETRIC GROUPS*

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1. Introduction. During the past ten years a number of papers on the solutions of $x^d=1$ in symmetric groups have appeared. In these papers asymptotic expansions for the number of elements whose period is a divisor of d have been obtained; the expansions in turn have been based on recursion formulas for the finite case. Jacobsthal [4] obtained a recursion formula and an asymptotic expansion for the case when d is a prime. Chowla and Herstein, together with Moore [1], and later with Scott [2], obtained properties of the recursion for d taken equal to 2, and then for d taken as any positive integer. The recursion formula for positive integral d, obtained by Chowla, Herstein and Scott, is equivalent to equation (3) given below.

More recently, general results in the asymptotic case have been obtained by Moser and Wyman [5], who have also extended the investigation of T_n, the number of solutions of $x^2=1$ in the symmetric group. By using Hermite polynomials and functions of a complex variable, Moser and Wyman have obtained refinements of expansions suggested in [2], together with further expansions for both the symmetric group and the alternating group.

The result obtained in the present paper is a complete expansion formula (in the finite case) for the number of elements of given period in finite symmetric groups. This result was first announced in 1946; it first appeared in print in 1953 [3].

2. Elements of period r in S_k. Let $S(k)$ be the set of all 1-1 mappings of the first k integers upon themselves, that is, permutations of $1, \cdots, k$, and let $S(k-1)$ be the subset of all mappings in $S(k)$ which leave k invariant, that is,

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* Presented to the Mathematics Seminar, University of California, Los Angeles, May 22, 1946, and, in revised form, to the Southern California Section of the Association, March 14, 1953.

The author wishes to thank Professor Leonard Carlitz for editorial advice in the preparation of this paper.

which map k onto k. Then $S(k)$ is a realization of the symmetric group of degree k and order $k!$, and $S(k-1)$ is the symmetric group of degree $k-1$ as a subgroup of $S(k)$.

Let $N(k; r)$ represent the number of elements in $S(k)$ of period r, with the convention that $N(0; 1)=1$. We shall prove the

THEOREM. $N(k; r)=\sum_s \{(k-1)!/(k-s)!\} \sum_b N(k-s; b)$, *where* $[s, b]=r$.

We require the

LEMMA. *If* $S(k)$ *be separated into cosets mod* $S(k-1)$, *such that* $S(k)=S(k-1)+\sum_{i=1}^{k-1} X(i)$, *where* $X(i)$ *is the right coset of* $S(k-1)$ *containing all the mappings in* $S(k)$ *which carry* k *into* i, *then each* $X(i)$ *contains the same number of elements of given period.*

Proof of the lemma. Let p be a permutation in $S(k-1)$ which carries i into j, then $p^{-1}X(i)p=X(j)$. Thus $X(i)$ and $X(j)$ are conjugate subsets of $S(k)$, and as such they contain the same number of elements of given period.

As a consequence of the lemma we have the

COROLLARY. $N(k; r)=N(k-1; r)+(k-1)N'(k; r)$, *where* $N'(k; r)$ *is the number of elements of period* r *in one of the cosets of* $S(k-1)$.

Proof of the theorem. Let $M(k; r)$ be the number of elements in $S(k)$ whose period is a divisor of r; let $S(k-1)$ be the subgroup of permutations in $S(k)$ leaving k invariant, and let $X(k-1)$ be the right coset of permutations in $S(k)$ mapping k upon $k-1$.

By the lemma we find

$$M(k; r) = M(k-1; r) + (k-1)M'(k; r), \tag{1}$$

where $M'(k; r)$ is the number of permutations in $X(k-1)$ whose period is a divisor of r.

The permutations from $X(k-1)$ have the form

$$(k, k-1, j_3, j_4, \cdots, j_s)\cdot p = c\cdot p,$$

where $s \mid r$, and p is a permutation of the integers other than $k, k-1, j_3, j_4, \cdots, j_s$ such that (period of p)$\mid r$.

The cycle c can be selected in exactly the following number of different ways, $(k-2)(k-3)\cdots(k-s+1)$, since j_3 is selected from among $k-2$ integers, j_4 from among $k-3, \cdots, j_s$ from among $k-(s-1)$ integers.

Having selected c, we may select permutation p in exactly $M(k-s; r)$ ways, as a permutation on $k-s$ symbols, whose period is a divisor of r. Thus we find

$$\begin{aligned} M(k; r) &= M(k-1; r) \\ &\quad + (k-1)\sum_{s\mid r}(k-2)(k-3)\cdots(k-s+1)M(k-s; r), \end{aligned} \tag{2}$$

where $s \geqq 2$.

If the vacuous product is defined to be 1, then we have

$$M(k; r) = \sum_{s|r} (k-1)(k-2) \cdots (k-s+1)M(k-s; r),$$

which gives us

$$M(k; r) = \sum_{s|r} \frac{(k-1)!}{(k-s)!} M(k-s; r). \tag{3}$$

(Formula (3) is equivalent to formula (I), [2], the result obtained by Chowla, Herstein and Scott.)

To compute $N(k; r)$ we employ the number-theoretic combinatorial principle for selecting a defect-free subset from a given set of objects. Thus we find

$$N(k;r) = M(k;r) - \sum_{p|q} M(k; r/p) + \sum_{pq|r} M(k;r/pq) - \sum + \sum \cdots, \tag{4}$$

where $p, q, \cdots$, are distinct prime divisors of r.

The expansion (4) can be condensed by means of the Möbius function, $\mu(a)$, to the form

$$N(k; r) = \sum_{a|r} \mu(a)M(k; r/a). \tag{5}$$

Substituting for $M(k; r/a)$ in (5) from (3), we obtain

$$N(k; r) = \sum_{a|r} \mu(a)\left[\sum_{s|r/a} \frac{(k-1)!}{(k-s)!} M(k-s; r/a)\right]. \tag{6}$$

The sum in (6) can be expressed in the equivalent form

$$N(k; r) = \sum_{s|r} \frac{(k-1)!}{(k-s)!} \left\{ \sum_{b|r}\left[\sum_{c|b} \mu(c)M(k-s; b/c)\right]\right\}, \tag{7}$$

where $[b, s] = r$.

(Proof of the equivalence of (7) and (6) is quite detailed; the details are omitted in this paper.) Substituting from (5) into (7), we obtain the result announced in the theorem.

References

1. S. Chowla, I. N. Herstein, and W. K. Moore, On recursions connected with symmetric groups I, Canad. J. Math., vol. 3, 1951, pp. 328–334.

2. S. Chowla, I. N. Herstein, and W. R. Scott, The solutions of $x^d = 1$ in symmetric groups, Norske Vid. Selsk. Forh., Trondheim, vol. 25, 1952, pp. 29–31.

3. R. B. Herrera, On the number of elements of a given period in finite symmetric groups, this MONTHLY, vol. 60, 1953, pp. 509–510.

4. E. Jacobsthal, Sur le nombre d'elements du groupe symétrique dont l'order est un nombre premier, Norske Vid. Selsk. Forh., Trondheim, vol. 21, 1948, pp. 49–51.

5. L. Moser and M. Wyman, On solutions of $x^d = 1$ in symmetric groups, Canad. J. Math., vol. 7, 1955, pp. 159–168.

A SIMPLER MATRIC APPROACH TO LINEAR DIFFERENTIAL EQUATIONS

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In a recent paper, Barrett and Wylie [2] discuss systems of linear differential equations by using a bit of matric theory. It seems desirable to observe that matric theory has much more to offer to such a discussion than is brought out in [2]. The purpose of this note is to remind our readers that the Smith canonical form of a matrix with polynomial elements [1, p. 92] is just the tool one needs to provide a complete discussion of the systems considered in [2].

Consider the equation $AX=F$, where A is a *rectangular* matrix of polynomials in the differential operator D with constant coefficients, while X and F are vectors of unknown and known functions of t, respectively. According to Smith, there are nonsingular matrices P and Q whose elements are polynomials in D and such that PAQ has nonzero entries only on its diagonal. Then, clearly, $PAQQ^{-1}X=PF$ is a set of linear differential equations for the components of the unknown vector $Y=Q^{-1}X$, each of which involves just one component of Y. If we assume that these equations can be solved, then we can solve $AX=F$ by computing $X=QY$.

There are many texts (for example, [3, p. 82]) which tell us how to *compute* P and Q, and it is clear that we need not compute P when $F=0$. It should be emphasized that this procedure applies equally well if the entries of A happen to be independent of D. In this case the procedure is closely related to the familiar one for solving a system of linear algebraic equations.

The examples cited in [2] are, of course, very easily handled by the procedure of Smith. For the one on page 475, we find that

$$PAQ = \begin{bmatrix} 1 & 0 \\ 0 & f(D) \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & g(D) \\ 0 & 1 \end{bmatrix}$$

with $f(x)=x^4+2x^2-8x+5$ and $g(x)=(1/8)(x^3-x^2-5x-3)$. If y_2 is the general solution of $f(D)y=0$, we obtain $x_1=g(D)y_2$, $x_2=y_2$ as the general solution of $AX=0$. In the example on page 478, we may take P and Q independent of D and reduce the example to the solution of

$$(D^2-4)y_1 = (1/6)(2f_1+f_2), \qquad (D^2-1)y_2 = f_1+2f_2.$$

We conclude with the observation that a complete discussion of systems of linear differential equations with constant coefficients cannot avoid the subject of invariant factors.

References

1. A. A. Albert, Introduction to Algebraic Theories, Chicago, 1941.

2. L. C. Barrett and C. R. Wylie, Jr., A simple matrix approach to linear differential equations, this MONTHLY, vol. 63, 1956, pp. 472–478.

3. R. R. Stoll, Linear Algebra and Matrix Theory, New York, 1952.

ON CONTINUOUS DIFFERENTIABILITY

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It is well known that the derivative of a function differentiable at each point of an interval may exhibit discontinuities at points of the interval. An example frequently cited is the derivative of the everywhere continuous function defined, when $x \neq 0$, by $f(x) = x^2 \sin x^{-1}$; here $f'(x)$ exists for all x but is discontinuous at $x = 0$. Not so well known, however, is the simple relationship between differentiability and continuous differentiability which is the object of this note. The author has never seen the theorem in print, and he is quite surprised not to have found any one who claims to have previously known it—although several individuals have produced almost immediate proofs upon hearing a statement of it.

THEOREM. *Let the derivative $f'(x)$ exist on an open interval I, of which $a \leqq x \leqq b$ is a closed subinterval. A necessary and sufficient condition that $f'(x)$ be continuous on $a \leqq x \leqq b$ is the "uniform differentiability" of $f(x)$ on $a \leqq x \leqq b$—namely, that for each $\epsilon > 0$ there exist an x-independent $\delta > 0$ such that*

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon \tag{1}$$

whenever $0 < |h| < \delta (a \leqq x \leqq b,\ x+h \in I)$.

Proof of sufficiency. By hypothesis, if $\{h_i\}$ is any sequence of nonzero numbers (with $x+h_i \in I$ for all x on $a \leqq x \leqq b$) that converges to zero, then the sequence of *continuous* functions defined by $\{[f(x+h_i) - f(x)]/h_i\}$ converges *uniformly* to $f'(x)$ on $a \leqq x \leqq b$. The limit $f'(x)$ is therefore continuous on $a \leqq x \leqq b$.

Proof of necessity. The mean-value theorem provides:

$$\frac{f(x+h) - f(x)}{h} - f'(x) = f'(x+\theta h) - f'(x) \qquad (0 < \theta < 1) \tag{2}$$

for each x on $a \leqq x \leqq b$ and $x+h \in I$. Since $f'(x)$ is by hypothesis continuous, and therefore uniformly continuous, on $a \leqq x \leqq b$, there exists for each $\epsilon > 0$ an x-independent $\delta > 0$ such that

$$|f'(x+\theta h) - f'(x)| < \epsilon \tag{3}$$

whenever $0 < |\theta h| < \delta$ and, therefore, whenever $|h| < \delta$. The required uniformity of differentiability (1) follows from (2) and (3).

ON TAYLOR'S THEOREM WITH REMAINDER

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1. Introduction and summary. In this note I prove Taylor's theorem, with the Schlömilch-Roche form of the remainder, in the following form:

THEOREM A. *If*

(1) $f(x)$ *is continuous in* $a \leqq x \leqq b(=a+h)$,

(2) $f^{(n-1)}(x)$ *is continuous in* $a \leqq x < b$,

(3) $f^{(n)}(x)$ *exists in* $a < x < b$,

then, for each $\nu < n$,

$$f(b) = f(a) + hf'(a) + (h^2/2!)f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^\nu}{(n-1)!(n-\nu)}f^{(n)}(a+\theta h),$$

for some θ *such that* $0<\theta<1$.

Ordinarily this theorem is proved under the more restrictive assumptions obtained by replacing the interval $a \leqq x < b$ in (2) by the closed interval $a \leqq x \leqq b$. Stolz (see [1]) proved the theorem in the form stated above, but, as W. H. Young [3] and Hobson [1] have noted, Stolz's proof breaks down whenever ν is a positive integer. His proof made use of Cauchy's formula. The present proof, based on the usual proof of the theorem in its ordinary form, makes use of an extension of Rolle's theorem and one of l'Hospital's rules.

2. Preliminaries. The following forms of Rolle's theorem (see [1]) and of l'Hospital's rule [2] are used in proving Theorem A.

THEOREM 1. *If*

(1) $F(x)$ *is continuous in* $a \leqq x < b$,

(2) $F(b-0)$ *either does not exist uniquely or exists uniquely and equals* $F(a)$,

(3) $F'(x)$ *exists in* $a < x < b$,

then $F'(\xi)=0$ *for some* ξ *such that* $a<\xi<b$.

THEOREM 2. *If*

(1) $g(x)$, $G(x)$ *are continuous in* $a<x<b$,

(2) $g'(x)$, $G'(x)$ *exist finitely in* $a<x<b$,

(3) $G(b-0)$ *exists uniquely and is infinite*,

(4) $g'(x)/G'(x) \to l$ *(finite or infinite) as* $x \to b-0$,

then $g(x)/G(x) \to l$ *as* $x \to b-0$.

3. Proof of Theorem A. For $a \leqq x < b$, let

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - \frac{(b-x)^{n-\nu}}{(n-1)!(n-\nu)}K,$$

where K is a constant so chosen as to make $F(a)=0$. Now $F(b-0)$ either exists uniquely or does not so exist. The conditions of Theorem 1 are therefore satisfied if the following lemma holds.

LEMMA 1. *If $F(b-0)$ exists uniquely, then $F(b-0)=0$.*

Assuming the truth of this lemma we can now apply Theorem 1 to obtain that $F'(a+\theta h)=0$ for some θ such that $0<\theta<1$. Since, for $a<x<b$,

$$F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) + \frac{(b-x)^{n-\nu-1}}{(n-1)!}K,$$

Theorem A then follows.

Remark. $F(b)=0$ under the conditions ordinarily assumed in the proof of Taylor's theorem. Complications arise in our proof since nothing is known about $f'(b-0), f''(b-0), \cdots$.

We now proceed to the proof of Lemma 1. This we prove by repeated application of

LEMMA 2. *For $k=0, \cdots, n-1$, let*

$$\phi_k(x) \equiv a_{0,k}f(x) + a_{1,k}(b-x)f'(x) + \cdots + a_{k,k}(b-x)^k f^{(k)}(x).$$

If, for some $r(0<r<n)$ and for some finite constants $a_{0,r}, \cdots, a_{r,r}$, $\phi_r(b-0)$ exists uniquely (finite or infinite) and $\phi_r(b-0)\neq a_{0,r}f(b)$, then finite constants $a_{0,r-1}, \cdots, a_{r-1,r-1}$ exist such that $\phi_{r-1}(b-0)$ exists uniquely and is infinite.

Proof. It can be verified that $(b-x)\phi'_{r-1}(x)=\phi_r(x)-a_{0,r}f(x)$ if

$$\begin{aligned} a_{r-1,r-1} &= a_{r,r}, \\ a_{r-2,r-1} &= a_{r-1,r} + (r-1)a_{r-1,r-1}, \\ &\cdots\cdots\cdots\cdots \\ a_{0,r-1} &= a_{1,r} + a_{1,r-1}. \end{aligned}$$

These equations define finite constants $a_{r-1,r-1}, \cdots, a_{0,r-1}$ uniquely in terms of $a_{r,r}, \cdots, a_{1,r}$. With $\phi_{r-1}(x)$ determined as above we now set $g(x)=\phi_{r-1}(x)$, $G(x)=-\log(b-x)$, $l=\phi_r(b-0)-a_{0,r}f(b)$. Then, noting that $g'(x)/G'(x) =(b-x)\phi_{r-1}(x)=\phi_r(x)-a_{0,r}f(x)\to l$ as $x\to b-0$, we obtain, by Theorem 2, that

$$\frac{g(x)}{G(x)} = \frac{\phi_{r-1}(x)}{-\log(b-x)} \to l \qquad (x\to b-0).$$

Since $l\neq 0$ and $-\log(b-x)\to\infty$ as $x\to b-0$, it follows from the above that $\phi_{r-1}(b-0)$ exists uniquely and is infinite. We thus have Lemma 2.

To prove Lemma 1 we assume that $F(b-0)$, which exists uniquely, is not zero and obtain a contradiction. Our assumption gives us that $\phi_{n-1}(b-0) =f(b)-F(b-0)\neq f(b)$, on setting $a_{s,n-1}=1/s!$ $(s=0, \cdots, n-1)$. Repeated application of Lemma 2 (with $r=n-1$, $r=n-2, \cdots, r=1$) now gives us that the functions $\phi_{n-2}(x), \phi_{n-3}(x), \cdots, \phi_0(x)$ defined by Lemma 2 are such that each of them has a unique infinite limit as $x\to b-0$.

But now, $\phi_0(x)=a_{0,0}f(x)$, so that $\phi_0(b-0)=a_{0,0}f(b)$, which is finite. We thus

have a contradiction and this shows that $F(b-0)$, if it exists uniquely, is zero. This completes the proof of Lemma 1 and thus of Theorem A.

References

1. E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, vol. 2, Cambridge, 1926, pp. 200–204.
2. A. E. Taylor, L'Hospital's rule, this MONTHLY, vol. 59, 1952, pp. 20–24.
3. W. H. Young, Note on a remainder form of Taylor's theorem, Quart. J. Math., vol. 40, 1909, pp. 146–153.

A DETERMINANT INEQUALITY FOR UNIVALENT FUNCTIONS

S. D. BERNARDI, New York University

Let $w=f(z)=z+a_2z^2+a_3z^3+\cdots\in(S)$ denote the class of functions regular and univalent for $|z|<1$. Form

$$F(z,\alpha)=\left[z^{-1}f(z)\right]^{-\alpha/2}=1+b_1(\alpha)z+b_2(\alpha)z^2+\cdots, \tag{1}$$

having the coefficient relation

$$nb_n(\alpha)+\sum_{i=1}^{n}(\alpha i/2+n-i)a_{i+1}b_{n-i}(\alpha)=0;\qquad n=1,2,\cdots,b_0(\alpha)=1. \tag{1a}$$

Then, by Prawitz [1, p. 273] it follows that

$$\sum_{n=1}^{\infty}\{(2n-\alpha)/\alpha\}\,|\,b_n(\alpha)\,|^2\leqq 1,\ \text{for all real }\alpha. \tag{2}$$

The inequality (2) which is a generalization of the so-called "area principle" has been used [1, p. 268] a great deal in the classical analysis of the theory of univalent (schlicht) functions. Numerical upper bounds for certain coefficients $b_n(\alpha)$ for special values of α have been estimated in order to find upper bounds for certain coefficients [1, p. 274] a_n of $f(z)$. Our results are summarized in the following theorem.

THEOREM. *Let $f(z)\in(S)$. Let $F(z,\alpha)$ be defined as in* (1). *Let*

$$z=f^{-1}(w)=w+\sum_{n=2}^{\infty}(c_{n-1}/n)w^n \tag{3}$$

where $f^{-1}(w)$ denotes the function inverse to f. Then

(a) $c_{n-1}=b_{n-1}(2n),\qquad n=2,3,\cdots$

(b) $\left|\dfrac{b_{n-1}(2n)}{n}\right|\leqq H(n)=\dfrac{1\cdot3\cdot5\cdots(2n-1)}{1\cdot2\cdot3\cdots(n+1)}2^n,\qquad n=2,3,\cdots$

(c) $\sum_{n=3}^{\infty}(n-2)\,|\,b_n(4)\,|^2\leqq 18.$

Proof. Applying Lagrange's formula [3, p. 125] for the reversion of series, we have

$$z = f^{-1}(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)}\right)^n\right]_{z=0}.$$

A formal computation yields

$$\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)}\right)^n\right]_{z=0} = (n-1)!b_{n-1}(2n), \qquad n = 1, 2, \cdots; b_0(2) = 1.$$

We thus obtain

$$z = f^{-1}(w) = \sum_{n=1}^{\infty} (1/n)b_{n-1}(2n)w^n. \tag{4}$$

Comparing (3) and (4) we get

$$c_{n-1} = b_{n-1}(2n); \qquad n = 1, 2, \cdots. \tag{5}$$

It has been shown by Löwner [1, p. 286] that the coefficients of the inverse function $z=f^{-1}(w)$ of a univalent function $f(z)$ are majorized by the coefficients of the inverse function $K^{-1}(w)$ of the Koebe function $w=K(z)=z/(1-z)^2$. Since

$$\begin{aligned} z = K^{-1}(w) &= w + \sum_{n=2}^{\infty} (-1)^{n-1}\frac{(2n)(2n-1)(2n-2)\cdots(n+2)}{n!}w^n \\ &= w + \sum_{n=2}^{\infty}(-1)^{n-1}H(n), \end{aligned} \tag{6}$$

it follows from (3) and (6) that $|c_{n-1}/n| \leqq H(n)$. Substituting from (5) we obtain (b) of the theorem. Letting $\alpha=4$ in (2) and using $|b_1(4)| = |-2a_2| \leqq 2H(2)=4$ yields (c) of the theorem, and also shows that the inequality (b) is sharp since $|a_2| \leqq 2$ is a well known sharp result. This completes the proof of the theorem.

If we let $\alpha=2k$, the coefficient relation in equation (1a) becomes $nb_n(\alpha) + \sum_{i=1}^{n} (ik+n-i)a_{i+1}b_{n-i}(\alpha)=0$; $n=1, 2, \cdots, b_0(\alpha)=1$. In determinant notation this is equivalent to

$$\frac{k}{n!}\begin{vmatrix} a_2 & 1 & 0 & 0 & \cdots & 0 \\ 2a_3 & (k+1)a_2 & 2 & 0 & \cdots & 0 \\ 3a_4 & (2k+1)a_3 & (k+2)a_2 & 3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ na_{n+1} & ((n-1)k+1)a_n & ((n-2)k+2)a_{n-1} & ((n-3)k+3)a_{n-2} & \cdots & (k+n-1)a_2 \end{vmatrix}$$
$$=(-1)^n b_n(2k)=\Delta.$$

Thus, the inequality relation (b) of the theorem offers certain upper bounds for the determinant Δ. The special case of the determinant Δ obtained by taking

$k=1$ and restricting the coefficients a_n to be rational integers is considered in [4]. Another special case of Δ obtained by letting $k=1$ and restricting the coefficients a_n to be integers in an imaginary quadratic field is considered in [2]. It seems reasonable to suppose that the inequality (b) acting upon Δ plus the imposition of special properties on the coefficients a_n should be a source of many special results.

References

1. S. D. Bernardi, A survey of the development of the theory of schlicht functions, Duke Math. J., vol. 19, 1952, pp. 253–287.
2. S. D. Bernardi, Note on an inequality of Prawitz, Duke Math. J., vol. 23, 1956, pp. 385–392.
3. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford, 1935.
4. Viktors Linis, Note on univalent functions, this MONTHLY, vol. 62, 1955, pp. 109–110.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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TRANSFORMATION OF COORDINATES

C. S. OGILVY, Hamilton College

In a note, *Finding the cartesian equation of a locus*, this MONTHLY, vol. 63, pp. 661–663, the cartesian equation of $r=\sin(\theta/3)$ is found by a new method involving a certain amount of laborious reduction. The remark is made that the "example would be extremely difficult to carry out without the aid of a special device." Actually the cartesian equation can be found by the following simple procedure.

In the standard formula $\sin 3\phi = 3\sin\phi - 4\sin^3\phi$, write $\phi=\theta/3$. Then $\sin\phi = \sin(\theta/3) = r$, yielding $\sin\theta = 3r-4r^3$. Multiplying through by r, $y = 3(x^2+y^2) - 4(x^2+y^2)^2$.

A METHOD FOR SOLVING $\phi(x)=n$

L. L. PENNISI, University of Illinois, Chicago

This note reports a method for finding all the values of x, when given an integer n, such that

$$\phi(x) = n, \tag{1}$$

where $\phi(x)$ is the Euler ϕ-function of x.

Let

$$d_1, d_2, \cdots \tag{2}$$

be all the divisors of n satisfying the condition that neither d_i nor n/d_i is equal to $2k+1$, $k>0$.

For each d_i, let

$$E_{i,1},\ E_{i,2},\ \cdots \tag{3}$$

denote all the solutions of the equation $\phi(y)=d_i$ having the form p^e, where p is a prime number.

Remark. Such solutions need not necessarily exist. For example, $\phi(y)=50$ has no solutions of the form p^e. Also $E_{i,j}/d_i=p^e/p^{e-1}(p-1)=p/(p-1)\leqq 2$. Therefore, $E_{i,j}\leqq 2d_i$, that is, the solutions of $\phi(y)=d_i$ of the form p^e cannot be greater than $2d_i$.

Let

$$n = d_{i_1}\cdot d_{i_2}\ \cdots$$

be a factorization of n whose factors satisfy the condition (2). Also, let

$$E_{i_1,j_1},\ E_{i_2,j_2},\ \cdots$$

be relatively prime, where the E's are as in (3).

Then

$$x = E_{i_1,j_1}\cdot E_{i_2,j_2}\ \cdots \tag{4}$$

is a solution of equation (1).

For, since the E's are relatively prime, we have

$$\phi(x) = \phi(E_{i_1,j_1})\cdot\phi(E_{i_2,j_2})\ \cdots = d_{i_1}\cdot d_{i_2}\ \cdots = n.$$

Every solution of equation (1) *has the form given in* (4). For, suppose that x_0 is a solution of $\phi(x)=n$. Then, factoring x_0 into prime powers we get $x_0=p_1^{e_1}\ p_2^{e_2}\cdots$. Hence $n=\phi(x_0)=\phi(p_1^{e_1})\cdot\phi(p_2^{e_2})\cdots$. Consequently, $\phi(p_1^{e_1})$, $\phi(p_2^{e_2}),\cdots$, are divisors of n and neither $\phi(p_i^{e_i})$ nor $n/\phi(p_i^{e_i})$ has the form $2k+1$, $k>0$. Hence, $\phi(p_1^{e_1})=d_{i_1}$, $\phi(p_2^{e_2})=d_{i_2},\cdots$, where the d's are as described in (2). Therefore, $p_1^{e_1}=E_{i_1,j_1}$, $p_2^{e_2}=E_{i_2,j_2},\cdots$, where the E's are as described in (3), and are relatively prime. Therefore $x_0=E_{i_1,j_1}\cdot E_{i_2,j_2}\cdots$.

EXAMPLE. Solve $\phi(x)=12$.

Using (2), we find $d_1=1$, $d_2=2$, $d_3=6$, $d_4=12$. Using (3), we have:

$\phi(y) = d_1 = 1$ implies that $y = 1$, $y = 2$. Take $E_{1,1} = 1$, $E_{1,2} = 2$.

$\phi(y) = d_2 = 2$ implies that $y = 3$, $y = 2^2$. Take $E_{2,1} = 3$, $E_{2,2} = 4$.

$\phi(y) = d_3 = 6$ implies that $y = 7$, $y = 3^2$. Take $E_{3,1} = 7$, $E_{3,2} = 9$.

$\phi(y) = d_4 = 12$ implies that $y = 13$. Take $E_{4,1} = 13$.

Using (4), we find all the values of x to be:

$$x = E_{1,1} \cdot E_{4,1} = 1 \cdot 13 = 13.$$
$$x = E_{1,2} \cdot E_{4,1} = 2 \cdot 13 = 26.$$
$$x = E_{2,1} \cdot E_{3,1} = 3 \cdot 7 = 21.$$
$$x = E_{2,2} \cdot E_{3,1} = 4 \cdot 7 = 28.$$
$$x = E_{2,2} \cdot E_{3,2} = 4 \cdot 9 = 36.$$
$$x = E_{1,2} \cdot E_{2,1} \cdot E_{3,1} = 2 \cdot 3 \cdot 7 = 42.$$

A NEW PROOF THAT NO PERMUTATION IS BOTH EVEN AND ODD

J. L. BRENNER, Stanford Research Institute

This note contains an informal proof that no permutation (on a finite set) is both even and odd. The parallel formal (abstract) proof would begin with postulates which "permutation" must satisfy, and would derive the same conclusion from these postulates. We know one other informal proof which can be made formal in this sense [**1**, p. 36]; the postulates which would be needed to make that proof abstract seem to us to be less simple. A third proof [**3**, p. 8] is very short, but not abstract; in this third proof, the objects on which the permutations act play an essential role in the argument. In the first two proofs this is not the case; the objects on which the permutations act do not even appear in the formal versions of the proofs.

To carry through the proof, we need to use the following facts: (1) Every permutation can be written as a product of transpositions; (2) mutually disjoint transpositions commute; (3) a transposition is its own inverse; (4) if a_r, a_s are distinct, then $(a_s, a_t)(a_r, a_t) = (a_r, a_t)(a_s, a_r)$; (5) if all the symbols $a_t, a_{t+1}, \cdots, a_{r-1}, a_r$ $(r > t)$ are distinct, then the first and last transpositions in the product

$$(a_t, a_{t+1})(a_{t+1}, a_{t+2}) \cdots (a_{r-1}, a_r)(a_r, a_t)e$$

can be omitted. Now the informal proof.

No permutation on a finite set can be expressed as a product of an even, and also as a product of an odd number of transpositions. Thus, by (1), *every permutation can be classified as either even or odd.*

Proof. Assume that the permutation P can be written in two ways, first as a product of an even number, and then as a product of an odd number of transpositions. We equate these products, use (3), and have a representation of the identity e as a product of an odd number $2k+1$ of transpositions. We show that if such a representation exists, there must be another such representation in which the number of factors is two fewer; this is clearly a contradiction,* since e is not equal to a single transposition.

Of all the ways in which the identity can be written as a product of $2k+1$

* This is a contradiction since if there were a representation, there would be one in which the number of factors is a minimum.

	Sin	Cos	Tan	Cot	Csc	Sec
Sin^{-1}	x	$\frac{\pi}{2}-x$	*	*	—	—
Cos^{-1}	$\frac{\pi}{2}-x$	x	*	*	—	—
Tan^{-1}	*	*	x	$\frac{\pi}{2}-x$	*	*
Cot^{-1}	*	*	$\frac{\pi}{2}-x$	x	*	*
Csc^{-1}	—	—	*	*	x	$\frac{\pi}{2}-x$
Sec^{-1}	—	—	*	*	$\frac{\pi}{2}-x$	x

and then looking up the value of the angle whose sine is the given value of Tan x. Geometrically we could construct the unit circle; draw the tangent at the point (1, 0); draw the angle x and let the terminal side intersect the tangent; let the line $y = \text{Tan } x$ intersect the circle and connect the point with the origin. This line is the terminal side of the angle Sin^{-1} (Tan x). The situations indicated by asterisks in the second table are similar. In these cases, functions of x exist over parts of the interval $0 \leqq x \leqq \pi/2$.

SPEAK UP

The excellent *Suggestions to students on talking about mathematics*, this MONTHLY, vol. 64, 1957, pp. 16–18, omits one of the most important suggestions that should be made: SPEAK UP!

So many speakers, notable men whose talks have been eagerly expected, fail to deliver, because they speak in a conversational tone, drop their voices on points of especial interest, and talk to the blackboard instead of to their hearers.

I once saw a distinguished man deliberately turn away from the microphone furnished and let his moderate voice die out over the first few rows.

Another began by saying he would divide his subject into two parts; he drew a vertical line on the blackboard, added nothing more, but kept his eye upon it and talked to it throughout his lecture.

These bad habits seem to be on the increase among learned people. At recent scientific meetings that I have attended, not one speaker in five has taken enough trouble to make his material audible to his whole audience.

W. R. RANSOM
Tufts University

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1275 [1957, 432]. *Corrected*

Solve for x:

$$\int_0^x s^{8/3}(1-s)^{4/3}ds = \int_0^1 t^{8/3}(1+t)^{-6}dt.$$

E 1276. *Proposed by C. S. Ogilvy, Hamilton College*

Find the greatest right circular cylinder coaxal with and inscribed in the solid formed by rotating around the y-axis the area bounded by the two axes, the parabola $y=9x^2-28x+24$, and the parabola's minimum ordinate.

E 1277. *Proposed by Alexander Oppenheim, University of Malaya, Singapore, Malaya*

For each $p>0$ there is a greatest q and a least r such that

$$\frac{q \sin x}{1+p\cos x} \leqq x \leqq \frac{r \sin x}{1+p\cos x}$$

for $0 \leqq x \leqq \pi/2$. Determine q and r as functions of p.

E 1278. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

If a hexagon $ABCDEF$ is inscribed in a conic, and if P is any point either on the conic or on the Pascal line of the hexagon, then

$$(\sin APB \sin CPD \sin EPF)/(\sin BPC \sin DPE \sin FPA) = \text{constant}.$$

E 1279. *Proposed by L. R. Ford, Jr., and D. R. Fulkerson, The RAND Corporation, Santa Monica, Calif.*

In the February 1957 issue of *Scientific American* the following problem appeared: "A carpenter, working with a buzz saw, wishes to cut a wooden cube, three inches on a side, into 27 one-inch cubes. He can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape. Can he reduce the number of necessary cuts by rearranging the pieces after each cut?" Generalize this to find the number of cuts necessary to dissect an $n\times n\times n$ cube into n^3 one-inch cubes.

E 1280. *Proposed by V. K. Narayanan, Puthenchanthai, Trivandrum, South India*

If $x=24$, then $x+1$ and $2x+1$ are perfect squares; if $x=40$, then $2x+1$ and $3x+1$ are perfect squares; if $x=8$, then $x+1$ and $3x+1$ are perfect squares. Is there a positive integer x such that all three $x+1$, $2x+1$, $3x+1$ are perfect squares?

SOLUTIONS

Cones and Trihedral Angles

E 1246 [1957, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relation between the radius of the base and the altitude of a right circular cone in which a trihedral angle can be inscribed whose face angles are all equal to a given angle 2α.

Show that if two trihedral angles whose face angles are all equal to 2α and $2\alpha'$, respectively, are inscribed in two right circular cones having a common base, then a necessary and sufficient condition for the radius of the common base to be a mean proportional between the altitudes of the cone is that

$$\sin^2\alpha + \sin^2\alpha' = 3/4.$$

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Let r and h denote the radius of the base and altitude, respectively, of the right circular cone. The specified trihedral angle and base of the cone form a regular pyramid whose lateral edge is $(r^2+h^2)^{1/2}$ and whose base has sides of $\sqrt{3}r/2$. The bisector of a face angle forms a right triangle in which

$$\sin^2\alpha = 3r^2/4(r^2+h^2),$$

the required relationship.

If $\sin^2\alpha+\sin^2\alpha'=3/4$, then

$$r^2/(r^2+h^2) + r^2/(r^2+h'^2) = 1,$$

which leads easily to $r=(hh')^{1/2}$. The converse follows just as readily.

Also solved by Leon Bankoff, A. L. Epstein, Hazel Evans, P. L. Falb, Michael Goldberg, Cornelius Groenewoud, Vern Hoggatt, A. R. Hyde, I. M. Isaacs, J. D. E. Konhauser, Andrew Kraus, Josef Langr, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy, D. S. Passman, P. A. Penzo, P. W. A. Raine, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, Chih-yi Wang, R. H. Wilson, Jr., and the proposer.

The relationship of the first part of the problem may be put into the form

$$h/r = [(3/4)\csc^2\alpha - 1]^{1/2}.$$

An instance of the second part of the problem is $\alpha=45°$ and $\alpha'=30°$.

An Identity

E 1247 [1957, 42]. *Proposed by C. W. Topp, Fenn College*

Prove that

$$\sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\binom{kn}{n} = (-1)^{n+1}n^n.$$

Solution by J. E. Darraugh and F. D. Parker, Clarkson College. Since

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^i b^{n-1-i},$$

we have

$$[1-(x+1)^n]^n = (-x)^n\left[\sum_{i=0}^{n-1}(x+1)^{n-1-i}\right]^n.$$

But

$$[1-(x+1)^n]^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(x+1)^{ni}.$$

Since the coefficients of x^n in the two expansions are equal, we obtain

$$(-1)^n n^n = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{kn}{n},$$

from which the desired result immediately follows.

Also solved by W. A. Al-Salam, Leonard Carlitz, C. C. Conley, M. P. Drazin, N. J. Fine, Virginia S. Hanly, J. H. Hodges, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, L. A. Ringenberg, D. A. Robinson, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by J. L. Brown, Jr.

Several solvers obtained the desired result by setting $a=0$ and $b=n$ in the more general identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{n} = (-1)^n b^n, \qquad n \geqq 0.$$

This identity may be found, established by finite difference methods, in H. W. Gould, *Some generalizations of Vandermonde's convolution*, this MONTHLY [1956, 84–91]; an alternative demonstration occurs in I. J. Schwatt, *An Introduction to the Operations with Series*, Univ. of Pa. Press, 1924, p. 104.

In Sec. 48 of Jordan, *Calculus of Finite Differences*, occurs the probability problem, "If an urn contains m balls numbered 1, 2, $\cdots$, m, and if n balls are drawn with replacement, what is the probability that the sum of the numbers drawn shall be less than x?" In connection with the case $x=nm+1$ appears the identity

$$(-m)^n = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{km}{n},$$

which, for $m=n$, reduces to the result of the given problem.

Two Linear Simultaneous Systems

E 1248 [1957, 42]. *Proposed by Leo Moser, University of Alberta*

(a) The ten numbers $s_1 \leqq s_2 \leqq \cdots \leqq s_{10}$ are the sums of the five unknown numbers $x_1 \leqq x_2 \leqq \cdots \leqq x_5$ taken two at a time. Determine the x's in terms of the s's.

(b) Show that if $s_1 < s_2 < \cdots < s_6$ are six distinct numbers formed by taking the sums of four numbers two at a time, then there exist four other numbers which give the same sums when added in pairs.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. (a) Obviously $s_1 = x_1 + x_2$, $s_2 = x_1 + x_3$, $s_9 = x_3 + x_5$, $s_{10} = x_4 + x_5$. Also $\Sigma s_i = 4\Sigma x_i$. These conditions suffice to give the solution

$$x_1 = s_1 + s_2 + s_{10} - (\Sigma s_i)/4,$$

$$x_2 = (\Sigma s_i)/4 - s_2 - s_{10},$$

$$x_3 = (\Sigma s_i)/4 - s_1 - s_{10},$$

$$x_4 = (\Sigma s_i)/4 - s_1 - s_9,$$

$$x_5 = s_1 + s_9 + s_{10} - (\Sigma s_i)/4.$$

(b) If one set is $\{x_i\}$, then another set is $\{S - x_i\}$, where $S = (\Sigma x_i)/2$. These will be distinct because $s_3 \neq s_4$ implies $S - x_4 \neq x_1$, where $S - x_4$ and x_1 are the least of their respective sets.

Also solved by J. L. Baker, Underwood Dudley, A. L. Epstein, N. J. Fine, Michael Goldberg, A. R. Hyde, V. F. Ivanoff, P. W. M. John, Joe Lipman, R. L. London, D. C. B. Marsh, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, E. D. Schell, and Gustavus Simmons. Late solution by A. S. Gregory.

Part (a) of the problem, with a solution, may be found in the Dec. 1955 issue of the *Newsletter* of the Canadian Mathematical Congress.

Cubes as Sums of Cubes

E 1249 [1957, 43]. *Proposed by C. M. Sandwick, Sr., Easton High School, Easton, Pa.*

Find an integer less than 1000, the cube of which may be represented as the sum of the cubes of three positive integers in five distinct ways.

Summary of the findings of Leon Bankoff, Los Angeles, Calif. Bankoff used the methods of Vieta, Euler, and Ramanujan to generate 216 primitive solutions of $a^3 + b^3 + c^3 = d^3$ ($d < 1000$). From these he selected 74 suitable for forming imprimitives that could be collected into sets of five or more equal sums of 3 cubes. He ultimately found 47 integers that meet the requirements. The most striking example is 870^3, which can be expressed as the sum of 3 cubes in nine distinct ways:

$$\begin{aligned} 870^3 &= 17^3 + 687^3 + 694^3 = 537^3 + 564^3 + 687^3 = 235^3 + 485^3 + 810^3 \\ &= 200^3 + 540^3 + 790^3 = 260^3 + 550^3 + 780^3 = 380^3 + 480^3 + 790^3 \\ &= 225^3 + 630^3 + 735^3 = 330^3 + 450^3 + 810^3 = 435^3 + 580^3 + 725^3. \end{aligned}$$

The other 46 integers are here grouped with regard to the number of ways their cubes can be expressed as the sum of 3 positive cubes:

8 ways: 348, 420, 522, 540, 630, 696, 720, 738, 756, 840, 900, 990.

7 ways: 174, 360, 450, 494, 792, 984.

6 ways: 246, 270, 396, 432, 504, 516, 580, 648, 810, 812, 864.

5 ways: 180, 216, 252, 261, 324, 369, 378, 600, 609, 618, 684, 690, 783, 820, 828, 972, 996.

Particular solutions were found also by M. Barnebey, W. J. Blundon, Michael Goldberg, John Leech, D. C. B. Marsh, Alan Wayne, and the proposer. Late solution by D. M. Brown.

Some allied references are: Heath, *Diophantus of Alexandria* (2nd ed.), p. 329 *et seq.*; Hardy and Wright, *An Introduction to the Theory of Numbers* (3rd ed.), p. 199 *et seq.*; Davenport, *The Higher Arithmetic*, p. 163; *Trans. Camb. Phil. Soc.* 22 (1920) 402; Dickson, *History of the Theory of Numbers*, vol. II, pp. 552–554; Dickson, *Introduction to the Theory of Numbers*, art. 36.

Centers of Similitude

E 1250 [1957, 43]. *Proposed by N. A. Court, University of Oklahoma*

Through a point G two secants GAD, GBC are drawn meeting a given circle (H) in the points A, D; B, C. Show that the points $E=(AB, CD)$, $F=(AC, BD)$ are the centers of similitude of the two circles orthogonal to (H) and having for centers the harmonic conjugates of G for the pairs of points A, D; B, C, respectively.

Solution by the proposer. The diagonal triangle EFG of the complete quadrangle $(Q)=ABCD$ inscribed in (H) is polar for (H) [Cremona, *Projective Geometry*, art. 260], and therefore EF is the polar of G for (H). Thus the points $I=(AD, EF)$, $J=(BC, EF)$ are separated harmonically from G by the pairs of points A, D; B, C, respectively, and the vertices E, F are conjugate with respect to circle (H).

Let (I), (J) be the circles drawn with I, J as centers orthogonal to (H). The circle (EF) having E, F for diametrically opposite points is also orthogonal to (H) [proposer's *College Geometry*, 2nd ed., art. 387]. We thus have three circles, (I), (J), (EF), orthogonal to the same circle (H) and having their centers on the same line $EIFH$, whence the three circles are coaxal [*ibid.*, art. 458]. Moreover, from quadrangle (Q) it results that the pairs of points I, J; E, F are harmonic, whence (EF) is the circle of similitude of circles (I), (J) [*ibid.*, art. 481], and therefore E, F are their centers of similitude.

Also solved by D. C. B. Marsh (analytically) and Beckham Martin (synthetically). Late solution by Victor Thébault.

Martin showed that if circles are drawn with centers at I and J and cutting (H) under angles α and β, respectively, then E, F will be the centers of similitude of these two circles if and only if $(\cos\alpha)/(\cos\beta)=IE/EJ$. It now follows that if $\alpha=\pi/2$, we must also have $\beta=\pi/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4748. *Proposed by Alexander Oppenheim, University of Malaya*

Suppose that x and y are defined by the two series

$$x = \frac{1}{a_1} + \frac{1}{a_1^2}\frac{1}{a_2} + \frac{1}{a_1^2 a_2^2}\frac{1}{a_3} + \cdots,$$

$$y = \frac{1}{a_1} + \frac{1}{a_1^3}\frac{1}{a_2} + \frac{1}{a_1^3 a_2^3}\frac{1}{a_3} + \cdots,$$

where the a_i are integers such that $a_i \geqq 2$ $(i=1, 2, \cdots)$. Prove that x and y are both rational or both irrational.

4749. *Proposed by Joseph Lehner, Los Alamos Scientific Laboratory*

Is it true that for every set E of positive measure contained in the interval $(0, 1)$, the numbers C_n have the property $|nC_n| < M$, $n=1, 2, \cdots$, where

$$C_n = \int_E e^{2\pi i n x} dx,$$

and M is a constant depending only on E?

4750. *Proposed by Leonard Carlitz, Duke University*

Find the denominator of

$$\binom{i}{m} = \frac{i(i-1)\cdots(i-m+1)}{m!}$$

where $i = \sqrt{-1}$ and the fraction is reduced to lowest terms in the field $R(i)$.

4751. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the Feuerbach hyperbola of a triangle ABC, with orthocenter H and incenter I, is tangent to the circle ABC, then it is also tangent to the circles BCH, CAH, ABH, and its focal axis is the perpendicular bisector of IH. (The Feuerbach hyperbola is the equilateral hyperbola determined by the four points, A, B, C, and I.)

4752. *Proposed by M. S. Klamkin, AVCO, Research and Development, Lawrence, Mass.*

Determine a set of n distinct, nonzero terms such that their geometric mean is the geometric mean of their arithmetic and harmonic means.

SOLUTIONS

Divergent Integral and Series

4670 [1956, 47, 190]. *Proposed by K. L. Chung, Syracuse University*

If $f(x)$ is continuous and nonnegative in $[0, \infty)$, and $\int_0^\infty f(x)dx = \infty$, then there exists an $h>0$ such that $\sum_{n=1}^\infty f(nh) = \infty$.

Editorial Note. As pointed out by the proposer, solution I [1957, 119] is incorrect because h there depends upon b and cannot be regarded as fixed while $b \to \infty$.

Minimal Weakly Prime Ideal

4691 [1956, 347]. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring R is any ideal I having the property that either $aR \subset I$ or $Ra \subset I$ implies that a is in I. Give an elementwise characterization of the unique minimal weakly prime ideal of R.

II. *Solution by the proposer.* In a former solution [1957, 375] it is claimed that a certain set $W(I)$ is weakly prime and that $aR \subset W(I)$ implies $a \in W(I)$. This is not so. From $aR \subset W(I)$ one can conclude that $R^n a r R^m \subset I$, where the integers m and n depend on r. It can happen that no m and n work for all $r \in R$. A valid solution may be obtained as follows.

If $\{x\} = \{x_1, x_2, \cdots, x_n, \cdots\}$ is a sequence of elements in R, let

$$x^n = x_1 \cdot x_2 \cdot \cdots \cdot x_n, \qquad {}^n x = x_n \cdot \cdots \cdot x_2 \cdot x_1.$$

An element a in R is called *strongly nilpotent* if for every pair $(\{x\}, \{y\})$ of sequences in R there exist integers m and n such that ${}^m x a y^n = 0$. We prove the

THEOREM. *The unique minimal weakly prime ideal of R is the set S of all strongly nilpotent elements of R.*

We first prove the following lemma.

LEMMA. An element a of R is in S if and only if $aR \subset S (Ra \subset S)$.

Proof. If a is in S and r is in R, and if $(\{x\}, \{y\})$ is a pair of sequences in R, then let $\{z\} = \{r, y_1, \cdots, y_n, \cdots\}$. By definition of S, there exist integers m and n such that ${}^m x a z^n = 0$. Hence, also, ${}^m x(ar)y^n = 0$. We conclude that ar is in S for every r in R.

Conversely, if ar is in S for every r in R, and if $(\{x\}, \{y\})$ is a pair of sequences in R, then let $\{z\} = \{y_2, y_3, \cdots, y_n, \cdots\}$. By definition of S, there exist integers m and n such that ${}^m x(ay_1)z^n = 0$. Hence ${}^m x a y^{n+1} = 0$ and we conclude that a is in S.

Proof of theorem. It is clear that S is closed under addition and, in view of the lemma, that S is a weakly prime ideal of R.

If I is any weakly prime ideal of R and a is not in I, then $RaR\not\subset I$. Hence there exist elements x_1, y_1 in R such that x_1ay_1 is not in I. By the same reasoning, there exist elements x_2, y_2 in R such that $x_2x_1ay_1y_2$ is not in I. Continuing, there exists a pair $(\{x\}, \{y\})$ of sequences in R such that ${}^mxay^n\neq 0$ for all integers m and n. Clearly a is not in S, and we conclude that $S\subset I$.

Inferior and Superior Limits

4692 [1956, 347] Corrected. *Proposed by A. E. Currier, United States Naval Academy*

Given $f(x)$ defined by

$$f(x) = -\frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

prove that

$$-\frac{3+8\sqrt{6}}{750} < \liminf_{x\to 1^-} f(x) < -\frac{1}{750}, \tag{1}$$

$$\frac{1}{750} < \limsup_{x\to 1^-} f(x) < \frac{3+8\sqrt{6}}{750}. \tag{2}$$

Editorial Note. The proposer effects the solution of the problem by an approximation of Cesàro means of the third order of $\sum a_\nu$ defined by $f(x)=\sum_{\nu=0}^{\infty} a_\nu x^\nu = -\frac{1}{2}+\sum_{n=0}^{\infty}(-1)^n x^{2^n}$. The somewhat lengthy details are available for any interested reader.

An Extremal Property of the Normal Distribution

4699 [1956, 497]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let $f(x)$ be a frequency function of finite variance σ^2 and such that $f(x) \cdot \log f(x)$ is summable ($x \log x$ is defined to be $=0$ if $x=0$). Show that

$$\int_{-\infty}^{\infty} f(x) \log f(x) dx \geqq \log \frac{1}{\sigma\sqrt{2\pi e}}$$

with equality if and only if $f(x)$ is equal almost everywhere to a normal function

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}.$$

Solution by Edgar Reich and Chih-yi Wang, University of Minnesota. Since $f(x)$ is a frequency function of finite variance σ^2 we have

$$f(x) \geqq 0, \quad \int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_{-\infty}^{\infty} xf(x)dx = m, \quad \int_{-\infty}^{\infty} (x-m)^2 f(x)dx = \sigma^2.$$

Then by the familiar inequalities for the arithmetic, geometric and harmonic means, we have for any function $g(x)$, $g(x) \geqq 0$.

$$(1) \qquad \left[\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\, dx\right]^{-1} \leqq \exp\left\{\int_{-\infty}^{\infty} f(x) \log g(x) dx\right\} \leqq \int_{-\infty}^{\infty} f(x) g(x) dx.$$

Let the normal function of the given form be denoted by $N(x)$. Then by substituting either $g(x) = f(x)/N(x)$ in the left inequality of (1) or $g(x) = N(x)/f(x)$ in the right inequality of (1), we obtain a result from which the required inequality follows easily. Equality holds if and only if $g(x) = C$, a constant, almost everywhere. But by the relations of normalization

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} N(x) dx = 1,$$

we find $C = 1$.

Also solved by N. J. Fine, C. H. Kraft and Ingram Olkin, Morton Kupperman, the proposer, and one whose solution is unsigned.

Editorial Note. The problem is given, with indicated solution, in Woodward, *Probability and Information Theory, with Applications to Radar*, p. 25. The negative of the integral in the above proposal is termed the entropy of the distribution given by $f(x)$. For a closely related problem see Moriguti, *A lower bound for the probability moment of any absolutely continuous distribution with finite variance*, Ann. Math. Statist., vol. 23, 1952, pp. 286–289.

Power Series with Preassigned Zero

4701 [1956, 498]. *Proposed by Paul Erdös, Technion Mathematics Department, Haifa, Israel*

Let α be complex, $|\alpha| < 1$. Prove that there exists a power series $\sum a_k z^k$ with rational integral coefficients, $|a_k| \leqq [1/|\alpha|^2]$, such that $\sum a_k \alpha^k = 0$.

Solution by the proposer. Write $\beta = [1/|\alpha|^2]$, and assume that $a_k \neq 0$ for some values of k. Consider the $(\beta+1)^n$ polynomials $\sum_{k=0}^{n-1} a_k z^k$, $0 \leqq a_k \leqq \beta$. Clearly $|\sum a_k z^k| < C$ for $|z| \leqq 1$. We can assume that all the $(\beta+1)^n$ points $\sum_{k=0}^{n-1} a_k \alpha^k$ are distinct, for otherwise there would be a polynomial $\sum b_k \alpha^k = 0$ with $|b_k| \leqq \beta$. Now we use the lemma: *If there are n distinct points* $|z_i| \leqq C_1$, $1 \leqq i \leqq n$, *then there are two of them* z_{i_1} *and* z_{i_2} *for which* $|z_{i_1} - z_{i_2}| < C_2 n^{-1/2}$. (To see this, consider circles of center z_i and radius $\frac{1}{2} C_2 n^{-1/2}$; for large C_2 they have to overlap.) From the lemma it follows that there exist sets of coefficients a_k, $a_k' \leqq \beta$ such that

$$\sum_{k=0}^{n-1} (a_k - a_k')\alpha^k < C_3/(\beta + 1)^{n/2}.$$

Let l be the smallest subscript with $a_l \neq a_l'$, then upon dividing through by α^l we obtain a polynomial $\sum_{r=0}^{n-1} b_r z^r$, $b_0 = a_l - a_l' \neq 0$, for which

$$\sum_{r=0}^{n-1} b_r \alpha^r < \frac{C_3}{\alpha^l(\beta + 1)^{n/2}} < \frac{C_3}{\alpha^n(\beta + 1)^{n/2}}\,.$$

Now since $\beta+1>1/|\alpha|^2$ we have $C_3/\alpha^n(\beta+1)^{n/2}\to 0$. Thus it follows that for every ϵ there exists an n_0 so that for every $n>n_0$ there exists a polynomial $|\sum_{r=0}^{n-1} b_r\alpha^r|<\epsilon$, $|b_r|\leqq\beta$.

Now let $f_n(z)$ be a sequence of such polynomials with $|f_n(\alpha)|\to 0$ as $n\to\infty$. A power series $\sum c_k z^k$ will be called good if every partial sum $\sum_{k=1}^{m} c_k z^k$ is the beginning of infinitely many polynomials $f_n(z)$, *i.e.*, for infinitely many n, $f_n(z)=c_0+c_1z+\cdots+c_mz^m+\cdots$. Since there are only finitely many polynomials $\sum_{k=0}^{m} c_k z^k(|c_k|\leqq\beta)$ and infinitely many polynomials $f_n(z)$, a good power series exists from the convergence of $\sum_{k=0}^{\infty} c_k\alpha^k$; it then follows that $\sum_{k=0}^{\infty} c_k\alpha^k=0$.

Editorial Note. H. Graetzer proved this result (Journal of the London Mathematical Society, (1947), pp. 90–92) without demanding that the a_k be bounded. E. G. Strauss has a simple proof for the case $1/\alpha^2$ integral. L. Carlitz has submitted a short proof under the hypothesis that $|a_k|\leqq 2|\alpha|^{-2}$. For a related result see problem no. 4498 [1953, 634].

A Doubly Infinite Array of Sets

4702 [1956, 498]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Lexington, Mass., and W. E. Weissblum, Massachusetts Institute of Technology*

Let

$$\begin{array}{c} A_{11}, A_{12}, A_{13}, \cdots \\ A_{21}, A_{22}, A_{23}, \cdots \\ \cdots\cdots\cdots \end{array}$$

be an infinite square array of sets of real numbers in (0, 1). If $A_{kn}\to 0$ for each fixed k, does it follow that there exists a diagonal sequence, $A_{1n_1}, A_{2n_2}, A_{3n_3}, \cdots$, whose limit is 0?

Solution by Jerome Keisler, California Institute of Technology. The answer is no, and I shall proceed to construct an array in which there exists no diagonal sequence whose limit is the null set 0. Without loss of generality we may take the $A_{i,j}$ to be sets of real numbers in the closed interval $I_0=\{a\leqq x\leqq b\}$, where $0<a<b<1$, with a, b constant.

For any closed interval $I=\{c\leqq x\leqq d\}$, define $H_1(I)$ as the closed interval $\{c\leqq x\leqq(c+d)/2\}$, *i.e.*, the left half of I. Define $H_2(I)=H_1(I-H_1(I))$, and in general $H_n(I)=H_1(I-\bigcup_{i=1}^{n-1} H_i(I))$. We have $H_n(I)\subset I$. The sequence $H_1(I)$, $H_2(I), \cdots, H_n(I), \cdots$ consists of non-overlapping closed intervals, and any real point p lies in at most two of these intervals. Finally if I is the union of the closed intervals $I_j=\{a_j\leqq x\leqq b_j\}$, such that $a_1<b_1<a_2<b_2<\cdots<a_j<b_j<\cdots$, define $H_n(I)=\bigcup_j H_n(I_j)$.

Now take as the first-row sequence of sets $A_{1,j}$, the intervals $A_{1,n}=H_n(I_0)$. In general, take as the kth row sequence $A_{k,j}$, the sets $A_{k,n}=\bigcup_j H_n(a_{k-1,j})$. The infinite square array of sets $A_{i,j}$ is now inductively defined. For each k, we have

$\lim_n A_{k,n}=0$ since each real point lies in at most two $A_{k,n}$.

Now consider any subsequence $A_{1,n_1}, A_{2,n_2}, \cdots$. Define $S_1=A_{1,n_1}$, $S_2=H_{n_2}(A_{1,n_1})\subset A_{2,n_2}$, and, in general, $S_k=H_{n_k}(S_{k-1})\subset A_{k,n_k}$. We thus have a sequence $\{S_i\}$ with the properties for all k,

1) $S_k \subseteq A_{k,n_k}$, 2) $S_k \subset S_{k-1}$ $(k \geqq 2)$,

3) S_k is a closed interval, 4) the length of $S_k \to 0$ as $k \to \infty$.

Thus S_k is a sequence of nested intervals, and there exists a unique point q which is in every S_k. Then q is in every A_{k,n_k}, and $q\in\lim_k A_{k,n_k}$. Then $\lim_k A_{k,n_k} \neq 0$ for any subsequence A_{k,n_k}.

Also solved by L. A. Rubel, B. I. Penkov and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. II: Contributions to Probability Theory. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. x+246 pp. $6.50.

This book is the second of five volumes of papers contributed to the Third Berkeley Symposium, held under the auspices of the Statistical Laboratory and the Department of Statistics of the University of California at Berkeley. Most of the papers in this volume were presented in sessions held during July and August, 1955. The contributions were at the invitation of the Statistical Laboratory.

The present volume consists of 15 articles by a group of eminent scholars. No effort was made to unify the topics treated, but rather the papers represent current individual investigations undertaken by the authors. The result is a very good cross-section of modern probability theory as it is unfolding today, and in particular reflects strikingly the diminishing interest in limit theorems associated with sums of random variables and the vigorous growth of stochastic processes in general.

It is of course impossible to give here a review of each of the papers individually—they will undoubtedly be reviewed separately in, *e.g.*, *Mathematical*

Reviews. The following is a reproduction of the table of contents:

DONALD A. DARLING
University of Michigan

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. IV: Contributions to Biology and Problems of Health. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. vii+179 pp. $5.75.

This book is one of five volumes reporting the proceedings of a symposium held during December 1954 and July–August 1955. The material contained in vol. IV may best be summarized by reproducing the table of contents:

II. Contributions to Problems of Health
- M. S. Bartlett, *Deterministic and stochastic models for recurrent epidemics.*
- A. T. Bharucha-Reid, *On the stochastic theory of epidemics.*
- Chin L. Chiang, J. L. Hodges, Jr., and J. Yerushalmy, *Statistical studies in medical diagnoses.*
- Jerome Cornfield, *A statistical problem arising from retrospective studies.*
- David G. Kendall, *Deterministic and stochastic epidemics in closed populations.*
- William F. Taylor, *Problems in contagion.*

Of special interest to mathematicians is the range of mathematical concepts and techniques employed by the several authors. The following were noted (and others were no doubt used if not simply missed by the reviewer): partial differentiation, differential equations, Laplace transforms, eigenfunctions, difference equations, probability and distribution theory, Markoff chains, Monte Carlo methods, method of least squares, statistical theory of estimation and hypothesis testing, decision functions, stochastic processes.

No brief review can do justice to the skill and imagination shown by the authors in applying the techniques mentioned above to the difficult problems encountered in the biological sciences and in the allied fields of health and medicine. It is not that the techniques themselves are difficult, but rather that the expression of the problems in mathematical terms is not easy. Perhaps the greatest contribution of the mathematicians and statisticians has been in helping the biologist state his problems so that a mathematical formulation is possible. This reformulation of the problems in mathematical language results in two major benefits: (1) Once the problems are expressed in mathematical terms, solutions are frequently easily obtained. Of course, in so expressing the problems, simplifications, desirable or otherwise, have often been made. (2) The research workers in biology, medicine and public health have been forced to re-examine their assumptions and theories in the light of the questions posed by the mathematicians and statisticians.

It is gratifying to note that mathematics and mathematicians are aiding in the biological sciences. This reviewer has long thought that mathematicians have concerned themselves too much with the physical sciences and too little with the biological sciences. In recent years, however, as shown by the volume being reviewed here and other publications, it is evident that mathematicians have become aware of the varied opportunities to apply their knowledge in an area previously ignored. It is hoped that the degree of cooperation between the researcher in the biological sciences and the mathematician will continue to grow in the years ahead.

BERNARD OSTLE
Montana State College

Elements of Pure and Applied Mathematics. By Harry Lass. New York, International Series in Pure and Applied Mathematics, McGraw-Hill, 1957; ix +491 pp. $7.50.

This book is an outgrowth of courses which the author has taught, and is designed so that it may be used as a textbook in various courses from the junior level onward.

The material covered is indicated by the titles of the ten chapters. These are: (1) Linear equations, determinants, and matrices, (2) Vector analysis, (3) Tensor analysis, (4) Complex variable theory, (5) Differential equations, (6) Orthogonal polynomials, Fourier series, and Fourier integrals, (7) The Stieltjes integral, Laplace transforms, and calculus of variations, (8) Group theory and algebraic equations, (9) Probability theory and statistics, (10) Real variable theory.

Flexibility for the use of the book as a text is provided by the independence of the chapters; this is not strictly true of the last chapter since frequent reference to it is made in the rest of the work. Exercises are provided at the end of sections. The comprehensive choice of topics is a good one. As is, however, apt to be the case in a work of such extensive scope, some topics are treated more briefly than one might like. An early example of this is the omission of the Laplace development by minors in the treatment of determinants; the reader instead is urged to read this topic in a conventional book. Even so, the definition of determinants using permutations finds so little application either for hand or machine calculation of determinants, or for deducing theorems, that the recursive definition of a determinant in terms of its minors or cofactors would seem preferable.

Throughout the book, references are made to the last chapter, that dealing with real variable theory. This chapter consists of a development of the real number system beginning with the Peano postulates, an introduction to the theory of sets of real numbers, to cardinal numbers, a discussion of limits, continuity, and topics generally treated in courses in advanced calculus. It is regrettable that rigor appears lacking at some crucial points. We give two examples: The field of rational numbers is derived from the ring of integers by postulating a *rational function*, instead of the construction using ordered pairs. It would appear that this is not a suitable substitute, as it more or less begs the question. In order to postulate a function, we must assume its range, in this case the rational numbers themselves. Limits are treated in section 10.12 by loosely introducing an infinitesimal as "a variable which approaches zero as a *limit*." The discussion of limits is then carried on in terms of infinitesimals. From this discussion the reader would have difficulty in giving valid proofs of the fundamental limit theorems, as he is directed to do. An acceptable definition of continuity follows the treatment of limits.

A. A. GRAU
University of Oklahoma and
Oak Ridge National Laboratory

History of Analytic Geometry. By C. B. Boyer, Scripta Mathematica, New York, 1956, ix+291 pp. $6.00.

Here is an integrated survey of the history of analytic geometry from the earliest glimmerings in ancient Mesopotamia and Egypt up through the death in 1868 of the subject's most powerful and prolific contributor, Julius Plücker. Leaning largely upon the earlier, but either incomplete or inconveniently inaccessible, works of Loria, Tropfke, Wieleitner, and Coolidge, Professor Boyer has in scholarly fashion sketched the fascinating story of the emergence and development of this important branch of mathematics, filling in many lacunae and correcting many errors perpetrated by early writers and perpetuated by subsequent historians of mathematics.

There is no unanimity of opinion among historians of mathematics as to who invented analytic geometry, nor even as to what age should be credited with the invention, and this matter certainly cannot be settled without an agreement as to the defining characteristics of the subject. Professor Boyer steers an admirably restrained course here, and though he bespeaks his own views he gives a fair account of existing opinions. A bird's-eye view of the book's chronological trend is obtained from the titles of the nine chapters: I, The Earliest Contributions; II, The Alexandrian Age; III, The Medieval Period; IV, The Early Modern Prelude; V, Fermat and Descartes; VI, The Age of Commentaries; VII, From Newton to Euler; VIII, The Definitive Formulation; IX, The Golden Age. These chapters are sown with 470 footnotes of considerable worth to the more deeply interested reader. At the end of the book appears a list of 76 important primary sources (arranged roughly in chronological order) and a list of 142 secondary works (arranged alphabetically by authors). The value of these references has been enhanced by accompanying each with a brief indication of the nature of the material involved. The book closes with a good index.

Professor Boyer's book is published as Numbers Six and Seven of the Scripta Mathematica Studies, and thus enjoys the neat appearance for which these studies are known. Some readers of the book will recall that goodly portions of the manuscript appeared earlier in the pages of *Scripta Mathematica*. It hardly seems fair to attempt a criticism of the book in regard to areas of omission, for it must be admitted that Professor Boyer has well chosen his material so as not to exceed a volume of modest size. Nor would it be fitting to comment on observed typographical errors; they are not serious and are well within expected limits.

Teachers and students of analytic geometry, and all interested in the history of mathematics, owe Professor Boyer a very real debt of gratitude for performing the patient, painstaking, and time consuming research that undoubtedly went into the preparation of this fine book.

HOWARD EVES
University of Maine

Advanced Real Calculus. By Kenneth S. Miller. Harper, New York, 1957. viii+185 pp. $5.00.

The book is directed principally at the junior or senior mathematics major. The student is led from the Dedekind cut through the notions of differentiation and Riemann integration of functions of one variable to the corresponding notions for functions of two variables.

Concentration is on the ϵ, δ method with complete details of the method presented in almost all instances. Unfortunately, while stressing the ϵ, δ method, there is a lack of precision with several other notions. For example, a function of one real variable is described but not defined, while no attempt is made to define a function of two real variables. A "number" is a Dedekind cut of the rational numbers. However, with no further discussion of isomorphism or identification, certain "numbers" are called rational numbers. The conception of mathematical elegance that is given the student is that of showing that a quantity is less than ϵ, rather than a fixed multiple of ϵ.

A novel feature of the book is that complete solutions to many of the exercises in the book are given in an appendix.

PAUL CIVIN
University of Oregon

The Leibniz-Clarke Correspondence, together with Extracts from Newton's *Principia* and *Opticks*, edited with Introduction and Notes by H. G. Alexander. Philosophical Library, New York, 1956. 200 pp. $4.75.

This correspondence exchanged during the years 1715–1716, at the suggestion of Caroline, Princess of Wales, in which Samuel Clarke acted as the authorized champion of Sir Isaac Newton, is one of the most important controversies in the history of modern philosophy and science. First published by Clarke in 1717 in a bilingual edition for which he had translated Leibniz's French papers into English, the complete English text is here reprinted for the first time in a separate and extremely careful and competent edition, accompanied by extracts from Newton's and Leibniz's writings which have a direct bearing on the problems under discussion.

So fundamental are the issues involved in this conflict between two thinkers of equal stature that they have continued to be debated up to the present time. The nature of space and time and the measurement of force, together with many correlated questions, are thoroughly discussed in these letters by philosopher-scientists, of whom each fought from inside his own majestic system of physics, mathematics, and metaphysics. No understanding, compromise, or mutual approach could therefore appear possible, when Leibniz's death put an end to the correspondence. But the later development of science, at least as long as classical

mechanics dominated the scene, seemed to decide in favor of the Newtonian system which eclipsed Leibniz's idealistic solution during almost two centuries.

In his introduction, H. G. Alexander has masterfully epitomized the conflicting doctrines and traced the development of the problems involved through the contributions of Berkeley, Maclaurin, Euler, Boscovitch, Kant, and other scientists up to Ernest Mach and Einstein. His conclusion stating that "in the light of modern physics, it is perhaps best to call it [the controversy] a drawn contest," can hardly be challenged by the unpredjudiced student who will appreciate this authoritative guidance in the study of an historically and systematically central issue.

PAUL SCHRECKER
University of Pennsylvania

Numerical Integration of Differential Equations. By A. E. Bennett, W. E. Milne, and H. Bateman, Dover, New York, 1956, 108 pp. $1.35.

This little book is a reproduction of the monograph, of the same title, prepared in 1931 for the National Research Council, with no addition or deletion of material.

Much has happened in this field since 1931, including and in particular the development of high-speed calculating machines. There is little here of interest to the "programmer," and no guidance at all respecting practical methods for solving the various classes of partial differential equations. Chapter 4, devoted to this topic, is in fact mainly restricted to a brief account of classical semi-analytic methods such as those of Rayleigh and Ritz and "least squares," the use of Fourier series, and some comments on integral equations. Chapter 2, similarly, has little of immediate practical value, being an historical account of methods of successive approximation and a justification of step-by-step methods of integration. Neither of these chapters gives worked examples.

Chapters 1 and 3, however, have much useful information in little space. In Chapter 1 are collected nearly all the useful finite-difference formulae for interpolation, differentiation and integration, both with differences and with ordinates only, and with valuable comments on their use and misuse. Chapter 3 gives almost all known methods, with numerical illustration, for step-by-step solution of ordinary differential equations.

For the scholar, and particularly the historian, there is a wealth of information, including some five hundred references of original material and additional reading! It is doubtful whether many readers will take advantage of this opportunity to delve into the past, but Chapters 1 and 3 probably justify the rather modest investment in this slim volume.

L. FOX
University of California, Berkeley

Numbers: Fun and Facts. By J. Newton Friend. Charles Scribner's Sons, New York, 1954. xi+208 pp. $2.95.

This is a book whose title adequately describes the contents. The only requirements for reading and enjoying it are a knowledge of elementary arithmetic and a fascination for problem solving. The purpose of the book as given in the preface is, "to show how interesting, indeed fascinating, is the study of numbers, their origin, and peculiarities, and of the traditions, legends and superstitions that have in the course of ages, collected around them."

The author discusses: one-to-one correspondence, counting by various methods, Roman and Hindu-Arabic numerals, origin of numerical symbols, kinds of numbers and superstitions about various numbers. There is also a short description of gematria and numerology. The discussions of one-to-one correspondence, concrete and abstract numbers, the number π and number sense are very well done.

There are many problems in this attractive book. In the prologue the author states that several of them are from what appears to be a 17th century monastic manuscript that he found in an old shop. There are problems based on different ways of writing numbers including mirror writing and palindromes. There are match stick, coin and age problems. One chapter consists of problems where the digits are replaced by other symbols. The author gives an analysis of the solution for several problems.

There are a few typographical errors, one wrong answer and several incomplete answers. One will be surprised to read on p. 25, "The sole purpose of the 0 (in 50) is to show that 5 occupies the second place." On p. 69, 0/0 is treated as an absurdity instead of an indeterminate form.

MARION P. EMERSON
Southwest Missouri State College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

THIRD U. S. NATIONAL CONGRESS OF APPLIED MECHANICS

The third U. S. National Congress of Applied Mechanics will be held at Brown University, Providence, Rhode Island, June 11–14, 1958. All research workers in the field are cordially invited to submit papers constituting original experimental or theoretical contributions to Applied Mechanics, including mechanics of rigid bodies and deformable solids, mechanics of fluids and gases, thermodynamics and heat transfer. Instructions to authors of papers may be obtained from the Organizing Committee. It is expected

that papers accepted by the Editorial Committee with the advice of recognized authorities and presented at the Congress, will be published in full in the Proceedings of the Congress.

To be considered for presentation at the Congress, complete papers and manuscripts must be submitted to the Chairman of the Editorial Committee before January 1, 1958; to be scheduled for presentation the final manuscript of a paper must have been accepted before May 1, 1958. To avoid delays caused by overburdening reviewers and editorial staff, authors are urged to submit manuscripts well ahead of the deadline of January 1, 1958.

The papers will be grouped by subject, and 30 minutes will be allotted for presentation and discussion of each paper. Arrangements will also be made for general lectures by outstanding authorities on subjects of general interest to members of the Congress. Facilities will be provided for informal discussions and social contact.

Inquiries regarding the Congress should be addressed to D. C. Drucker, Secretary of the Congress, Brown University, Box D, Providence 12, R. I.

POLISH SCHOLARSHIPS FOR FOREIGN STUDENTS

Poland's Ministry of Higher Education is offering a number of scholarships at Polish universities for qualified foreign graduate students. The awards, which run from six months to two years, will be made for advanced study and research in mathematics, Polish linguistics and literature and Polish history. Slavic philology is included in the program. Highly qualified advanced undergraduates may also apply for the awards.

The Polish scholarships include double-room dormitory accommodations and a monthly sum of about 2400 zlotys. This sum is adequate to cover living costs, tuition and ordinary personal expenses. The awards do not cover transportation to and from Poland although in exceptional cases assistance may be given towards the cost of the return trip. If such aid is required, it must be indicated in the scholarship application.

Application forms are available from the Cultural Attaché of the Polish Embassy, 2640 Sixteenth St., N.W., Washington 9, D. C. Applications should be accompanied by a transcript of the candidate's university or college record, a biographical sketch, a statement of chief academic interests and of the work the applicant hopes to do in Poland. In addition, there should be a letter of recommendation from a faculty member under whom the applicant has done his major course work, copies of diplomas or academic certificates and three recent passport-size photographs.

OPPORTUNITY FELLOWSHIPS OF JOHN HAY WHITNEY FOUNDATION

The competition for Opportunity Fellowships is open to any citizen of the United States (including residents of territories) who has given evidence of special ability and who has not had full opportunity to develop his talents because of arbitrary barriers, such as racial or cultural background or region of residence. Candidates are expected to be mature enough to have given *positive evidence of superior promise*, yet young enough to have their careers before them; in general to be between the ages of 22 and 35 and to have completed their general education. While the Committee of Award has full discretion to take all factors into account and make awards outside the above ages and qualifications, candidates under 35 are given decided preference.

The fellowships are open not only for academic study (graduate) but for any kind of training or experience (journalism, industry, labor, the arts, *etc.*) which may be most useful in developing varied talents and varied forms of leadership.

Awards are expected normally to range from \$1,000 to \$3,000 depending on the nature of the proposed project and the financial need of the candidate. It is hoped that in many cases funds from other sources may supplement these awards (for example fellowships for study in foreign countries, additional scholarship aid from universities, pay-

ments for certain types of apprentice work, *etc.*). Awards are for a full year of serious work, *not for incidental or temporary projects.* In special cases grants may be renewed for a second year or more.

Awards are made annually by a special Committee on the basis of formal written applications by the candidates on forms provided by the John Hay Whitney Foundation. Completed applications must be filed not later than November 30 so as to assure ample time for processing applications, assembling references, and making selections. Awards are announced in April or May. Communications should be addressed: Opportunity Fellowships, John Hay Whitney Foundation, 630 Fifth Ave., New York 20, New York.

RESEARCH POTENTIAL AND TRAINING IN THE MATHEMATICAL SCIENCES

The Committee on the Survey announces the completion of the SURVEY OF RESEARCH POTENTIAL AND TRAINING IN THE MATHEMATICAL SCIENCES. The Survey began in January 1955, and nearly thirty mathematicians have served on its Committee and four Subcommittees during the past two and one-half years.

The Survey has produced three documents. One of these is the Report on a Conference on Undergraduate Mathematics Curricula. The Conference was held at Hunter College on October 12–13, 1956 and was attended by 25 representatives of various colleges and universities. There were formal reports by various speakers covering many aspects of, and subjects related to, undergraduate education in mathematics, and these formal reports have been incorporated into the conference report. This document has been circulated widely. A small number of copies have been deposited with the National Science Foundation and are available on request.

The other two documents constitute the Final Report. Part I is a 163 page report on the organization of the Survey and the results of two large scale data-gathering activities of the Committee. The first activity consisted of interviews of the principal Ph.D. granting institutions. Fifty-nine departments of mathematics were visited, and three others completed the interview schedule without a visit. The data presented as obtained by these interviews are concerned with such matters as facilities, libraries, promotion and leave policies, administration, and salaries. The remainder of the report presents a set of tables giving the composite life story of 1851 mathematicians who received the Ph.D. from 1915 to 1954, as derived from their replies to a very penetrating questionnaire.

Part II of the Final Report consists of the reports of the four Subcommittees, and the 28 resulting recommendations of the Committee. We note that the Subcommittee on Undergraduate Colleges recommended the extended support of prize competitions in mathematics, urged that an attempt be made to call to the attention of industry the need for support of mathematics in undergraduate colleges, and requested that the National Science Foundation publish annually an up-to-date record of the mathematics faculties of all colleges and universities with the degrees held, and the date and place these degrees were conferred.

The report of the Subcommittee on Research Environment discussed such matters as facilities, salaries, leaves and fellowships, and made a number of recommendations on these matters.

The Subcommittee on Non-Teaching Opportunities recommended the exploration of further methods for establishing understanding among academic mathematicians of the nature of non-academic mathematics and made recommendations on various other related matters. The report also called attention to the recent action of the American Mathematical Society in adding the SIAM Journal to the group of journals receiving a subsidy from the American Mathematical Society, and suggested the possibility that the Journal may provide another desirable publication outlet for some of the research results of non-academic mathematicians.

The final report, from the Subcommittee on Publications, recommended the establishment of a journal for expository articles, and the establishment of prizes for mathematical books, recommended the subsidization of mathematical publications by the National Science Foundation on a permanent basis, and recommended that the volume of translations published by the American Mathematical Society be at least doubled.

The Final Report has been distributed to the chairmen of about 110 departments of mathematics, to Committee and Subcommittee members, and to the officers of various mathematical organizations. A limited number of additional copies are available, and can be obtained by writing to the Program Director for Mathematics, National Science Foundation, 1520 H Street, Washington 25, D. C.

A. A. ALBERT, *Chairman, Committee on the Survey*

PERSONAL ITEMS

Professor H. W. Brinkmann, Swarthmore College, was the representative of the Association at the inauguration of President Frederick de Wolfe Bolman, Jr., of Franklin and Marshall College on April 6, 1957.

Professor G. K. Kalisch, University of Minnesota, represented the Association at the inauguration of the Very Reverend James P. Shannon as President of the College of St. Thomas on May 8, 1957.

Professor Marguerite Lehr, Bryn Mawr College, was the representative of the Association at the Convocation commemorating the Hundredth Anniversary of the National Education Association in Philadelphia, Pennsylvania, on July 3, 1957.

Professor S. S. Wilks, Princeton University, represented the Association at the sesquicentennial dinner of John Wiley and Sons, Inc. in New York City on May 28, 1957.

Dr. Max Herzberger, Head, Optical Research, Eastman Kodak Company, Rochester, New York, has been honored by election to the Bavarian Academy of Science.

Professor Emeritus W. C. Krathwohl, Director of Tests, Institute for Psychological Services, Illinois Institute of Technology, received an Alumni Distinguished Service Award on May 3, 1957.

Catholic University of America: Mr. D. B. Tomiuk, Graduate Assistant, has been appointed Instructor; Associate Professor J. N. Rice has retired; Dr. R. G. Laha, Indian Statistical Institute, Calcutta, has been appointed Research Associate; Professor Eugene Lukacs who is on leave of absence is lecturing at the Sorbonne, Paris, France.

Fresno State College: Mrs. Margaret Finn, University of Southern California, Assistant Professors W. C. Guenther, Arizona State College, Tempe, J. G. Marica, Humboldt State College, E. S. Robbins, Arizona State College, Tempe, and Mr. Peter Yff, University of Illinois, have been appointed Assistant Professors; Dr. S. J. Bryant, Mr. D. J. Ewy, Dr. V. E. Howes, and Dr. T. C. Kipps have been promoted to Assistant Professors.

Los Angeles City College: The Seventh Annual William B. Orange Mathematics Prize Competition for students in Los Angeles high schools was held on May 17, 1957 with 156 students from 33 local high schools participating.

Massachusetts Institute of Technology: Dr. H. P. McKean, Jr., Princeton University Dr. M. L. Minsky, Society of Fellows, Harvard University, and Assistant Professor D. B. Ray, Cornell University, have been appointed Assistant Professors; Mr. R. T. Dames, Willow Run Research Center, University of Michigan, Mr. A. M. Garsia, Stanford University, and Mr. R. A. Kunze, University of Chicago, have been appointed C. L. E. Moore Instructors; Mr. D. J. Newman, AVCO Corporation, has been appointed Instructor; Professor Armand Borel, Institute for Advanced Study, will be Visiting Professor during the spring semester; Professors Lennert Carleson, Mathematical Institute of the University of Uppsala, and S. S. Chern, University of Chicago, are Visiting Professors during the fall semester; Dr. M. S. Longuet-Higgins, National Institute of Oceanography, Wormley, England, is Visiting Professor during the academic year 1957–

58; Assistant Professor Jürgen Moser, New York University, is Visiting Assistant Professor during the academic year; Assistant Professor Barrett O'Neill, University of California at Los Angeles, is guest of the Department during the fall semester and will be Visiting Assistant Professor during the spring semester; Associate Professors Warren Ambrose and G. W. Whitehead have been promoted to Professors; Assistant Professors N. C. Ankeny and J. F. Nash have been promoted to Associate Professors; Associate Professor Kenkichi Iwasawa, who has been awarded a Guggenheim Fellowship, is on leave of absence at the Institute for Advanced Study; Professor H. P. McKean, Jr. has been awarded a Fulbright Grant and a National Science Foundation Post-doctoral Fellowship and is on leave at the University of Kyoto, Japan.

North Carolina State College of Agriculture and Engineering: Professor J. W. Cell has been appointed Chairman of the Department of Mathematics; Professor H. A. Fisher, former Chairman of the Department, has retired.

Ripon College: Associate Professor E. G. H. Comfort has been promoted to Professor; Mr. C. W. Larson has been promoted to Assistant Professor.

Wesleyan University: Dr. C. S. Coleman has been promoted to Assistant Professor; Dr. Frederic Cunningham, Jr., Lecturer, Bryn Mawr College, has been appointed Assistant Professor; Dr. R. G. Long, University of Washington, and Dr. John McKibben, University of Chicago, have been appointed Instructors; Professor R. A. Rosenbaum has been awarded a National Science Foundation Faculty Fellowship which he will hold during the academic year 1958–59 at Oxford University; Associate Professor Hing Tong is on sabbatical leave during 1957–58; Professor L. B. Williams, Reed College, is Visiting Professor during 1957–58.

Dr. A. G. Anderson, Mathematician, Glass Division Research Laboratory, Pittsburgh Plate Glass Company, Pittsburgh, Pennsylvania, is employed now as Chief Statistician at the General Tire and Rubber Company, Akron, Ohio.

Miss Allene Archer, Teacher, Thomas Jefferson High School, Richmond, Virginia, has been appointed Professor of Mathematics and Mathematics Education at Maryland State Teachers College, Towson.

Associate Professor B. H. Arnold, Oregon State College, has been granted a Fulbright Award and is teaching at Higher Teachers Training College, Baghdad, Iraq, during 1957–58.

Mr. A. F. Bakenhus, Student, University of Houston, has a position as Mathematician-Programmer for Service Bureau Corporation, Houston, Texas.

Mr. C. W. Barnett, Research Programmer, Louisiana State University, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Fort Worth, Texas.

Assistant Professor J. F. Blackburn, U. S. Air Force Academy, has a position as Education Coordinator with the International Business Machines Corporation, Cambridge, Massachusetts.

Dr. E. K. Blum, Naval Ordnance Laboratory, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor R. C. Boles, Mercer University, has been appointed Assistant Professor at North Carolina State College.

Assistant Professor Fred Brafman, Southern Illinois University, has been appointed Associate Professor at the University of Oklahoma.

Whitney Visiting Professor Louis Brand, Trinity College, has been appointed M. D. Anderson Professor of Mathematics at the University of Houston for 1957–58.

Mrs. Carolina del Mar Brennan, University of Philippines, is a graduate research mathematician at the University of California Computer Center, Berkeley.

Dr. R. J. Buehler, Project Associate, Naval Research Laboratory, University of Wisconsin, has been appointed Assistant Professor in Statistics at Iowa State College.

Dr. E. L. Buell, Technical Director, Aerial Measurements Laboratory, Northwestern University, has been appointed Professor at Worcester Polytechnic Institute.

Dr. J. O. Carter, Stanford University, has accepted a position as Applied Science Representative with International Business Machines Corporation, San Jose, California.

Dr. D. R. Childs has a position as a scientist for Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Assistant Professor H. E. Chrestenson, Whitman College, has been appointed Assistant Professor at Reed College.

Assistant Professor K. L. Cooke, State College of Washington, has been appointed Assistant Professor at Pomona College.

Mr. Walter Cornetz, Instrumentation Engineer, Grumman Aircraft Corporation, Bethpage, New York, is employed now as Principal Flight Test Engineer, Republic Aviation Corporation, Farmingdale, New York.

Professor W. H. H. Cowles, Chairman of the Department of Mathematics, Pratt Institute, has retired.

Dr. R. H. Crowell, Forrestal Research Center, has been appointed Lecturer at Massachusetts Institute of Technology.

Professor D. R. Davis, Chairman of Department of Mathematics, State Teachers College, Montclair, New Jersey, has been appointed Chairman of the Department of Mathematics of East Carolina College.

Mr. A. C. deWilde, Detroit Edison Company, has accepted a position as Senior Analyst with General Motors, Detroit, Michigan.

Mr. W. C. Dixon has accepted a position as Engineer with the Analysis and Programming Section, BIZMAC Engineering, Radio Corporation of America, New Jersey.

Mr. Carlos Fallon, Systems Engineer, Vitro Corporation of America, is now Chief Mechanical Design Engineer with Nems-Clarke, Silver Spring, Maryland.

Dr. G. E. Forsythe, Research Mathematician, Numerical Analysis Research, University of California at Los Angeles, has been appointed Professor at Stanford University.

Dr. J. W. Frick has been appointed Operations Analyst at Technical Operations, Inc., Monterey, California.

Associate Professor R. L. Garrett, Middle Georgia College, has been promoted to Professor.

Dr. Betty J. Gassner, Graduate Student, New York University, is employed as a research analyst at Remington Rand, New York, New York.

Mr. J. H. Gissel, Instructor, General Motors Institute, is a test engineer for Division of General Dynamics, Consolidated-Vultee Aircraft, San Diego, California.

Mr. B. T. Goldbeck, Jr., of the University of Oklahoma has been appointed to an assistant professorship at Texas Christian University.

Mr. M. L. Goldwater, Research Engineer, J. B. Rea, Santa Monica, California, is a senior engineer for Litton Industries, Beverly Hills, California.

Dr. Ulf Grenander, University of Stockholm, Sweden, has been appointed Professor of Mathematical Statistics and Probability in the Division of Applied Mathematics, Brown University.

Dr. R. M. Gundersen has been appointed Assistant Professor at Illinois Institute of Technology.

Dr. C. J. A. Halberg, Jr., University of California, Riverside, has been promoted to Assistant Professor.

Dr. Carl Hammer of UNIVAC European Computing Center has accepted a position with the Remington Rand International Corporation, New York, New York.

Mr. J. W. Haynes, Jr., Graduate Student, University of California, Berkeley, has a position as a mechanical engineer at Pearl Harbor Navy Shipyard, Honolulu, Hawaii.

Dr. R. T. Herbst, Associate Director, Mathematical Sciences Division, Office of Naval Research, U. S. Army, Durham, North Carolina, is now a member of the technical staff of Bell Telephone Laboratories, Winston-Salem, North Carolina.

Professor P. G. Hodge, Jr., Polytechnic Institute of Brooklyn, has been appointed Professor of Mechanics at Illinois Institute of Technology.

Mr. R. B. Jackson, Jr., of Duke University has been appointed to an assistant professorship at Davidson College.

Mr. W. H. Jones, Mathematician, Department of Defense, Washington, D. C., is now Associate Social Scientist in System Development Corporation, Santa Monica, California.

Professor H. K. Justice, University of Cincinnati, has been appointed Dean of the College of Engineering.

Dr. May R. Kinsolving, Harpur College, has been appointed Instructor at Cornell University.

Mr. A. G. Konheim, Student, Polytechnic Institute of Brooklyn, has been awarded a General Electric Honors Program Fellowship and is studying at Cornell University.

Professor Harry Langman, Chairman of the Department of Mathematics, Detroit Institute of Technology, has been appointed Professor and Chairman of the Department of Mathematics, Ohio Northern University, and Lecturer, University of Pittsburgh, Lima, Ohio.

Dr. Marguerite Lehr, Bryn Mawr College, is on sabbatical leave as visiting fellow at Princeton University.

Assistant Professor D. R. Lewis, State Teachers College, Mankato, Minnesota, is employed as a mathematician with Remington-Rand UNIVAC, St. Paul, Minnesota.

Associate Professor Viktors Linis, University of Ottawa, has been appointed Chairman of the Department of Mathematics.

Professor L. L. Lowenstein, Kent State University, has been appointed Professor at Arizona State College, Tempe.

Professor W. G. Madow, University of Illinois, has been appointed Consulting Professor at Stanford University and Senior Research Statistician at Stanford Research Institute.

Dr. M. E. Mahowald, U. S. Marine Corps, has accepted a position as a mathematician with the General Electric Company, Cincinnati, Ohio.

Dr. M. J. Mansfield, Purdue University, has been appointed Assistant Professor at Washington and Jefferson College.

Dr. J. R. Mayor, Director, Science Teaching Improvement Program, American Association for the Advancement of Science, has been appointed to the newly established position of Director of Education of the A.A.A.S.

Mr. J. S. McNair has been appointed Associate Professor, Teachers College at Plattsburgh, State University of New York.

Assistant Professor B. C. Meyer, University of Arizona, has been appointed Assistant Professor at the University of Colorado.

Professor Emeritus W. L. Miser, Vanderbilt University, recently at McKendree College, has been appointed Professor at Ohio Northern University.

Assistant Professor Irene P. Monahan, Keuka College, has been promoted to Associate Professor and Head of the Department of Mathematics.

Mr. H. G. Moore, Graduate Assistant, University of Utah, has been appointed Instructor at Purdue University.

Associate Professor L. T. Moore of Brooklyn College has retired.

Associate Professor Irene Nolan, West Virginia Institute of Technology, has been appointed Assistant Professor at Tennessee Polytechnic Institute.

Assistant Professor C. S. Ogilvy, Hamilton College, has been promoted to Associate Professor.

Mr. Jack Padgett, Armstrong College of Savannah, has been appointed Registrar of the College.

Associate Professor T. K. Pan, University of Oklahoma, has been promoted to Professor.

Assistant Professor W. H. Peirce, Michigan State University, has a position as Mathematical Analyst with the Electric Boat Division, General Dynamics Corporation, Groton, Connecticut.

Mr. R. J. Pipino, Instructor, Western Reserve University, is Research Staff Assistant at Johns Hopkins University.

Associate Professor R. M. Redheffer, University of California at Los Angeles, was on a sabbatical leave during 1956–57 and spent the fall semester as a National Science Foundation fellow in Göttingen, Germany and the spring semester as a Fulbright Research Scholar in Vienna, Austria.

Mr. A. T. Rice, Test Calculations Liaison Specialist, Investigations Section, General Electric Company, Lynn, Massachusetts, is now a member of the technical staff, Ramo-Wooldridge Corporation, Los Angeles, California.

Professor E. K. Ritter, Director, Rich Electronic Computer Center, Georgia Institute of Technology, has accepted a position as Manager, Mathematical Analysis Department, Lockheed Aircraft Corporation, Marietta, Georgia.

Dr. Azriel Rosenfeld, Junior Physicist, Fairchild Controls Corporation, New York, New York, has a position as Senior Design Engineer, Ford Instrument Company, Long Island City, New York.

Associate Professor Diran Sarafyan, Lamar State College of Technology, has been appointed Assistant Professor at the University of Florida.

Assistant Professor Alice T. Schafer, Connecticut College, has been promoted to Associate Professor.

Mrs. Martha M. Schneider, Student, University of Kentucky, is now a computer programmer for the General Electric Company, Schenectady, New York.

Mr. R. C. Scott, Lehigh University, has been appointed Instructor at Worcester Polytechnic Institute.

Associate Professor C. F. Sebesta, Duquesne University, has been appointed Head of the Department of Mathematics.

Professor Virginia Shuford, Rinehardt College, has been appointed Instructor at Lenoir Rhyne College.

Professor H. W. Smith, Oklahoma Agricultural and Mechanical College, has retired from this position and has been appointed Professor at the University of South Carolina.

Mr. P. H. Thrower, University of Texas, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Dallas, Texas.

Mr. Richard Tukovits, Student, Hunter College, is now an analyst at Babcock and Wilcox, New York, New York.

Dr. H. S. Valk, Research Assistant, Washington University, has been appointed Assistant Professor of Physics at the University of Oregon.

Dr. E. T. Welmers, Chief of Dynamics, Bell Aircraft Corporation, Niagara Falls, New York, has been appointed Director of the Lawrence D. Bell Research Center, Bell Aircraft Corporation.

Associate Professor Albert Wilansky, Lehigh University, has been promoted to Professor.

Assistant Professor Frances M. Wright, Harpur College, has been promoted to Associate Professor.

Mr. Burton A. Yale is a junior mathematician for Cornell Aeronautical Laboratories, Buffalo, New York.

Mr. C. P. Cebulla, Instructor, Hermantown High School, Duluth, Minnesota, died on October 21, 1956.

Mr. B. A. Chiappinelli of Canning, Sisson, and Associates, Los Angeles, California, died on December 20, 1956.

Associate Professor Emeritus A. A. Hatch, Lafayette College, died on May 7, 1957. He was a member of the Association for thirty-four years.

Assistant Professor D. K. Kazarinoff, University of Michigan, died on February 9, 1957.

Professor Emeritus Konrad Knopp of the Eberhard Karls University of Tübingen, Germany, died on April 30, 1957.

Assistant Professor A. G. Makarov, Rutgers University, died on March 25, 1957. He was a member of the Association for sixteen years.

Associate Professor Emeritus V. C. Poor, University of Michigan, died on March 14, 1957.

Mr. W. E. Voronovich, Lewis Machine Company, Cleveland, Ohio, died on February 18, 1957.

Professor R. B. Wildermuth, Capital University, died on September 28, 1956. He was a member of the Association for forty years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1957 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	J. C. Knipp, University of Pittsburgh
Indiana	Lamberto Cesari, Purdue University
Kentucky	R. S. Park, Eastern Kentucky State College
Metropolitan New York	Jewell H. Bushey, Hunter College
Nebraska	W. G. Leavitt, University of Nebraska
Northern California	George Pólya, Stanford University
Oklahoma	W. N. Huff, University of Oklahoma
Rocky Mountain	C. R. Wylie, Jr., University of Utah
Wisconsin	H. P. Evans, University of Wisconsin

As usual, a high percentage of votes was cast in the elections of sectional governors. In the case of four sections, votes were received from more than 50% of the membership. In the Oklahoma Section, the percentage of votes cast was 58%. In the Kentucky, Rocky Mountain, and Wisconsin Sections it was 54%.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1957 COMBINED MEMBERSHIP LIST

The Mathematical Association of America joins with the American Mathematical Society and the Society for Industrial and Applied Mathematics each year in publishing a Combined Membership List. Each member of the Association will receive a copy of the

1957 List as one of the privileges of membership. It is expected that the 1957 List will be ready for distribution in December 1957.

Prompt notice should be given to the office of the Association of all changes in position, rank, and address which have not been reported previously. Any errors in the 1956 List should also be reported. The final date for receipt of changes is October 15.

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 131 persons have been elected to membership by the Board of Governors on applications duly certified.

JONATHAN L. ALPERIN, Student, Harvard University.

JUDITH AMES, Student, Vassar College.

MARY ASTERITA, Res. Asst., Linde Air Products Co., New York, N. Y.

DUANE W. BAILEY, Student, State College of Washington.

LOUIS A. P. BALÁZS, Student, University of California.

RALPH BARGEN, B.S. (Bethel) Grad. Student, University of Wichita.

NYLES N. BARNERT, Student, Haverford College.

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C. DUANE ZIMMERMAN, Student, Emmanuel Missionary College.

procedure of successive partitioning, we obtain a matric algorithm for detecting all the cliques in any group. This is equivalent to determining all the maximal complete subgroups in any directed graph using the adjacency matrix.

9. *Ambiguous points of a function defined inside a sphere*, by Professor George Piranian, University of Michigan.

There exists a function $f(x, y, z)$, continuous in the sphere $x^2+y^2+z^2<1$ and possessing the following property: for every real number A and every point P on the surface $x^2+y^2+z^2=1$, the sphere contains a rectifiable path $C(A, P)$ from the origin to P such that $f(x, y, z)\to A$ as $(x, y, z)\to P$ along $C(A, P)$.

F. A. BEELER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-fourth annual meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State Teachers College, Cedar Falls, Iowa, April 26–27, 1957. Professor F. W. Lott, Chairman of the Section, presided at the April 26 afternoon session and Professor A. H. Blue, Vice-Chairman, presided at the April 27 morning session. Total attendance was 63, including 37 members of the Association.

Routine business was considered and a committee report was made on arrangements for a joint meeting with the Iowa Association of Mathematics Teachers.

It was voted that the Chairman should appoint a committee to study the problems of examinations (contests), to meet with other groups interested in sponsoring the examination, and to report to the Executive Committee of the Section for a decision regarding the matter. It was also voted that the group express itself as being in favor of such a contest.

The following officers were elected: Chairman, Professor A. H. Blue, Cornell College; Vice-Chairman, Professor A. T. Craig, The State University of Iowa; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Report of the committee on cooperation among college and high school teachers of mathematics*, by Professor O. C. Kreider, Iowa State College, Professor M. F. Smiley, State University of Iowa and Professor I. H. Brune, Chairman of Committee, Iowa State Teachers College, presented by Professor Brune.

Members of the Iowa Section of the Mathematical Association of America met with the Iowa Association of Mathematics Teachers on November 2, 1956. Some high school teachers met with the Iowa Section on April 26, 1957. Members of both organizations have been invited to a conference on mathematics at the State University of Iowa in October 1957.

The committee recommended that members of the Iowa Section: (1) maintain membership in the IAMT; (2) meet whenever possible with high school teachers; (3) contribute items for the IAMT Newsletter; (4) speak at meetings of high school teachers; (5) sponsor meetings for both groups of teachers.

2. *A trick in solving a class of boundary value problems*, by Professor Don Kirkham, Iowa State College.

In a class of boundary value problems it is found that the solution $f(x)$ reduces to a series $\sum A_m \sin(m\pi x/b)$, $m=1, 2, \cdots, \infty$, defined for $0<x<a$, $a<b$, b being the length of the boundary, and the coefficients A_m to be determined. In this same type of problem the boundary condition is satisfied independently of the series for $a<x<b$ and therefore one may use the trick of setting the series equal to itself for $a<x<b$ in the form $\sum A_m \sin(m\pi x/b) = \sum A_p \sin(p\pi x/b)$, $p=1, 2, \cdots, N$, $N\to\infty$. Then A_m may be found from a series of N simultaneous equations which arise. The trick works also for Bessel-Fourier series *etc.*

3. *Some "solutions" of inconsistent linear systems*, by Professor C. E. Langenhop, Iowa State College.

Let A be an n by r matrix, $n>r$, and let c and x be n and r dimensional column vectors, respectively. The system (1) $Ax=c$ is generally inconsistent. The least squares solution, $\bar{x}$, must satisfy $A'Ax=A'c$ (B'=transpose of B), which is always consistent. Let (2) $A_ix=c_i$ denote a system of r equations from (1) and denote the solution of (2) by $x^{(i)}$ if it exists. Then if A has rank r it was shown that $\bar{x}=|A'A|^{-1}\sum_i |A_i|^2x^{(i)}$, where $|B|$ =determinant of B and the sum is over all distinct systems (2), the summand being interpreted as a zero if $|A_i|=0$. Various extensions of this relation were also presented.

4. *Filters and equivalent nets*, by Professor M. F. Smiley, State University of Iowa.

This presentation was an attempt to convey the essential technical idea in pedagogical terms by proposing the question as to which of the nets, Basic Mathematical Concepts or Basic Mathematical Skills, is the finer one to catch college freshman.

5. *The cardinal number of residual sets*, by Professor U. R. Kodres, Iowa State College, introduced by the Chairman.

An elementary construction is used to prove the known result that every residual set has the cardinal number of the continuum.

6. *Remarks on the convolution of real functions in Laplace transform theory*, by Professor C. G. Maple and Professor Bernard Vinograde, Iowa State College, presented by Professor Vinograde.

A real function is of class T if it is: (1) of exponential order and (2) sectionally continuous except for a possible infinite discontinuity of order $t^{-\alpha}$, $(0<\alpha<1)$, at $t=0$. It is shown that functions of class T possess absolutely convergent Laplace transforms and furthermore the convolution of two functions of class T is also of class T.

7. *Mathematical training for the exceptional student*, by Professor Fred Robertson, Iowa State College.

The author discusses the program in mathematics for these students as conducted at the Iowa State College since 1946. Tables showing accomplishments in some lines are given. Some results and evaluations of the program are obtained.

8. *An analysis of error in the learning of algebra*, by Professor Vivian Strand and Professor D. A. Yos, Burlington High School and College, presented by Professor Yos, introduced by the Chairman.

The errors which learners of mathematics make are a clue to their thoughts. Some of the more common of these errors have been listed and described. The erratic pattern of error, common errors in addition and division, errors in the addition of unlike fractions, and the compounding of error in the solution of simultaneous equations are treated. The psychology of fear is considered to be a very important factor in initiating these errors.

9. *A remark on a certain sufficient statistic*, by Professor A. T. Craig, State University of Iowa.

In a regular case of estimation let X denote a continuous type random variable having probability density function $f(x;\theta)=\exp[p(\theta)K(x)+S(x)+q(\theta)]$. Let $Y=K(X)$. It is proved that $E(Y)=b+cp(\theta)$, $c>0$, is both necessary and sufficient for Y to be normally distributed. Hence, if $X_1,\cdots,X_n$ denote a random sample of $n>1$ values of X, then $Z=\sum_i K(X_i)$ is a sufficient statistic for θ, and Z will be normally distributed if and only if Y is normally distributed.

10. *On quadratic forms whose distributions are free of the population mean*, by Professor R. V. Hogg, State University of Iowa.

Let $Q_1, Q_2, \cdots, Q_k$ be k nonnegative real symmetric quadratic forms in n random values of a normally distributed variable with mean m. A necessary and sufficient condition that Q_i, $i=1, 2, \cdots, k$, has a distribution free of m is that their sum $Q=\sum_i^k Q_i$ has this property.

11. *A modern approach to elementary analysis*, by Professor H. C. Trimble, Iowa State Teachers College. (By invitation).

While mathematics has grown up in the past one-hundred fifty years, the mathematics taught to high school pupils and college freshmen is seldom more modern than Euler. Experience at Iowa State Teachers College suggests that portions of modern mathematics are teachable. Ninth graders readily assimilate modern ideas of variable, unknown, parameter, relation, function, and the like. College freshmen resist these ideas for a time, and then progress rapidly. Is it true that "mathematics is teachable just because it is good mathematics?"

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-second annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Kansas, Lawrence, Kansas, on April 13, 1957 in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 72 persons registered including 40 members of the Association. Professor W. R. Scott, Chairman, presided at the sessions.

The following officers were elected for one year terms: Chairman, Professor L. E. Laird, Kansas State Teachers College, Emporia; Vice-Chairman, Professor P. S. Pretz, St. Benedict's College; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State Teachers College, Pittsburg.

At the business meeting, the section went on record as opposing the sponsorship of high school contests at this time; however, the Executive Committee was requested to investigate the situation and present a report at the 1958 meeting.

The following short papers were presented:

1. *Implicit functions in analog computation*, by Professor P. G. Kirmser, Kansas State College.

The basic components of electronic analog computers are high gain amplifiers and function generators which perform the operations Gx and $f(z, x)$ given (z, x). Using an amplifier and function generator together enables the solution of $f(z, x)=0$, $|x|<K$, $|z|<K$ to be obtained for each x by scanning the (z, x) plane along the line x beginning at $(0, x)$. If z is multiple valued, several scannings must be made from different starting points to insure completeness of solution. A sufficient condition for stability of operation is that $f(z, x)$ given (z, x) is computed more rapidly than Gx. Generalizations lead to the solution of simultaneous implicit equations the form of which is limited only by the practical construction of adequate function generators.

2. *Operations research*, by Professor A. M. Feyerherm, Kansas State College.

The speaker gave a general outline of the methodology used in operations research. The growing interest in the use of mathematical tools and techniques to attack problems whose solutions require knowledge from a variety of disciplines was discussed. Interest in solving such problems carries various implications for future mathematical activities.

3. *Axiomatic algebra for freshmen*, by Professor W. C. Doyle, Rockhurst College.

Postulational algebra for college freshmen with a weak background presents serious pedagogical difficulties. A series of lessons was presented for discussion along with the results of their experimental use in the classroom. The natural numbers are defined in terms of their commutative, distributive, and closure properties. It is then postulated that equations of the form, $a+x=b$ and

$a \cdot x = b$, have solutions for any pair of elements a and b. A series of 23 theorems are proved to show the existence of identity elements, inverses, and their properties. The additive inverse of a is written $\underset{=}{a}$ (instead of $-a$); the multiplicative inverse is written $/a$. The theory is accompanied with a copious supply of manipulative problems.

4. *Simple circuits*, by Dr. R. W. Gibson, Wakeeney, Kansas.

Simple circuits, by electrical analogy, are defined as finite sets of independent (*SPST*) switches, represented by line segments with only endpoints in common (topological graphs); two junctions are terminals. Routes between terminals correspond to terms in the expression by the algebra of sets applied to the switches. Factorization into subcircuits has a corresponding modified factorization of the expression. The factorization into prime subcircuits is unique (parallel and series retained unfactored) and a prime subcircuit has a unique expression. An inverse of a circuit is open instead of closed (switch continuity); factorization applies; a coplanar circuit has a readily constructed inverse.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 27, 1957 at Berea College, Berea, Kentucky. Professor T. M. Wright, Berea College, presided at both the morning and afternoon sessions. There were 40 persons in attendance, including 28 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor J. C. Eaves, University of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor W. H. Spragens, Jr., University of Louisville.

By invitation of the Committee, Professor H. D. Brunk of the University of Missouri delivered an hour address at the afternoon session entitled, "Sensitivity Experiments, Stochastic Approximation, and Estimation of Ordered Parameters." An abstract of this address follows:

Sensitivity experiments (yielding "all-or-none" data) were contrasted with ordinary random sampling. Generalizations were illustrated by examples, including one representing a problem in statistical "discrimination." The examples were discussed from the distinct points of view of stochastic approximation and estimation of ordered parameters.

The following papers were presented:

1. *On bounded univalent functions*, by Professor V. F. Cowling, University of Kentucky.

In this paper results analogous to those obtained by Hardy, *Quart. J. Math.* vol. 44, 1913, pp. 147–160, for functions regular and bounded in the unit disk were obtained for that subclass of these functions which are in addition univalent in the unit disk.

2. *Finite difference approximations to partial differential equations*, by Professor W. C. Royster, University of Kentucky.

In solving partial differential equations by numerical methods the topics of convergence and stability are of prime importance. In this paper a discussion of these topics was given for a parabolic differential equation.

3. *Graphs of Chebyshev polynomials*, by Professor W. H. Spragens, University of Louisville.

The Chebyshev polynomials may be represented graphically as plane projections of trigonometric curves drawn on a circular cylinder. Through this representation, many of their properties are readily referred to familiar properties of the trigonometric functions. Among such properties are odd-or even-ness, nature and distribution of zeros and extrema, and orthogonality.

4. *Problems in matrix differential equations*, by Professor T. J. Pignani, University of Kentucky.

This note pointed out the relationship between a linear differential equation and its associated matrix differential equation. Further, bounds for the solution of an interface problem were developed as this problem is expressed in the *J. Elisha Mitchell Sci. Soc.*, vol. 72, 1956, no. 1. Throughout this note open problems were cited.

5. *A paragraph in the topology of algebra*, by Professor J. G. Horne, University of Kentucky.

An application of the Heine-Borel theorem yields the following theorem. *Let f_0 and h_0 be continuous functions over a closed interval I and suppose that for each x in I a neighborhood U and a continuous function h can be found such that $f_0(t) = h(t)h_0(t)$ for all t in U; then there is a continuous function h_1 defined over I such that $f_0(t) = h_1(t)h_0(t)$ for all t in I.*

This result can be formulated and proved in such a way as to generalize to a wide class of ideal structure theorems. The clues needed are these: 1) agree to discuss compactness (the Heine-Borel property) of a set X with respect to a collection T of its subsets, without any concern for whether T is a topology; *e.g.*, if S is a commutative ring and X is a family of its ideals, let T be the collection of subsets $O(f)$ of X which contain the fixed element f of S; 2) agree that the members M of X have the property that if f is in M then there is an e in M such that $fe = f$. The stipulation that X be T-compact yields some interesting results.

6. *Finite projective geometries*, by Professor A. W. Goodman, University of Kentucky.

This talk was an expository talk in which the speaker presented a few of the results contained in the paper of the same title by O. Veblen and W. H. Bussey, *Trans. Amer. Math. Soc.*, vol. 17, 1906, pp. 241–259.

7. *On rearrangements of infinite series*, by Mr. K. E. Stoll, University of Kentucky.

This was an expository talk based on a paper with a similar title by E. Steinitz, *J. Reine Angew. Math.*, vol. 143, 1913, pp. 128–175.

8. *Report on the Kentucky lecture program*, by Professor J. C. Eaves, University of Kentucky.

This paper is a summary of the program carried out by the elected lecturer of the Kentucky Section of the Mathematical Association of America during the two year period 1955–57. The lecturer visited more than one hundred schools during this period and spoke to between 10,000 and 15,000 high school and college students on mathematics and on the need for mathematicians, on the present demand for mathematicians and the various phases of the work that they would perform both in industry and in the teaching field. A series of questions usually asked at these meetings was discussed together with the answers used. It was also brought out that the attitude of many high school mathematics students changed considerably after these visits. It is the belief of the lecturer that more programs of this kind should be set up and especially that the Kentucky Lecture Program should be continued.

V. F. COWLING, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-third annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 26, 1957, at Lincoln, Nebraska, in conjunction with the meetings of the Nebraska Academy of Sciences. Professor J. M. Earl of the University of Omaha presided. There were 52 persons in attendance, including 31 members of the Association.

At the business meeting, the following resolution was passed:

The Nebraska Sections of the Mathematical Association of America and the National

Council of Teachers of Mathematics will jointly sponsor the national mathematics contest within Nebraska, sharing the responsibility of publicizing, administering, and grading the examinations and of making the awards. The problems of carrying out the program as provided by the national organization will be handled by a joint committee of the two sections.

The following officers were elected for 1957–1958: Chairman, Professor Edwin Halfar, University of Nebraska: Vice-Chairman, Professor J. M. Earl, University of Omaha; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *Some derived curves*, by Professor H. L. Rice, University of Omaha.

As the point P moves along a curve whose equation is given in polar coordinates, the point N (the intersection of the normal from P and the subnormal) and the point T (the intersection of the tangent at P and the subtangent) trace their respective loci. Let the radii vectors of the points P, N, and T be designated p, R, and r, respectively. Then $R \cdot r = p^2$. This relation leads to some interesting observations regarding inverse points, harmonic properties, and geometric illustrations of indeterminate forms and their limits.

2. *A study on the use of science counselors*, by Mr. M. L. Keedy, University of Nebraska.

Nebraska is one of four states which have received a grant, administered by the American Association for the Advancement of Science, to conduct a study on the use of traveling consultants to high school teachers of mathematics and the sciences. Thirty-nine selected high schools are participating in the Nebraska study. The study shows promise of: (1) focusing the attention of students, teachers, and school administrators on the importance of mathematics and the sciences; (2) raising the level of instruction in these subject-matter areas; and (3) helping to establish better cooperation between schools and colleges.

3. *The mathematics profession*, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

It is now generally agreed that engineering is a profession. In increasing degree, much of the content of theology, law, medicine, and engineering is mathematical; this implies that mathematics is a profession. Lawyers, doctors, and engineers must be examined and licensed by the state before practising. This legal standing might be advantageous to the growing number of mathematical consultants, available to all mathematicians (in whom, however, the amateur tradition remains strong).

4. *Generalization of groups*, by Professor Edwin Hewitt, University of Washington. (Address provided under the Visiting Lectureship Program of the Mathematical Association of America.)

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The forty-first annual meeting of the Ohio Section of the Mathematical Association of America was held at The University of Cincinnati, Cincinnati, Ohio, on April 20, 1957. Professor P. V. Reichelderfer, Chairman of the Section, presided at the morning and afternoon sessions. There were 61 persons in attendance including 53 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Samuel Selby, University of Akron; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Andrew Sterrett, Jr., Denison University; Program Committee: Chairman, Professor E. B. Leach, Case Institute of Technology, Professor Samuel Selby, University of Akron, and Professor H. E. Tinnappel, Bowling Green State University.

The Section voted that it is interested in sponsoring the Association high school mathematics contest in this region, and authorized the establishment of a standing committee to assume local responsibility.

The following papers were presented:

1. *On the geometrical meaning of a derivative*, by Professor P. V. Reichelderfer, The Ohio State University. (Chairman's address.)

Let T be a continuous transformation from a bounded domain in n-space into a bounded portion of n-space which is essentially of bounded variation. If C is a maximal model continuum for the point TC under T then for every sufficiently small open n-cube Q containing TC there is a unique component D of its inverse under T containing C, for which the topological index $u(x, T, D)$ is defined almost everywhere and integrable. The limit of the ratio of the integral of $u(x, T, D)$ to the measure of D as Q closes down on TC is described.

2. *Eigenvalue computation by successive suppression*, by Professor R. F. Rinehart, Case Institute of Technology.

Known methods of iteration provide in rather simple fashion the dominant eigenvalue(s) and corresponding eigenvector(s) of a square matrix over the complex field. Methods of calculating the remaining eigenvalues and vectors are less elementary. By appropriate employment of the Frobenius theorem on the eigenvalues and vectors belonging to a polynomial in a matrix, it is possible to replace successively the dominant eigenvalue(s) by zero(s), and, in effect, to put a new eigenvalue in the position of dominance. This permits the re-application of the iterative (or other) process to obtain ultimately all eigenvalues and vectors.

3. *A serial numbering system for permutations*, by Mr. A. J. Gruber, Student, Kent State University.

The paper presents a simple system for establishing a bi-unique correspondence between the set of $n!$ permutations of n distinct objects taken n at a time, and the set of natural numbers from 1 to $n!$ Given one of the permutations of n objects, it is possible to calculate the number from 1 to $n!$ associated with the permutation; and, conversely, given n and an integer from 1 to $n!$, it is possible to calculate the corresponding permutation. Some of the algebraic properties of this system and some of the problems solved with the system were also discussed.

4. *Calculation of the probability of a bivariate normal distribution over a circular region*, by Mr. H. E. Fettis and Dr. H. L. Harter, Aeronautical Research Laboratory, Wright Air Development Center, Wright-Patterson Air Force Base, presented by Mr. Fettis.

Various analytical expressions are derived for obtaining the probability of a distribution with a density given by

$$z(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp \left\{ -\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]\right\},$$

where the region is the one bounded by the xy-plane, the surface defined by the above expression, and a cylinder of radius K_x. These are well suited to either a desk or automatic digital computer.

Numerical results are presented giving those values of K for which $P(K, \sigma_x, \sigma_y) \equiv \int z(x, y)dxdy = .5$, where the integral is taken over the region $(x^2+y^2)^{1/2} \leq K\sigma_x$. These calculations were carried out on the Burroughs E-101, a small-scale digital computer.

5. *Synthesis of electric circuits as a problem in mathematical logic*, by Professor Alonzo Church, Princeton University.

Propositional calculus is used in treating circuits in which action time of the elements may be neglected. Analogue of this for sequential circuits is *restricted recursive arithmetic*, a system related to Skolem's *recursive arithmetic*. A given condition which a circuit is to satisfy may be expressed in

logistic notation, with aid of quantifiers *etc.* if necessary. The *synthesis problem* is then to find an algorithm for obtaining recursion equivalences defining the required circuit. The general solution has been found in important special cases and solution of others seems hopeful. Adaptation of the method to asynchronous circuits is possible but cumbrous.

6. *A new approach to the teaching of elementary mathematics*, by Professor B. H. Gundlach, Bowling Green State University.

A college course in mathematics for elementary teachers and the program for in-service teachers which forms an intrinsic part of it are described. Basic concepts used are those of transformation and invariance. The desirable trait to be developed in students is transform ability. The psychological approaches of association and Gestalt theories are combined to form the dynamics of the advocated approach. Emphasis is placed upon (1) a mathematically *continuous* program, (2) the *creative* aspects of elementary mathematics, and (3) *construction* of problems rather than solving. The classrooms of the in-service teachers form the life laboratories for the college trainees.

7. *Mathematics in the Cincinnati Public High Schools*, by Miss Mildred Keiffer, Cincinnati Board of Education.

All pupils in Cincinnati Public Schools are required to study mathematics in grades 7 and 8 and a minimum of one year in mathematics in grades 9 through 12. The latter must be algebra unless the pupils score below the critical score on an eighth grade mathematics test. All college preparatory students are required to study a minimum of two years of mathematics, algebra and plane geometry. Advanced standing classes for selected pupils start in grade 10. Approximately 45% of all high school graduates study three or more years of mathematics, and about 60% of all college-bound students study mathematics for three or more years.

The acute shortage of teachers with an adequate background of mathematics is a real problem. For example, the Cincinnati Public Schools must fill 25 vacancies for next year, several of which are for advanced mathematics.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The semi-annual meeting of the Oklahoma Section of the Mathematical Association of America was held at the University of Arkansas, Fayetteville, Arkansas, April 12–13, 1957. Professor O. P. Sanders, Chairman of the Section, presided. There were 42 persons in attendance including 27 members of the Association.

Copies of the employment register from recent meetings were available. Several informal luncheon discussions were arranged by congenial groups. The section experimented by permitting more time for individual papers at this meeting. The minimum time allotted was 30 minutes. Time was also arranged for informal discussion.

The ladies of the staff and distaff of the University of Arkansas served homemade sandwiches, cookies, rolls and coffee during the refreshment breaks each day.

The following discussion groups and papers were presented:

Friday, April 12:

1. Informal luncheon and discussion.

First session for contributed papers, Professor O. P. Sanders, Chairman, presiding.

2. *Functions with positive real part*, by Professor Harold Shniad, University of Arkansas.

A polynomial $f(z)$ of degree n of the form $f(z)=(1-z)\sum_{k=0}^{n-1} a_k z^k$ is shown to have positive real part in the unit circle $|z|\leq 1$ if and only if a related polynomial of degree $(n-1)$ has positive real part in the unit circle. The related polynomial $g(z)$ is given by

$$g(z) = \sum_{k=0}^{n-1} a_k + 2\sum_{k=0}^{n-1} a_j \cdot z^k.$$

Applicability of this result to extremal problems in coefficients of univalent functions is discussed.

3. *Topological vector lattices*, by Mr. J. C. Bradford, University of Oklahoma.

A locally convex topology on a vector lattice is said to be *compatible* if it has a basis consisting of ordered closed sets, *i.e.* sets for which $x \in V$, $|y| \leqq x \Rightarrow y \in V$. It is found that every total order-closed subspace of the order dual is a topological dual for some compatible topology. The finest compatible topology is the *bornological* topology whose basic system of bounded sets is the class of all order-bounded sets. Hence the finest compatible topology is the *Mackey* topology for the order dual. If there is a *complete* bornological compatible topology, it is unique and is the finest compatible topology. (This generalizes a theorem of C. Goffman, *Proc. Nat. Acad. Sc.* U.S.A., vol. 42, 1956.)

4. *A new kind of truth table*, by Professor W. E. Stuermann, University of Tulsa.

This truth table consists of an array of line segments (a line segment is assigned to each variable and its negation) arranged so that a "compartment" is defined for each of the possible sets of values which can be assigned to the variables. The use of the table was demonstrated for functions in disjunctive normal form, in conjunctive normal form, and in a form where some clauses are conjuncts of variables (or their negations). The technique is simpler and quicker, especially for functions of more than two variables, than is the use of the conventional truth table. The paper also mentions the use of this procedure as a graphical method for simplifying truth functions.

5. *Business Meeting.*

6. *Three hyperbolas associated with a triangle*, by Professor Emeritus N. A. Court, University of Oklahoma. (This MONTHLY, vol. 64, 1957, pp. 241–247.)

7. *Current availability of institutes and other supported programs for training teachers of mathematics*, by Professor J. H. Zant, Oklahoma State University of Agriculture and Applied Science.

Government money through the National Science Foundation to the amount of nine million dollars is being used to support high school and college teachers of Mathematics and Science in 95 Summer Institutes and 16 Academic Year Institutes for the period June 1957 to June 1958. In addition numerous grants from industry and private foundations are being used for similar purposes. Mathematics content, as well as science content, for these programs vary widely from review courses in traditional subject matter to thoroughly modern courses and sets of lectures in the various fields. This financial support and the opportunity for the study of content on an advanced level furnish a challenging and stimulating experience for teachers in service who wish to improve their competence in the sciences. Too little is being done to show teachers just how modern concepts in Mathematics and Science can be used in actual classroom situations. Steps are already being taken to remedy this situation. However, with the rapid expansion of these programs there is a real danger that not enough competent scientists who have ability in subject matter and interest in teaching problems can be found to man the programs.

8. Informal discussion and time for examination of the Employment Register.

An Arkansas barbecued chicken dinner was served at the Faculty Club, following which discussions and demonstrations of the mechanics of rolling spheres and distribution functions for normal cards were presented, along with an open discussion of current trends in college mathematics.

Saturday, April 13:

Second session for contributed papers, Professor O. P. Sanders, Chairman, presiding.

9. *Application of the continuously-compounded interest law to the dissipation of moisture*, by Mr. K. C. Cartwright, Vandervoort, Arkansas, introduced by the Secretary.

It is believed that when initial conditions sufficient for the solution of differential equations are known but the solutions of the equations too difficult, then it is possible, in some cases, to rely

on the continuously-compounded interest law to obtain a result sufficiently accurate for technical purposes. One such problem is the dissipation of moisture over a farm field subsequent to irrigation. Only end conditions are necessary and the need for physical constants needed for differential equations solution is eliminated. It is pointed out that cases of application are frequent enough to keep the continuously-compounded interest law always in mind for possible application when necessary.

10. *Dynamical trajectories and union curves*, by Professor C. E. Springer, University of Oklahoma.

In a paper, *Dynamical trajectories and geodesics*, Ann. of Math., vol. 30, 1929, pp. 591–606, Eisenhart showed that the trajectories of a mechanical system of n degrees of freedom correspond to the geodesics of a Riemannian space of $n+2$ dimensions in case the potential function involves the time, and that the Riemannian space has $n+1$ dimensions in case the potential function does not involve the time. Professor Springer developed the equations of union curves and showed that for the case in which the potential function does not involve the time, the dimensionality $n+1$ may be reduced to n if the geodesics are replaced by union curves with respect to a certain congruence.

11. *Some properties of non-continuous transformations*, by Professor O. H. Hamilton, Oklahoma State University of Agriculture and Applied Science.

Special types of noncontinuous transformations are defined and discussed. Among these are connectedness maps and peripherally continuous transformations. It is shown that under certain conditions a connectedness map is peripherally continuous, and that under certain other conditions a peripherally continuous transformation of a compact continuum leaves a point of that continuum fixed. Some ways in which noncontinuous transformations may arise from continuous transformations are discussed. Several examples of noncontinuous functions and transformations are given.

12. *Existence of surfaces in a certain conformal correspondence*, by Professor T. K. Pan, University of Oklahoma. (Presented by title).

Let two surfaces S and $\bar{S}$, associated with unit vector fields v and $\bar{v}$ respectively, be in conformal correspondence such that: (1) the curve of v corresponds to the curve of $\bar{v}$, (2) the asymptotic line of v corresponds to the asymptotic line of $\bar{v}$, and (3) the geodesic curvature of v along any curve C on S is equal to the geodesic curvature of $\bar{v}$ along the corresponding curve $\bar{C}$ on $\bar{S}$. This paper investigates the existence of these surfaces. Exterior differential calculus and the method of moving trihedrals are employed. It is found that the general solution depends on six arbitrary functions of one variable. Singular solutions and Cauchy problem are discussed.

13. *Boolean algebra and simple switching circuits*, by Professor W. E. Stuermann and Professor Simon Green, University of Tulsa.

This paper begins by developing Boolean algebra as an abstract system. The postulates are listed and certain derivable theorems are presented. The system is then restricted to a two-valued algebra. The peculiar characteristics of this kind of algebra are cited. The system is then interpreted in terms of switching circuits. Methods of simplifying functions are applied to the problem of simplifying circuits. The relations between switching operations, circuits and Boolean functions are described.

14. *On mappings on spheres*, by Professor J. W. Keesee, University of Arkansas, introduced by the Secretary.

A combinatorial lemma of Tucker and Fan is used to prove several theorems about sphere mappings.

15. Informal luncheon and discussion.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, on April 26–27, 1957. Professor R. B. Lyon, Chairman of the Section, presided at the afternoon session on April 26, and also at the morning session on April 27. There were 60 persons in attendance including 39 members of the Association.

The following officers were elected: Chairman, Dr. R. C. Hildner, Sandia Corporation, Albuquerque; Vice-Chairman, Professor J. H. Butchart, Arizona State College, Flagstaff; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *The equivalence of the method of Stodola to the use of Green's function in solving Sturm-Liouville systems*, by Professor L. C. Barrett, Arizona State College, Tempe, and Professor C. R. Wylie, Jr., University of Utah, presented by Professor Barrett.

The practical solution of eigenvalue problems by iterative procedures based on the so-called method of Stodola and on the use of "influence-functions," *i.e.* Green's functions, is commonplace in applied mathematics. However, it does not seem to be well known that these two methods are in fact completely equivalent, at least in the case of quite general second and fourth order Sturm-Liouville differential systems. It is the purpose of the present paper to point out this equivalence.

2. *The cardinal number of a metric space defined on* (0, 1), by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

Assuming that $2^{\aleph_0}=c$, it is proved that a certain metric space—based upon equivalence classes of subsets of (0, 1)—has the cardinal number $2^c=f$.

3. *Sides, angles, and faces of progressions*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart extended the property that if the sides of a triangle form an arithmetic progression then the join of the centroid and the incenter is parallel to a side. Using position vectors, he showed that the join of the centroid and the orthocenter (circumcenter, symmedian point) is parallel to a side if the tangents of the angles (sines of twice the angles, the squares of the sides) form an arithmetic progression. Also, the join of the centroid and the incenter of a tetrahedron is parallel to a face if the faces form an arithmetic progression.

4. *A characterization of conics*, by Mr. Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At a point V of maximum curvature on a conic γ of eccentricity e, all j-th derivatives ρ_j of the radius of curvature ρ with respect to the arc-lengths vanish for j odd; for $j=2n$, $\rho_j=[3e^2P_j(e^2)]\rho^{1-j}$ where $P_j(e^2)$ is a polynomial in e^2 of degree $(j-2)/2$ with coefficients integers. For $j=2$, $\rho\rho_2=3e^2$.

5. *Periodic sets for second order differential equations with periodic forcing functions*, by Professor E. D. Nering, University of Arizona.

Let $D(x)=\sin t$ be a second order differential equation for x as a function of t with the periodic forcing function $\sin t$. For any given solution $x(t)$ of this equation, the set of points $[x(t_n), \dot{x}(t_n)]$ where $t_n=2n\pi$, n running through the set of integers, is called a periodic set. Using an analogue computer these points can be easily plotted. Periodic sets for some elementary nonlinear second order differential equations have been plotted, in particular for Daffing's equation. The resulting patterns are strikingly systematic and have not yet received a satisfactory explanation. Some conjectures which they suggest can be easily verified for the linear case.

6. *Some remarks on monomial groups*, by Mr. A. B. Gray, Jr., New Mexico College of Agriculture and Mechanic Arts, introduced by the Secretary.

Let $\sum_m(H)$ be the monomial group of H over U where H is a group and U is a finite set.

The normal subgroups of $\sum_m(H)$ are known (see: O. Ore, *Theory of Monomial Groups*, Trans. Amer. Math. Soc., vol. 51, 1942, pp. 15–64). The set of all monomial substitutions with permutation component in the alternating group is a subgroup $\sum_{A,m}(H)$ and the normal subgroups have been determined for $m \geq 5$, $m=2$. (See: R. B. Crouch, *Monomial Groups*, Trans. Amer. Math. Soc., vol. 80, pp. 187–215.) The problem of determining the normal subgroups when $m=3$, 4 is discussed.

7. *Preliminary report on the definition of the system concept*, by Dr. D. O. Ellis, Litton Industries, Beverly Hills, California.

Although numerous theories of special systems and a few of relatively general systems have been proposed, there is, in the writer's knowledge, no theory encompassing those things generally referred to as systems and/or operations and meeting the basic criteria: (1). potential of specialization to any desired degree, (2). avoidance of restricting notions inherently "non-realizable physically" (*e.g.*, the infinite tape concept of Turing's theory), and (3). sufficient generality to permit coverage of all presently known and predicated types of information processing.

Presented in this communication is a tentative definition of the concept upon which such a theory would be based. It is believed the definition permits satisfaction of the three criteria above and yet is not so general as to be meaningless. The notions employed are elementary ones from set theory and lattice theory.

8. *Optimum rocket motion and the satellite problem*, by Professor L. C. Barrett, Arizona State College, Tempe, and Dr. Herbert Knothe, Holloman Air Development Center, Holloman, New Mexico, presented by Professor Barrett.

This paper is concerned with optimum problems of the following type:

"Let a rocket, with an arbitrary initial velocity, be required to expend its fuel continuously during the course of a given flight time. Moreover, suppose that at the instant the flight time terminates the rocket is to have a prescribed final velocity. Of all paths the rocket may pursue in fulfilling these conditions we desire the one(s) over which a minimum of fuel is required."

A path thus characterized is defined as an optimum trajectory corresponding to the prescribed velocity conditions. In addition to giving a necessary and sufficient condition for optimum rocket motion in a constant gravity field, we also find necessary conditions for optimum rocket motion in a general central force field.

9. *The mathematics curriculum and the computer*, by Dr. H. R. J. Grosch, General Electric Company, Tempe, Arizona.

10. *The geometry of variation of parameters*, by Professor R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.

The effect of perturbation of a homogeneous second order differential equation is studied through the change in shape of its solution in the phase plane. The variation of parameters technique for finding a particular integral of the perturbed equation can be interpreted as a change of coordinates in this phase plane, and the usual conditions imposed on the parameters are easily explained in this setting.

11. *Ways and means for the solution of boundary value problems in engineering*, by Dr. M. A. Dengler, AiResearch, Phoenix, Arizona.

Most of the nonlinear partial differential equations in engineering can be linearized by maintaining the coefficients of the second derivatives of the unknown function as functions of the independent variables of the system, and following a scheme of consecutive iterations. The functions indicated are kept constant during one phase of the iteration and changed from phase to phase. In this perspective also nonlinear equations of the second order can be subdivided into the categories of elliptic, parabolic and hyperbolic equations. Methods for solving all three types by finite

differences and matrix iteration were illustrated. Methods are specifically suited to modern means of high speed computation.

12. *Model making and the computer*, by Dr. H. R. J. Grosch and Dr. F. B. Thompson, General Electric Company, Tucson, Arizona.

13. *Subordination in complex variable theory and a theorem on minimal surfaces*, by Mr. J. H. Evans, Los Alamos Scientific Laboratory, introduced by the Secretary.

A definition of subordinate functions is given, and theorems concerning them by G. M. Goluzin and E. Reich are stated. A definition of minimal surfaces is stated and a convenient notation for their representation is presented. Then the following theorem is proved for pairs of minimal surfaces whose coordinate functions are subordinate: *If f_j and F_j are the coordinate functions of two minimal surfaces given in conformal representation and f_j is subordinate to F_j, then the minimal surfaces are plane surfaces and the F_j surface is a subset of the f_j surface.*

14. *On two lemmas of F. Riesz*, by Professor Oswald Wyler, University of New Mexico.

The theory of Lebesgue integration can be based on two simple lemmas on integrals of step functions, as shown in: F. Riesz and B. Sz.-Nagy, *Functional Analysis*, New York 1955 (*cf.* pp. 30 and ff.). A proof of these two lemmas is given for the most general case that a measure μ is defined as a countably additive set function in a ring $\mathfrak{R}$ of sets, a step function being defined as a linear combination of characteristic functions of sets in $\mathfrak{R}$.

15. *Guessing on multiple choice tests*, by Professor R. F. Graesser, University of Arizona.

If a student guesses at random on a multiple choice test, the expected value of his score is zero, when his score is obtained by the usual formula. It can be proven that the odds never favor a guesser on a true-false test. However, if there are 3, 4, or 5 answers from which to choose, then an investigation of 190 special cases yields the empirical rule that the odds favor the guesser if, and only if, the number of questions plus one is divisible by the number of choices.

16. *Principal ideal domains with no division algorithm*, by Professor D. W. Dubois and Dr. Arthur Steger, University of New Mexico, introduced by the Secretary.

It is already known that the ring of algebraic integers in $K(\sqrt{-m})$ (K is the field of rationals) with m a positive, square-free integer, is a principal ideal domain when $m=19$, 43, 67, and 163, but the domain is not euclidean according to the norm. The case $m=19$ was known to Dedekind. In this paper we show that none of the above domains is euclidean by *any* function. The methods do not apply to real quadratic fields.

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The spring meeting of the Texas Section of the Mathematical Association of America was held at the University of Houston, Houston, Texas, on April 26–27, 1957. Professor Don Cude and Professor Martin Wright presided at the regular sessions. There were 120 persons in attendance, including 80 members of the Association.

At the business meeting the following officers were elected for the year 1957–58: Chairman, Professor Martin Wright, University of Houston; Vice-Chairman, Professor E. K. McLachlan, Baylor University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were presented:

1. *Some loci of the Lemoine point of a triangle*, by Professor C. P. Benner, University of Houston.

Let P be any point in the plane and let L be the Lemoine point of a certain triangle associated

with P. If P is allowed to vary according to specified restrictions, the Lemoine point describes, among other loci, the four-leaf rose, the lemniscate, the cissoid of Diocles, and the bifolium.

2. *Wronskian determinants and linear dependents*, by Professor W. T. Guy, Jr., University of Texas.

Students seem to have a rather difficult time learning the correct relationships between Wronskian determinants and linear dependence. An alternative determinant is suggested which might be considered as a more natural criterion for linear dependence.

3. *8-Rings in minimal maps*, by Mr. B. T. Goldbeck, Jr., Texas Christian University.

For an irreducible ring of eight regions, three sets of criteria are established. These criteria are Kempe equalities (E), primary inequalities (P), and secondary inequalities (S). Each peripheral 8-ring, not known to be reducible, was tested by these criteria. The entire class of peripheral 8-rings were shown to be irreducible, (E), (P) and (S). Algebraic cases were constructed, which though undrawable, also satisfied these 1,180 criteria.

4. *Prime divisors of sequences of integers*, by Professor L. K. Durst, Rice Institute.

The Fibonacci numbers are given by the recursion $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$. For each n except 6 and 12 there is a prime p such that $p \mid F_n$ but $P \nmid F_m$, $0 < m < n$. The Fibonacci sequence F_n is a special Lehmer sequence P_n defined by the recursions $P_{2n} = P_{2n-1} - MP_{2n-2}$, $P_{2n+1} = LP_{2n} - MP_{2n-1}$, $P_0 = 0$, $P_1 = 1$, $L > 0$. Suppose $(L, M) = 1$ and $K = L - 4M > 0$. An index n greater than 2 is called *exceptional* if $p \mid P_n$ implies $p \mid \prod_{m=1}^{n-1} P_m$. For these Lehmer sequences the following results are proved. The only exceptional indices are 6 and 12. The index 12 is exceptional only when $L = 1$, $M = -1$ or $L = 5$, $M = 1$. The index 6 is exceptional if and only if K is odd, $L = 2^{N+2} - 3K > 0$, $M = 2^N - K$. Compare M. Ward, *Ann. of Math.* (2), vol. 62, 1955, pp. 230–236.

5. *Some formulas and congruences involving Bernoulli numbers and Fibonacci numbers*, by Mr. R. P. Kelisky, University of Texas.

The polynomials $F_n(x)$ which have the generating function $2e^{xt/2} \sinh (t/2)$ reduce to combinations of Fibonacci and Lucas numbers for certain values of x. Simple formulas expressing sums of products of binomial coefficients, Fibonacci numbers, and Bernoulli numbers in terms of Fibonacci and Lucas numbers may be obtained by multiplying the generating function for the sequence $F_n(x)$ by the well-known generating function for the Bernoulli polynomials. From these formulas are obtained several congruences with respect to a prime modulus involving the numbers of Bernoulli, Fibonacci, and Lucas.

6. *Analytic functions of a bicomplex variable*, by Professor G. B. Price, University of Kansas. (By invitation.)

Let a and b be complex numbers. It has been customary to call $a + jb$ a bicomplex number if addition, subtraction, multiplication, and division are formally the same as for the complex numbers, with $j^2 = -1$. The bicomplex numbers contain divisors of zero. This paper contains an outline of a theory of analytic functions of a bicomplex variable $x + jy$. The theory has striking similarities and dissimilarities with the theory of functions of a complex variable.

7. *A class of k-sets of polynomials*, by Professor F. J. Palas, Southern Methodist University.

A k-set of polynomials is defined as a sequence of polynomials such that $P_{kn}(x)$ is of degree exactly kn. A linear differential operator is developed which has the property that $L[P_{kn}(x)] = P_{k,n-1}(x)$. A class of k-sets is characterized by the generating function, $g(x, t) = f(t) \exp [p(x)u(t)]$, where $p(x)$ is a polynomial of degree k such that $p(0) = 0$, and $f(t)$ and $u(t)$ are power series. Several recurrence relations are developed and are used to classify the k-sets as B_k-type and C_k-type in analogy with Sheffer's B-type and C-type classes. The equivalence of these classes is shown.

8. *A generalization of Bernoulli numbers and Bernoulli polynomials*, by Professor L. L. Silverman, University of Houston.

To each system consisting of a function $f(x)$, analytic about the origin, and of a positive integer p, there is associated a function $B(x)$ which generates the generalized Bernoulli numbers. The Euler numbers, as well as the ordinary Bernoulli numbers, and other well-known sets of numbers are included as special cases. Corresponding to each system, $f(x)$ and p, there is also defined a set of polynomials, which includes as special cases well-known sets of polynomials. An interesting feature is the novel algebra connecting the generating functions and their corresponding associated functions $f(x)$.

9. *A non-reflecting solution of Maxwell's equations*, by Professor J. T. Hurt, Agricultural and Mechanical College of Texas, introduced by the Secretary.

The early examples of nonreflecting waves in nonhomogeneous media required the assumption of properties of the media which are physically impossible. A new example is given in a medium with properties which are physically possible.

10. *Axioms of a generalized projective geometry*, by Professor D. E. Edmondson, Southern Methodist University.

A generalization is given for the axioms of a projective geometry. The points constitute a partially ordered set. When there are no orderings among the points, the axioms become the usual Veblen-Young axioms of a projective geometry. From the axioms it follows that the lattice of subspaces is a modular lattice and that generalized projective geometries serve as representation spaces for modular lattices.

11. *Properties of the family of curves generated from a base curve $\rho_0=\rho_0(\theta)$ by the transformation $\rho_n=g^n(\theta)\rho_0$*, by Professor F. S. Nowlan, University of Illinois, and Mr. A. A. Aucoin, University of Houston, presented by Mr. Aucoin.

The family of curves $\rho_n=g^n(\theta)\rho_0$ generated from a base curve $\rho_0=\rho_0(\theta)$ is shown to have many algebraic invariants. Some of these invariants have simple geometric interpretation.

12. *On the Gaussian curvature of modular surfaces*, by Mr. Daniel Weiser, Rice Institute.

Consider an analytic function $w=f(z)$ which is, with its first two derivatives, not zero at z_0. If $K(z_0)$ is the Gaussian curvature at the point $P(z_0, |f(z_0)|)$ on the associated modular surface, then $K(z_0)$ is greater than, equal to, or less than 0 if and only if $\Re\{[f'(z)]^2/[f(z_0)f''(z_0)]\}$ is greater than, equal to, or less than 1. The principal result, however, is the evaluation of the Gaussian curvature at the zeros of $f(z)$ and its derivatives.

13. *Some probability distributions for sociometric matrices*, by Professor P. D. Minton, Southern Methodist University.

A "sociometric matrix" has elements representing data collected in certain sociological investigations. Probability distributions obtained under acceptable assumptions are required to analyse the data. Assuming elements 0 or 1 with the main diagonal null, and assuming random entries in each row with row totals fixed, probability distributions related to column totals are given. Equal and unequal row totals are considered. The joint distribution of two column totals and of the sum of two column totals and the conditional distribution of the difference of two column totals given their sum are obtained or approximations indicated.

14. *The plane in life and mathematics*, by Professor R. S. Underwood, Texas Technological College.

Although a large part of the heritage of mankind is stored on the flat pages of books, the branch of mathematics which uses a plane for illustrative purposes—namely, plane analytic geom-

etry—is decidedly skimpy in its coverage, being restricted to two variables. It is therefore contended that a plane analytic geometry for n variables deserves far more attention and priority as a goal than mathematicians as a group have thus far accorded it. The writer's lone trek on the road to this goal is reviewed in its broader aspects, and some specific new results are described.

15. *Mathematics in the general education program of colleges and universities in the United States*, by Professor W. I. Layton, Stephen F. Austin State College.

This study covers 150 colleges and universities in 42 states. It is an investigation of the mathematics required and recommended for the general education program. General mathematics is the course most frequently required and most highly recommended for this program. The mean of mathematics recommended is 4.82 semester hours. This is in contrast to a mean of 1.97 semester hours of required mathematics in colleges which have general education programs. The study also presents topics which might be included in college general mathematics.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-sixth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois State Normal University, Normal, Illinois on May 10 and 11, 1957. Professor F. E. Hohn, Chairman of the Section, presided at all sessions. There were 63 persons in attendance, including 55 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor C. T. McCormick, Illinois State Normal University; Vice-Chairman, Professor A. E. Hallerberg, Illinois College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University.

Professor Joseph Stipanowich reported on the work of the Committee on Contests and Awards, stating that 4800 students from 120 different high schools participated this year, this being 1000 more than a year ago and 2000 more than two years ago. The Section then voted to cooperate with the National Committee on Contests in Mathematics for High School Students to the extent permitted by the Illinois High School Association. Professor Stipanowich pointed out that this Association will not approve the publication of scores made by the students until the contest has been approved by the National Association of Secondary School Principals, but his committee is working to get the approval of this Association.

Professor A. E. Hallerberg reported for the Committee on the Strengthening of the Teaching of Mathematics. Following his request for guidance of the Committee's efforts the Section authorized the Committee to pursue the idea of initiating a summer work camp in mathematics for high school students in Illinois, part of the necessary funds to come from the Section's treasury.

The following papers were presented:

1. *One hundred years of mathematics in Illinois*, by Professor Emeritus E. H. Taylor, Eastern Illinois State College.

The paper dealt with the preparation of elementary teachers and the teaching of arithmetic, influences leading to the improvement of the teaching of mathematics, and the increasing importance of mathematics and its teaching.

2. *Some comments on the finite induction postulate*, by Professor L. D. Rodabaugh, Southern Illinois University.

The paper presented a point of view on the Peano Postulates, particularly the Finite Induction Postulate, which may prove helpful to persons teaching courses in part dependent upon these postulates. In particular it is believed that the examples here provided will be pedagogically beneficial.

3. *Topics from matrix algebra suitable for college freshmen*, by Professor Arnold Wendt, Western Illinois State College.

Because most college freshmen have never encountered any but the real and complex number systems, they think that study of the postulates for these systems is a waste of time since these properties are obvious. The speaker has found 2×2 matrices can serve as an example of a noncommutative system with zero divisors which is simple enough to understand yet impressive enough to increase appreciation of the role of postulates and definitions.

Matrices can also be introduced quite naturally during the study of determinants and simultaneous linear equations.

4. *Experiences of a wandering mathematician*, by Professor W. W. Sawyer, University of Illinois, introduced by the Chairman. (By invitation.)

This paper presented some recollections of the English school system, allowing for very rapid progress of mathematically inclined students, its weakness with non-academic students; the New Zealand system as a contrast, satisfactory for least academic students, very unsatisfactory for scholars; lack of flexibility in both systems; no reason seen why an educational system should not cater to all types; some experiences in teaching English pupils of low intelligence, and the use of apparatus to secure interest.

An account was given of a high school mathematics society in New Zealand and its effect in increasing the supply of mathematics teachers.

5. *A supplementary training program for mathematics teachers*, by Professor Joseph Landin, University of Illinois.

The speaker outlined the supplementary training program for secondary school mathematics teachers at the University of Illinois. The basic courses of this program deal with the construction of the real number system beginning with the elements of set theory; elements of logic; axiomatic systems; analysis of high school geometry; Hilbert's axioms. Additional courses of the program are: the foundations of the calculus; modern algebra and geometry. Pedagogical methods and new curriculum developments in secondary school mathematics are studied, attention being given to the work of the University of Illinois Committee on School Mathematics.

6. *Problems encountered in industry by a junior mathematician*, by Professor W. J. Thomson, Western Illinois State College.

Three problems which the speaker encountered while working in an industrial position were presented. These problems were given to junior mathematicians to solve. The problems included one in which experimental data was to be tabulated and the results presented in usable form. Another involved the solution of a differential equation by numerical integration, and the third was concerned with determining an estimate of the maximum possible rates and accelerations from functions which did not lend themselves to direct solutions.

7. *Sequential machines*, by Professor D. D. Aufenkamp, University of Illinois.

Roughly speaking, sequential machines are devices designed to make certain sequences of "outputs" correspond to given sequences of "inputs." A simple, abstract model of many sequential machines has been proposed by G. H. Mealy. This model assumes finitely many "inputs," "outputs," and "states." Its present output and next state are determined uniquely by the present input and the present state. A representation of the model as a directed graph provides a convenient starting point for investigations into problems of analysis and synthesis of such machines. The speaker discussed several of these problems, and indicated how they could be solved.

8. *Geometry in transition*, by Professor D. R. Bey, Illinois State Normal University.

Evidence that the content and objectives of geometry are undergoing a period of adjustment is readily apparent. For example, the number of theorems considered basic to plane geometry

has been reduced from 212 to 30 in the past 25 years. Results of studies lend support to the soundness of the reasons for the present swing away from the "critical thinking in life situations" approach to geometry. Other findings suggest adding solid geometry, analytic geometry and topics from "other geometries" at the 10th grade level. The studies also recommend that geometry offerings at the college level should give first priority to a course in projective geometry and should include courses in noneuclidean geometries and fundamental concepts of geometry.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Maryland, on May 4, 1957. The meeting was sponsored jointly by the Association and the Baltimore, Maryland and Washington, D. C. Sections of the Society for Industrial and Applied Mathematics. Professor R. C. Yates, Chairman of the Section, Dr. J. K. Sterrett, Chairman of the Washington, D. C. Section of SIAM, and Dr. Michael Aissen, Chairman of the Baltimore, Maryland Section of SIAM, presided at joint sessions. There were 104 persons in attendance, including 84 members of the Association.

The following officers were elected to serve for a period of one year: Chairman, Professor R. P. Bailey, U. S. Naval Academy; Vice-Chairmen, Professor M. G. Humphreys, Randolph-Macon Woman's College and Professor C. A. Spicer, Western Maryland College; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The following contributed papers were presented:

1. *The Bureau of Ships computer program*, by Mr. A. E. Smith, Navy Department, Washington, D. C.

The Bureau of Ships is presently operating two Univacs at the David Taylor Model Basin and two IBM-650's at naval shipyards. A wide variety of problems in both engineering and management fields have been solved. These were concerned with nuclear reactor design, magnetic minesweeping, vibration of plates, determination of ships lines, pipe stress analysis and allocation of electronic equipment to ships. The last problem was formulated as a linear programming problem.

2. *The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations*, by Dr. T. N. E. Greville, Social Security Administration, Washington, D. C.

It is not generally known that E. H. Moore generalized the notion of inverse of a matrix to include all nonzero matrices, rectangular as well as square. The general solution of any system of linear equations which has an exact solution can be conveniently expressed in terms of this pseudoinverse. The additional fact that his procedure gives a best solution in the sense of least squares when there is no exact solution was noted for a restricted class of systems of Bjerhammar, who was unaware of Moore's work and did not appreciate the generality of his result.

3. *An interesting fourth order differential system*, by Mr. C. H. Murphy, Jr., Aberdeen Proving Ground, Maryland.

Although the theory of linear differential equations with constant coefficients is well known, most of the examples and problems given in texts have characteristic equations which are either first or second order or are easily factorable. In this paper an important subset of the set of all fourth order differential systems is described. The solution of members of this subset can be obtained by the solution of at most two quadratic equations instead of the usual quartic equation.

4. *Solution of the integrodifferential equation of transfer by successive approximations*, by Dr. P. M. Anselone, Radiation Laboratory, Johns Hopkins University.

The equations

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - 1/2 \int_{-1}^{+1} I(\tau, \mu')d\mu', \int_{-1}^{+1} I(\tau, \mu')\mu' d\mu' = F \text{ (a constant)},$$

where $0 \leqq \tau < \infty$ and $-1 \leqq \mu \leqq 1$, plus certain auxiliary conditions, define a classical problem in transfer theory. The existence and uniqueness of the solution was established by E. Hopf, who obtained the Neumann series solution of a related integral equation. The Wick-Chandrasekhar technique for approximating I involves replacing the two integrals above by sums corresponding to the $2n$ point Gauss quadrature formula. The resulting problem is solved to yield an approximation, I_n, to I. The "double-Gauss" formula, in which the n point Gauss formula is applied separately to each of the intervals $-1 \leqq \mu \leqq 0$ and $0 \leqq \mu \leqq 1$, also yields an approximation. The principal result obtained is that the sequence $\{I_n: n \geqq 1\}$ corresponding to the double-Gauss formula converges to I uniformly on each compact subset of the domain of I.

5. *Matrix analysis for production scheduling and inventory control*, by Professor D. N. Chorafas, Catholic University of America.

With the use of high speed electronic data processing systems, mathematical techniques which seem to be very complicated or unduly involved became of importance and interest for the solution of engineering production problems.

Matrix analysis can be used to advantage for the solution of problems in Engineering Production. Commodity requirements for the initial, the intermediate and the final steps can be set in the form of a rectangular matrix. Then with a simple matrix multiplication engineering management is able to study the input-output requirements of any production system.

The speaker discussed the mechanics of the method from the conception of the model to the data processing through an electronic digital computer and evaluated the method with respect to its potentialities for future application in industry.

6. *Intermittent rotors*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department, Washington, D. C.

The shape of the least area which, when placed at random on a square lattice of points, always includes at least one of the points was shown by J. J. Shaffer and D. B. Sawyer to be a square to which has been added the areas included by two parabolic arcs, one on each of two opposite edges of the square. The speaker showed that this shape is one of a family of convex curves which may be rotated through all orientations in the plane while passing through at least three of the four vertices of a square. Extensions to a series of similar problems were indicated.

In addition, the following hour lectures were presented by invitation of the joint program committee:

1. *Geometry in the mathematics curriculum*, by Professor W. L. Chow, The Johns Hopkins University (auspices of MAA).
2. *Quaternions and Clifford numbers*, by Professor Marcel Riesz, Institute of Fluid Dynamics, University of Maryland (auspices of SIAM).

R. P. BAILEY, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The fortieth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, on Friday afternoon and evening and Saturday forenoon, May 3 and 4, 1957. Professor R. R. Gutzman, Chairman of the Section, presided at all three sessions. There were 116 persons registered for the meeting, including 68 members of the Association.

Officers elected at the meeting for 1957–1958 were: Chairman, Professor D. O. Patterson, Colorado State College; Vice-Chairman, Professor N. C. Hunsaker, Utah State Agricultural College; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *On generalized Legendre polynomials*, by Professor Arne Magnus, University of Colorado, introduced by the Secretary.

A recurrence formula is developed for the polynomials $P_k = P_k(\phi_1, \cdots, \phi_n)$ defined by $[1+\phi_1 t + \cdots + \phi_n t^n](m/n) = 1 + P_1 t + \cdots + P_1 t^k + \cdots$ and application made to the polynomial solutions of the partial differential equation $u_x \cdot v_y - u_y \cdot v_x = 1$ where $u = u(x, y)$ and $v = v(x, y)$.

2. *The Laplace transform in discontinuous solutions of a partial differential equation*, by Professor V. W. Bauman, Colorado School of Mines, introduced by the Secretary.

Using the Laplace transform to solve problems involving the equation (1) $Y_{tt}(x, t) = a^2 Y_{xx}(x, t)$, equation is assumed to be (2) $s^2 L\{y\} = a^2(\partial^2/\partial x^2)[L\{Y\}]$. In some problems the solution, $Y(x, t)$, or its partial derivatives of first order are only sectionally continuous. In this case the members of (2) are not the transforms of members of (1), both members of (2) having additional terms which involve the salti in the discontinuous functions. It was shown in this paper that the true transformed equation always reduces to equation (2) if the solution or its partial derivatives of first order have only finite discontinuities on the lines in the xt-plane, $t = C \pm x/a$.

3. *The economic index numbers of Divisia*, by Professor E. A. Davis, University of Utah.

An expository account of the "historical" index numbers, for prices and quantities of goods traded, due to Divisia (F. Divisia, *Economique Rationelle*, Paris, 1928) was presented. Relationships between these quantities and the index formulae of Laspeyres were noted and, in particular, a device for approximating the former by means of products of Laspeyres indexes was indicated.

4. *A mathematical analysis of fuel burnout in nuclear reactors*, by Captain A. W. Banister, United States Air Force Academy, introduced by the Secretary.

Mathematical analysis of nuclear reactors is necessary in order to provide quantitative information regarding certain design and operational features. One such item is the effect of fuel burnout in modern reactors operating at high power levels. An approach to this problem can be made by writing a differential equation descriptive of a generalized reactor volume element, and introducing perturbations in the reactor constants to simulate burnout. By making certain approximations justifiable on physical grounds, the equation can be transformed into a linear, non-homogeneous type, easily solvable by the operator method. The solution then provides quantitative information on changes in the power distribution function, and adjustments necessary to maintain level operation.

5. *A characterization of n-groups*, by Professor D. W. Robinson, Brigham Young University.

The generalized groups defined by W. Dörnte (*Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z., vol. 29, 1928, pp. 1–19) are systems of elements with a polyadic operation satisfying an extension of the associativity and solvability axioms for ordinary groups. This note points out that these systems can be characterized as well by replacing the solvability axiom with a generalization of the identity-inverse axiom for groups.

6. *Infinite exponentials*, by Professor W. J. Thron, University of Colorado.

For every $n > 1$ let $t_n(z) = e^{a_n z}$, where the a_n are complex numbers. Define $T_n(z)$ to be: $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$. Then $\{T_n(1)\}$ is called an infinite exponential. It is proved that an infinite exponential converges if $|a_n| \leq e^{-1}$ for all n.

7. *Speed-up college mathematics*, by Professor I. L. Hebel, Colorado School of Mines.

A review of the progress of a highly selected group of twenty entering freshmen who have been allowed to progress at an accelerated pace with a view to completing 21 semester hours of college mathematics (through Differential Equations) in three semesters. From the results of this first attempt to do something for the better-than-average student, the conclusion reached is that many additional students taking the "standard" courses could profitably be placed in the accelerated program. It is anticipated that about 50 of the next group of 300 entering freshmen will be assigned to a similar group.

8. *Antenna theory*, by Dr. James Wait, Boulder Laboratories, National Bureau of Standards. (Invited Address.)

The calculation of the radiation field of a flush mounted antenna in the tangent plane (the classical light-shadow boundary) is not readily treated by either geometrical optics or the residue-series. In the former case the field is indeterminate and in the latter case the convergence is extremely poor and would actually diverge in the illuminated region. Despite the fact that the harmonic series is cumbersome, it is valid in this transition zone between the illuminated and shadow regions of space. Therefore, it is desirable to attempt to adapt the harmonic-series representation to surfaces of large radius of curvature. This is the purpose of the present paper.

9. *Orthogonal functions whose k-th derivatives are also orthogonal*, by Professor F. M. Stein, Colorado State University.

Several sets of orthogonal functions possess the property that the k-th derivatives of these functions form sets of orthogonal functions, perhaps with a different weight function, $w_k(x)$, but over the same interval. This paper shows that the set $\{1, \cos nx, \sin nx\}$ possesses this property. Also, if the orthogonal functions are polynomials they must be those of Hermite, Jacobi, or Laguerre; and these are the only polynomials possessing this property under the definition of orthogonality that for $\{\phi_n(x)\}$,

$$\int_a^b w(x)\phi_n(x)\phi_m(x)dx = c_n\delta_{mn}.$$

10. *Introduction to SOMAC*, by Professor R. R. Gutzman, Colorado School of Mines.

The speaker explained the basic operating exponents of the analog computer. He discussed the methods of programming a differential equation of the type

$$a_0 \frac{d^2y}{dx^2} + a_1(dy/dx)^2 + a_2y = F(x),$$

and simultaneous linear algebraic equations. Different ways of generating $F(x)$ were considered.

11. *The relation of regular semigroups to groups*, by Mr. H. G. Moore, University of Utah.

In this paper theorems are given which relate regular semigroups and inverse semigroups with groups. A regular semigroup with cancellation is a group as is every regular semigroup generated by a single element. Cancellation may be relaxed slightly if certain other conditions are imposed on the regular semigroup. Every regular semigroup possesses subsets which are groups, and if not all the elements of the semigroup are idempotent it possesses nontrivial subsets which are groups.

12. *Likes and dislikes—when and why?*, by Professor O. M. Rasmussen, University of Denver.

A preliminary report of a survey in progress as an attempt to find when and why students like or dislike mathematics. Thus far it appears that grades 7, 9, and 10 are the critical ones with the teachers receiving credit or blame in most cases. Parental encouragement appears to be a factor

among those who now like mathematics. Information reported is from a survey of students at all levels in a university. Therefore, the results of changing conditions of the past five years are not included.

13. *The new Bachelor of Science Degree in Applied Mathematics at the University of Colorado*, by Professor L. W. Rutland and Professor J. R. Britton of the University, presented by Professor Rutland.

An announcement was made of the new degree being offered in the Department of Applied Mathematics at the University of Colorado. The curriculum includes study in the basic engineering sciences and applied mathematics subjects. The applied mathematics work includes intermediate differential equations, advanced calculus (*e.g.*, Taylor, *Advanced Calculus*), introduction to applied mathematics (*e.g.*, Wylie, *Advanced Engineering Mathematics*), statistics, algebraic methods, computing machines, and a senior seminar.

F. M. CARPENTER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, DePauw University, Greencastle, October 18, 1957.

IOWA

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA, State Teachers College, Mankato, October 5, 1957.

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Fairleigh Dickinson University' Rutherford, November 2, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA, Oklahoma City University, October 25, 1957.

PACIFIC NORTHWEST

PHILADELPHIA, November 30, 1957.

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

THE OTTO DUNKEL MEMORIAL PROBLEM BOOK

Edited by

HOWARD EVES AND E. P. STARKE

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PART II

The Otto Dunkel Memorial Problem Book

Edited by
HOWARD EVES and E. P. STARKE

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ONE DOLLAR A COPY

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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FOREWORD BY THE EDITORS

When Professor Otto Dunkel died in 1951, it was found that he left a bequest to the Mathematical Association of America. After deliberation, the Board of Governors of the Association voted that the income from Professor Dunkel's bequest be spent in publishing an *Otto Dunkel Memorial Problem Book*, based upon the better problems that have appeared over the years in the AMERICAN MATHEMATICAL MONTHLY. This was a very fitting decision, for Professor Dunkel had served for twenty-eight years as an editor of the MONTHLY Problem Department and is the one chiefly responsible for the Department's growth and high standards. A committee was formed, and in due course the present editors of the Problem Department were asked to assemble the book.

Although we approached the task with enthusiasm, the pressures of present-day college teaching bogged us down, and the assembling of the Problem Book progressed at a discouragingly slow pace. For this we offer our sincerest apologies. We fully realize the shortcomings of certain parts of the book—shortcomings that might not have been so serious had our thinking and work on the project not been so often interrupted for lengthy periods of time.

It was decided that the book should contain a brief biography of Professor Dunkel, a short account of some of the more human aspects of the Problem Department, a selection of the four hundred "best" problems proposed between 1918 (the year in which consecutive numbering of problems was begun, and the year in which Professor Dunkel joined the MONTHLY staff) and 1950, a classification of all problems proposed in this period, and, finally, an index of all these problems so that a reader could locate them and their solutions in the pages of the MONTHLY.

Professor P. R. Rider, a long time friend and colleague of Professor Dunkel, agreed to furnish the biography, and Dean C. W. Trigg, one of the country's noted problemists, agreed to write the human interest section. We deeply thank these men for their very kind assistance.

Since we found it difficult to construct a satisfactory definition of a "good" problem, we decided to secure the composite judgment of a number of mathematicians who, over the years, had shown interest in the MONTHLY Problem Department. These mathematicians sent us their selections of "best" problems, and from these selections we proceeded, largely on the basis of number of votes, to make the final choice. We owe far more to the generous cooperation of our

correspondents in this matter than is implied by the following simple listing of their names:

T. M. Apostol	Fritz Herzog	C. D. Olds
Leon Bankoff	J. B. Kelly	George Piranian
I. A. Barnett	M. S. Klamkin	O. J. Ramler
P. T. Bateman	A. E. Livingston	L. A. Ringenberg
Leonard Carlitz	George Lorentz	M. R. Spiegel
W. B. Carver	D. C. B. Marsh	C. W. Trigg
J. S. Frame	B. H. McCandless	Chih-yi Wang
S. H. Gould	Leo Moser	Albert Wilansky

The chief bugbear of the whole project was the formation of an appropriate classification of all problems proposed between 1918 and 1950, and it is undoubtedly this part of the book that has suffered most from our numerous interruptions. Probably no classification can be entirely satisfactory, but we do hope that the one we finally achieved may be of some use to teachers of courses in college mathematics, to writers of college textbooks, to mathematics clubs and seminars, and to individuals looking for interesting related problem material. The MONTHLY can rightfully boast of possessing one of the finest and most diversified problem departments on the collegiate level. There is evidence to the effect that a large number of research papers and theses for college degrees have had their origin in the MONTHLY Problem Department; it probably would be surprising if we could learn how many. In this connection, the listing of the 251 still unsolved MONTHLY problems for the given period may be of some interest and may stimulate some graduate and undergraduate research. For aid in the tedious task of classifying the nearly three thousand problems, the editors owe much to the expert assistance of Professors D. A. Kearns and S. H. Kimball.

With this we conclude our very inadequate explanation, apology, and thanks.

HOWARD EVES, University of Maine
E. P. STARKE, Rutgers University

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THE MONTHLY PROBLEM DEPARTMENTS, 1894–1954

C. W. TRIGG, Los Angeles City College

At its inception, the AMERICAN MATHEMATICAL MONTHLY was essentially a problem journal. Conceived and founded by B. F. Finkel at a time when it was difficult to obtain good articles from outstanding mathematicians, the early issues consisted largely of problems—proposed and solved. When Finkel's plan to start the publication began to crystallize, he invited J. M. Colaw to join him. He had never met Colaw, but had been impressed by his contributions, mostly problems, in the *School Visitor*, to which both were avid subscribers.

The editors. Under this joint editorship the MONTHLY was launched with the announced purpose of being "devoted to the solution of problems in pure and applied mathematics, papers on mathematical subjects, biographies of noted mathematicians, etc." The problems were classified under arithmetic, geometry, mechanics, and probability—edited by Finkel; and algebra, calculus, diophantine analysis, and miscellaneous—edited by Colaw. In the first year, the distribution of the problems was arithmetic—41, algebra—43, geometry—36, calculus—33, mechanics—19, diophantine analysis—23, probability—21, and miscellaneous—21. Solutions to 166 of the 237 proposals were published by the end of the year. At first, problems were numbered consecutively in each of the eight sections.

Finkel's interest in problems stemmed from a senseless one which an older half brother heard at the village grocery store and brought home for him to think about. It was: "There is a ball 12 feet in diameter on top of a pole 60 feet high. On the ball stands a man whose eye is 6 feet above the ball. How much ground beneath the ball is invisible to him?" Finkel, 15 years old, was unable to solve the problem with the rules of mensuration which had been made available to him by that time in his small Ohio country school. His teacher indicated that it might be solved by geometry, which Finkel did several years later.

This problem, which later found its way into Finkel's *Mathematical Solution Book*, so impressed Finkel in connection with his own introduction to mathematics that throughout his life he advocated the educational value of a good senseless problem as against that of the so-called practical problem which is so insisted upon by some modern educators. In the introduction to the first issue of the MONTHLY he said: "the solution of problems is one of the lowest forms of mathematical research, . . . yet its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem."

In 1903, L. E. Dickson replaced Colaw. In 1904, the arithmetic section was replaced by group theory, and the editorial assistance of Saul Epstein was obtained. In 1905, O. E. Glenn was added to the editorial staff to assist while Finkel was obtaining his Ph.D. Unsolved problems were printed in separate

groups. Multiple solutions were printed in the same and in several issues.

In August 1906, the University of Chicago assumed exclusive control of the MONTHLY with Finkel still editing the problem section. In 1909 the group theory section was abandoned, followed by the miscellaneous section in 1910, and the probability section in 1912. In 1911 the University of Illinois became a co-sponsor with Chicago, and G. E. Wahlin's services were obtained for the number theory (formerly diophantine analysis) section. In 1915 R. P. Baker joined the editorial staff of the problem section, and twelve other universities joined the sponsors.

In 1916 the MONTHLY was taken over by the newly formed Mathematical Association of America. In 1917 the present practice of indexing by problem number and by author was introduced, with bold face type indicating a solution published, italics a solution acknowledged, and ordinary type a problem proposed. The separate sections were abandoned in 1918 and proposals of all types were numbered consecutively beginning with number 2660.

In March 1919 Otto Dunkel joined Finkel as co-editor. The year 1920 was characterized by illuminating discussions and historical notes by Dunkel, R. C. Archibald, and H. S. Uhler. In August 1921, H. P. Manning was added to the staff to assist the other two editors in checking manuscripts for errors and in reading proof. He also contributed notes elaborating the subjects of some problems. This practice was abandoned in 1922.

In 1923 Manning was replaced by Norman Anning, who was replaced in 1924 by H. L. Olson. The present readable style was adopted in 1929.

In 1932 the Board of Governors, believing that the problems proposed were too difficult for most undergraduate college students, decided to split the department into an elementary and an advanced section. W. F. Cheney, Jr., initiated the elementary section in October, the first proposal being numbered E1. The old number sequence was retained in the advanced section.

The elementary section flourished under Cheney's guidance until 1939 when his total duties became so onerous that he requested relief. A fortunate replacement was available in H. S. M. Coxeter, fresh from revising Ball's *Mathematical Recreations*. He carried on until 1945 when the necessity for more research time caused him to be replaced by the present editor, Howard Eves, a keen problemist who was chosen because he had a tidy mind and a good style.

In 1934 Finkel became inactive in the problem department although he remained on the editorial board until his death on February 5, 1947. Dunkel assumed full responsibility of the department. He had a very wide range of interest and of talent in mathematics, and brought to his editorial duties a real scholarship in the subject. His editorial comments were masterful. He read every solution sent to him, and took time to point out oversights to solvers. His advancing age did not seem to impair his abilities. His copy continued to be neat, correct, and adequately marked for printing until the end. It gave Orrin Frink, who replaced Olsen in 1940, but little to do.

In March 1942 bold face titles for the problems were introduced at the suggestion of Editor-in-Chief L. R. Ford. These definitely brightened the appearance of the pages and made solutions easier to identify.

E. P. Starke, former problem editor of *National Mathematics Magazine*, and an ardent contributor to the MONTHLY department, fortunately was available to take over the advanced section in 1947 when Dunkel asked to be relieved. At that time Frink resigned, having convinced the editors that his services were no longer necessary nor desirable for efficient operation.

The MONTHLY has been particularly fortunate in its problem editors. Almost without exception they have had a real enthusiasm for problems, have enjoyed solving them themselves, and have realized the importance of problems in the growth and development of mathematics. They have had wide ranges of interest and mathematical ability and have supplemented the work of their colleagues nicely.

The problems. In the early years of the department the problems ranged from simple grade school exercises to very difficult graduate problems. The first problem printed in the MONTHLY in 1894 was "Arithmetic 1. Reduce 1/3 to 8ths." With the passage of time, the trend has been away from exercises toward problems of more significance.

As Algebra 43, Volume 1, page 433, (1894), F. M. Shields of Cooperwood, Mississippi, proposed a long involved time-rate-distance problem and offered "a city lot at St. Andrews, Florida, to the party sending the EDITOR the first correct answer." G. B. M. Zerr collected the prize.

In 1918, problem 2660, "Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed circle to the point of contact with an escribed circle, is equal to the side of the triangle between the two circles," was given in an examination at Cooper Union, New York City. Twenty-seven students submitted a correct answer.

In 1929, S. A. Corey, Des Moines, Iowa, in proposing 3402 (page 543), offered a prize of $10 to the person sending in a solution or comment most enlightening, in the judgment of the editors. The prize still is uncollected.

Throughout the evolution of the department the various editors have kept well attuned to their readers' tastes and to current mathematical styles as evidenced by their selection of proposals for publication. Problems whose difficulty lies principally in laborious computations have gradually disappeared. Interest in plane and solid geometry problems is currently waning. Problems best solved by table searching are proscribed. More proposals are appearing in abstract algebra and topology. A nice variety in the fields of mathematics, in difficulty of solutions, and in distribution of proposers is now evident in both departments.

Amateurs like the simpler number problems and the lighter recreational ones. Professionals show particular interest in problems connected with the field of analysis.

Some measure of popular interest is given by the number of solutions submitted to a problem. On the basis of this criterion, the three most popular problems have been,

E1111 [1954, 712], which had 70 solvers:

"Our good friend and eminent numerologist, Professor Euclide Paracelso Bombasto Umbugio, has been busily engaged testing on his desk calculator the $81 \cdot 10^9$ possible solutions to the problem of reconstructing the following exact long division in which the digits indiscriminately were replaced by x save in the quotient where they were almost entirely omitted.

$$\begin{array}{cccccccccccc}
 & & & & & & & & & 8 & & \\ \cline{5-12}
x & x & x &) & x & x & x & x & x & x & x & x \\
 & & & & & x & x & x & & & & \\ \cline{6-8}
 & & & & & & x & x & x & x & & \\
 & & & & & & & x & x & x & & \\ \cline{7-10}
 & & & & & & & & x & x & x & x \\
 & & & & & & & & x & x & x & x \\ \cline{9-12}
\end{array}$$

Deflate the Professor! That is, reduce the possibilities to $(81 \cdot 10^9)^0$."

E950 [1951, 566], which had 65 solvers:

"Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$."

E1042 [1953, 420], which had 62 solvers:

"A sequence $\{a_n\}$, where a_0 and a_1 are given numbers, is defined recursively by $a_{n+2}=(a_{n+1}+a_n)/2$. Find $\lim_{n\to\infty} a_n$."

Sources of problems are quite diverse. Many are by-products of research—incidental results which have interest for themselves. Some are constructed around particular relations for test purposes or for publication. Others are submitted without any knowledge that they can be solved, and others are clear examples of plagiarism offered without reference. These are the bothersome ones for the editor, who cannot be expected to recognize all old problems, nor has time to solve all proposals which are submitted without reference or solution.

Any editor will appreciate a proposal couched in non-ambiguous appealing language and accompanied by a solution necessitating facility and ingenuity and by such references as are known. He may publish it and hope for solutions of greater elegance. Also, he may hope that these solutions will not leave too much to be desired in organization, grammatical structure and legibility.

The contributors. During the sixty-one year span 31624 contributions—proposals and solutions—were offered by 3487 individuals. Of these, 1564 were content with a single contribution, 557 with two, 265 with three, 196 with four,

and 116 with five. The contributions of 565 others fell in the 6–25 range, 107 in the 26–50 range, 67 in the 51–100 range, 32 in the 101–200 range, and 13 in the 201–300 range. Five persons made more than 300 contributions.

The most ardent of these was the amazing G. M. B. Zerr who surpassed all rivals in quantity and quality of contributions in each of the 17 years from the inception of the MONTHLY until his untimely death in 1910, to amass the incredible total of 1697 contributions, both proposals and solutions. During this same period, this versatile mathematician, chemical engineer, professor and school administrator contributed with equal fervor to many other contemporary magazines. His closest rival, J. W. Scheffer, totaled 691.

During the first twenty years the number of problems proposed averaged 112 per year. This average dropped off to 79 during the middle period, and increased to 96 during the last twenty years. It was during the latter era that the indefatigable V. Thébault, whose original contributions to mathematical magazines on number properties and the geometry of the triangle and the tetrahedron have been international in scope, addressed his multitudinous contributions to the MONTHLY. The relatively few which the problem editors found room to use raised his total to 582. Following him, principally on the basis of solutions rather than proposals, were current editor E. P. Starke with 523 and C. W. Trigg with 333.

Faithfulness ranks with ardor in measuring the interest of a problem section. In this regard, none can surpass N. A. Court who contributed in each of 40 years and throughout that time encouraged his students to become problemists. Although not such a regular contributor, W. B. Carver's contributions span 56 years. W. E. Buker and B. F. Finkel contributed continuously for 24 years.

In its first year, 1894, the MONTHLY problem department attracted 123 contributors from 30 states. Twenty-four of these were from Ohio. The first foreign contributor was W. F. King of Canada in 1896; the first European, E. Woelffing of Germany in 1900; and the first Asian, T. Hayashi of Japan in 1907.

The number and geographical distribution of contributors varied with the fortunes of the MONTHLY, reaching a low of 46 contributors in 1912. Only 20 states and England were represented in 1911. A steady increase in popularity has been registered since that time, almost directly proportional to the number of subscribers. In 1954, there were 531 contributors from 45 states and territories and 20 foreign countries. The preponderant show of interest fluctuated among Ohio, Illinois, Pennsylvania, and New York until 1922, since when more contributors have lived in New York than in any other state or territory.

During their existence, the problem departments have received contributions from every one of the United States, and from Alaska, District of Columbia, Hawaii, and Puerto Rico; from Algeria, Egypt, and South Africa; from Australia and New Zealand; from Burma, China, India, Japan, Korea, Malaya, Philippine Islands, Singapore S. S., and Thailand; from Austria, Belgium, Czechoslovakia,

Denmark, England, France, Germany, Hungary, Ireland, Luxembourg, Netherlands, Norway, Poland, Portugal, Roumania, Scotland, Spain, Sweden, Switzerland, Turkey, Wales, and Yugoslavia; from British Honduras, Canada, Cuba, Mexico, and Panama; and from Argentina, Brazil, Chile, Uruguay, and Venezuela—a total of 98 political entities.

The problem departments have demonstrated their appeal to persons at all stages of mathematical development—from high school students to retired professors. D. J. Newman sent in his first contribution at age 16 as a student in Stuyvesant High School, New York City, and has kept at it ever since—all through Harvard and as an applied mathematician at Republic Aviation Corporation. R. D. Carmichael, of number theory fame, and many others had their interest in mathematics developed by the problem departments.

Among school teachers who have found an added stimulus in the MONTHLY problems are W. E. Buker, M. L. Constable, P. W. A. Raine, J. Rosenbaum, C. M. Sandwick, and A. Wayne. In the ranks of those eminent in their chosen specialties who were impelled to contribute frequently while graduate students or while actively engaged in university teaching are P. T. Bateman, E. T. Bell, A. A. Bennett, P. Erdös, L. E. Dickson, J. S. Frame, E. Hille, I. Kaplansky, D. H. Lehmer, O. Ore, G. Pólya, G. Szegö, H. S. Vandiver, M. Ward, J. H. M. Wedderburn, and scores more. Many other problemists upon becoming emeritus found continued problem-solving an effective weapon against vegetation, *e.g.*, N. Anning, W. B. Carver, D. L. MacKay, A. Pelletier, W. R. Ransom, and J. B. Reynolds.

Quite a few persons of varying mathematical backgrounds and with other means of livelihood have become problemists to enjoy an avocation wherein they may test their intellectual mettle. Typical enthusiasts have been: dentist L. Bankoff, nurseryman W. B. Clarke, lawyer M. Dernham, ballistics expert M. Goldberg, steel works manager P. Goormatigh, retired telephone engineer W. O. Pennell, automobile dealer P. A. Pizá, insurance inspector V. Thébault, clergyman G. W. Walker, and patent examiner P. Wiernicke.

This volume will come principally to the attention of those interested in problems and hence needing no indoctrination. Upon others we urge consideration of problem constructing and solving as an enjoyable mental exercise, one means of keeping mathematical skills in working order, a stimulus to new lines of investigation, and an ever-new never-boring avocation. If you should find some proposal in a problem department that stimulates you to a solution, send the solution to the editor. It is only in this way that he can know that he is meeting your interests.

Appreciation is hereby expressed for the aid given by previous and present editors in supplying much of the information recorded here.

2688 (*Frank Irwin*). With four quantities, a_1, a_2, a_3, a_4, we may, without changing their order, form the following complex fractions:

$$\frac{\dfrac{a_1}{a_2}}{\dfrac{a_3}{a_4}} \qquad \frac{\dfrac{a_1}{\dfrac{a_2}{a_3}}}{a_4} \qquad \frac{a_1}{\dfrac{\dfrac{a_2}{a_3}}{a_4}} \qquad \frac{\dfrac{\dfrac{a_1}{a_2}}{a_3}}{a_4} \qquad \frac{a_1}{\dfrac{a_2}{\dfrac{a_3}{a_4}}}.$$

But these are not all different values; the first and fourth are equal. Determine how many different rational functions of the quantities $a_1, a_2, \cdots, a_n$ may be obtained in this way, and which can be represented in more than one way as a complex fraction of the above kind, and which in only one way.

2752* (*R. E. Moore*). Test for convergence the series

$$\sum_{n=1}^{\infty} a_n, \qquad a_n = \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right]^2.$$

2758 (*Leonard Richardson*). Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin(2r+1)\psi}{\sin\psi}\,d\psi = \frac{\pi}{2},$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin\psi}\,d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

2759 (*J. L. Riley*). Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\psi(x)}{1-\phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\psi(y)}{[1-\phi(x)\phi(y)]^2}.$$

2774* (*Frank Irwin*). Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3\cdots n-1 & n \\ n & 1 & 2\cdots n-2 & n-1 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ 2 & 3 & 4\cdots\quad n & 1 \end{vmatrix}, \text{ and } \begin{vmatrix} a_1 & a_2 & a_3\cdots a_{n-1} & a_n \\ a_n & a_1 & a_2\cdots a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ a_2 & a_3 & a_4\cdots\quad a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots, a_n$ form an arithmetical progression.

* Provided with solution or note by Otto Dunkel. Wherever the asterisk appears hereafter in this section, it will have this meaning.

2784 (*T. H. Gronwall*). Show that all solutions in integers of

$$y^2 = 1 + x + x^2 + x^3 + x^4$$

are given by

$$x = -1, \quad 0, \quad 3;$$
$$y = \pm 1, \ \pm 1, \ \pm 11.$$

2804 (*T. H. Gronwall*). Show that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^4}}\int_0^x \frac{dx}{\sqrt{1-x^4}} = x + \sum_{n=1}^{\infty} \frac{3\cdot 7 \cdots (4n-1)}{5\cdot 9 \cdots (4n+1)}\, x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1-x^4}}\right)^2 = x^2 + \sum_{n=1}^{\infty} \frac{3\cdot 7 \cdots (4n-1)}{5\cdot 9 \cdots (4n+1)}\, \frac{x^{4n+2}}{2n+1}.$$

2838 (*Lewis Carroll*). A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?

2845* (*E. L. Post*). Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y_{x+1}^2}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x\to\infty} y_x \log x = 1.$$

2866 (*Norman Anning*). Equilateral triangles whose sides are 1, 3, 5, $\cdots$ are placed so that their bases lie corner to corner in a straight line. Show that the vertices lie upon a parabola and are all at integral distances from its focus.

2871 (*L. G. Weld*). Weight being disregarded, a package may be admitted to the parcel post if the length plus the greatest girth, measured transversely to the length, does not exceed 72 inches. What is the size of the smallest square window through which all admissible rectangular boxes can be passed?

2884 (*E. H. Moore*). Consider an $m\times n$ array A of numbers a_{st} and an $n\times m$ array B of numbers b_{ts}. Show that the system of mn equations:

$$\sum_{tu} a_{st}b_{tu}a_{uv} = 0 \quad (sv),$$

implies the equation:

$$\sum_{ts} a_{st}b_{ts} = 0.$$

The suffixes s, u have the range $1, 2, \cdots, m$ and the suffixes t, v have the range $1, 2, \cdots, n$.

2892 (*R. T. McGregor*). Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

2908 (*L. E. Dickson*). If f is a homogeneous polynomial in n variables and H is its Hessian determinant, prove that the Hessian of f^2 is cHf^n, where c is a constant.

2930 (*R. E. Gaines*). If in reducing p/q (p and q integers, $q>p$) to a decimal, the remainder $q-p$ ever appears, then the fraction will give a repeating decimal the number of digits in whose repetend will be exactly twice the number of digits in the quotient already obtained, and the remaining digits may, without further division, be obtained by subtracting the quotient already found from a succession of 9's.

2933* (*H. E. Dudeney*). With ruler and compasses only, divide an equilateral triangle into four rectilinear pieces which may be put together so as to form a square.

2936* (*J. P. Ballantine*). A person drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a minimum?

2938* (*C. F. Gummer*). If $a, b, \cdots, i$ are real numbers $\geqq 0$, and if

$$D(r) = \begin{vmatrix} a^r & b^r & c^r \\ d^r & e^r & f^r \\ g^r & h^r & i^r \end{vmatrix}$$

is equal to zero for five real values of r other than zero, prove that the determinant $D(1)$ has either two rows or two columns proportional or a single row or column of zeros.

2980* (*J. Rosenbaum*). Locate a point such that the sum of its distances from n given points shall be a minimum.

2991 (*E. J. Oglesby*). Sum the infinite series

$$S_2(x) = 1 + \frac{3x^2}{2!} + \frac{4x^4}{4!} + \frac{6x^6}{6!} + \cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

2994 (*R. M. Mathews*). Can the following construction be made without the use of a regulus? Construct a line which meets four given skew lines.

3011* (*E. T. Bell*). In a certain paper it is stated that "it is easy to prove that, if $p>0$ is an integer, the relation

$$a_1 \sin\frac{\pi}{2p} + a_2 \sin\frac{2\pi}{2p} + \cdots + a_{p-1} \sin\frac{(p-1)\pi}{2p} + a_p = 0$$

necessitates $a_1 = a_2 = \cdots = a_{p-1} = a_p = 0$, the a's being integers." Prove it.

3023 (*E. T. Bell*). The equation $x^p + y^p + z^p = 0$ is possible in integers x, y, z prime to the odd prime p, if

$$\frac{1}{2}\left[\frac{1}{2} N_2(p) - \frac{1}{3} N_3(p) + \frac{1}{4} N_4(p) - \cdots + \frac{1}{p-1} N_{p-1}(p)\right] + 1$$

is divisible by p, where $N_r(n)$ is the number of representations (order essential) of n as a sum of r square integers with roots $\neq 0$.

3032* (*Otto Dunkel*). If $a_1, a_2, \cdots, a_n$ are any real or complex quantities which satisfy the equation

$$x^n - na_1x^{n-1} + {}_nC_2a_2^2x_n^{n-2} + \cdots + (-1)^i {}_nC_ia_i^ix^{n-i} + \cdots + (-1)^n a_n^n = 0$$

where ${}_nC_i = n!/(n-i)!i!$, prove that $a_1 = a_2 = \cdots = a_n$.

3034 (*J. L. Riley*). If every root of the equation $f'(x) = 0$ be subtracted from every root of the equation $f(x) = 0$, find the sum of the reciprocals of the differences.

3040 (*William Hoover*). Given the radius, R, of a sphere rolling down two intersecting straight lines including the angle 2α and equally inclined to the horizon; show that the locus of the center of the sphere is an ellipse of semi-axes $R\csc\alpha$, R.

3043 (*O. D. Kellogg*). Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{dx^2} + \frac{\partial^2 V}{\partial y^2}\right) dxdy = 0$$

for all functions V with continuous second derivatives in T, and such that (a) the normal derivatives of V vanish on the boundary of T, or, (b) the boundary not being a smooth curve, the first derivatives of V at a point P approach 0 as P approaches any boundary point of T, U is harmonic in T, *i.e.*, U satisfies Laplace's equation $(\partial^2 U/\partial x^2) + (\partial^2 U/\partial y^2) = 0$.

3045* (*S. A. Corey*). If in the equation

$$
(1)\quad s!\left[\frac{1}{(s+1)!}+\frac{c_3}{4!(s-1)!}+\frac{c_5}{6!(s-3)!}+\frac{c_7}{8!(s-5)!}+\cdots+\frac{c_s}{(s+1)!2!}\right]=0
$$

$c_3, c_5, c_7, \cdots, c_s$ be given and retain such constant values that (1) is satisfied for all positive odd integral values of s, $(s>1)$, prove that if s be decreased by unity (so that $s=2n$), then the left member will become equal to $\pm B_n$, according as n is odd or even, B_n being Bernoulli's nth number. Also show how any one of the constants c may be found without first finding all the preceding constants.

3051 (*Norman Anning*). Given the sequence: $u_1=2$, $u_2=8$, $u_n=4u_{n-1}-u_{n-2}$, $(n=3, 4, 5, \cdots)$, show that

$$
(\pi/12) = \sum_{n=1}^{\infty} \operatorname{arccot} u_n^2.
$$

3053 (*J. Lense*). Let $a_1=a$, $a_2=a^{a_1}$, $a_3=a^{a_2}$, $\cdots$, $a_{n+1}=a^{a_n}$, $\cdots$. Discuss $\lim_{n\to\infty} a_n$ as a function of a. (The limit exists also for some values of $a>1$.)

3089* (*Norman Anning*). Given four points, O, A, B, C, on a straight line, to construct, with straightedge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

3092 (*N. A. Court*). What must be the relations between the coefficients of a cubic equation in order that its roots, considered as lengths, shall form a triangle?

3104* (*Otto Dunkel*). If $f(x)$ is a single-valued and continuous function of x in the interval $a\leqq x\leqq b$ which is not identically zero and which satisfies the inequality $0\leqq f(x)\leqq M$, show that

$$
0<\left[\int_a^b f(x)dx\right]^2-\left[\int_a^b f(x)\cos x\,dx\right]^2-\left[\int_a^b f(x)\sin x\,dx\right]^2
$$
$$
\leqq M^2(b-a)^4/12.
$$

3118 (*Harry Langman*). If $n>2$, and e is a primitive root of $e^n=1$, show that the determinant $|a_{ij}|$, of order $n-1$ and having the element a_{ij} equal to e^{ij}, has the value

$$
(-1)^{(n-1)(n-2)/4}n^{(n-2)/2}.
$$

3137 (*Harry Langman*). Show how to draw, using straightedge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

3140 (*C. C. MacDuffee*). Let f be any algebraic form of total degree $m>1$ in n variables, and $H(f)$ its Hessian. Let ϕ be any analytic function. Prove that

$$H[\phi(f)] = H(f)\cdot\left[\left(\frac{\partial\phi}{\partial f}\right)^n + \frac{mf}{m-1}\,\frac{\partial^2\phi}{\partial f^2}\left(\frac{\partial\phi}{\partial f}\right)^{n-1}\right].$$

In particular when $\phi(f)$ is f^r, we have Mrs. Ballantine's generalization of Problem 2908 [1923, 41]. Again, when $\phi(f)=\log f$, we have

$$H(f) = (1-m)f^n H[\log f],$$

which is a generalization of Exercise 22 of Sir Thomas Muir's *Budget of Exercises on Determinants*, this MONTHLY, 1922, p. 10.

3173 (*C. C. Camp*). Two parallel vertical walls stand upon horizontal ground. A ladder of length a has its foot at the bottom of the first wall and leans against the second. A ladder of length b has its foot at the bottom of the second wall and leans against the first. What must be the distance between the walls so that the ladders will cross at height h? When is a solution possible?

3177 (*Samuel Beatty*). If X is a positive irrational number and Y its reciprocal, prove that the sequences

$$\begin{array}{lll}(1+X), & 2(1+X), & 3(1+X), \cdots \\ (1+Y), & 2(1+Y), & 3(1+Y), \cdots\end{array}$$

contain one and only one number between each pair of consecutive positive integers.

3179* (*Otto Dunkel*). Is the following statement correct?

The intersections of two curves whose equations are given in polar coordinates (ρ, θ) are obtained by solving the two equations simultaneously for ρ and θ.

3207* (*C. N. Mills*). Prove that $\frac{1}{4}m^2\sqrt{3}$ is the maximum area of a triangle which can be formed with the lines a, b, c, subject to the condition that $a^3+b^3+c^3=3m^3$.

3231 (*R. B. Stone*). The root-mean-square of n numbers $x_1, x_2, \cdots, x_n$ is defined by the formula

$$R.M.S. = \left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\right)^{1/2}.$$

(a) For what values of n is the $R.M.S.$ of the first n integers also an integer? (b) For what values of a and n is the $R.M.S.$ of the n successive integers, a, $a+1, \cdots, a+n-1$, also an integer? (c) Under what conditions is the $R.M.S.$ of n integers also an integer?

3242 (*R. S. Underwood*). A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued until the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

3249* (*Otto Dunkel*). If $i!f_i = d^if(x)/dx^i$ and

$$D = \begin{vmatrix} f_r & f_{r+1} & \cdots & f_{r+n-1} \\ f_{r+1} & f_{r+2} & \cdots & f_{r+n} \\ \cdot & \cdot & \cdot & \cdot \\ f_{r+n-1} & f_{r+n} & \cdots & f_{r+2n-2} \end{vmatrix},$$

prove that $dD/dx = (r+2n-1)D'$, where D' is the determinant D with the subscripts of the last row increased by unity.

3251 (*J. V. Uspensky*). Show that the system

$$\phi(x, y) = x + y + \sum A_{i,k}x^iy^k = 1, \qquad x\phi_x' - ay\phi_y' = 0,$$

where $A_{i,k} > 0$, $a > 0$, $2 \leqq i+k \leqq n$, has one and only one solution in positive numbers.

3252 (*Laenas G. Weld*). Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1, y_1) to (x_2, y_2) in the least possible time.

3276 (*L. L. Silverman and J. Tamarkin*). Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z + \pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu} dt, \qquad \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

3296 (*J. Rosenbaum*). It is well known that the radius of the inscribed circle of a right triangle is equal to half the difference between the sum of the legs and the hypotenuse. Derive an analogous expression for the radius of the inscribed sphere of a right tetrahedron.

3305 (*Paul Wernicke*). (a) Prove that x^2+y^2, where x and y are integers, cannot be the square of an integer unless x or y is divisible by 3. (b) Modify the theorem for the case that both x and y are divisible by 3. (c) Generalize the theorem so as to cover other exponents, thereby proving a part of Fermat's greater theorem.

3309* (*Otto Dunkel*). If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is $x_i^2, x_iy_i, y_i^2, x_i, y_i, 1$, and if

$$(i, j, k) = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

show that

$$(1, 2, 3, 4, 5, 6) = (6, 1, 2)(2, 3, 4)(4, 5, 6)(1, 3, 5) - (1, 2, 3)(3, 4, 5)(5, 6, 1)(2, 4, 6)$$

is an identity in the 12 independent variables x_i, y_i.

3343 (*J. V. Uspensky*). Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z}\, dz = \pi - \frac{\pi}{2} \log 2\pi,$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z}\, dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

3345 (*B. F. Finkel*). A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

3348* (*A. C. Aitken*). Show that

$$\frac{\pi}{6} = \sum_{r=1}^{\infty} \operatorname{arccot} 2u_r^2,$$

where $u_r = 4u_{r-1} - u_{r-2}$ with $u_1 = 1$, $u_2 = 3$.

3367 (*Harry Langman*). Given any triangle. On each side construct an equilateral triangle externally. The centers of these triangles determine another equilateral triangle A. Similarly an equilateral triangle B is determined by constructing the equilateral triangle internally. Show that the difference between the areas of the triangles A and B is equal to the area of the given triangle.

3399* (*B. C. Wong*). Prove or disprove

$$\sum_{i=0}^{t} (-1)^i \binom{r+1}{i} \binom{2r-2i}{r} = r + 1,$$

where $t = r/2$ if r is even and $t = (r-1)/2$ if r is odd.

3408 (*J. V. Uspensky*). Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

3414* (*B. C. Wong*). Prove or disprove

$$\sum_{i=0}^{t} \left[\frac{(r-1)!(r-2i)}{(r-i)!i!}\right]^2 = \frac{(2r-2)!}{r!(r-1)!},$$

where $t = (r-2)/2$ if r is even and $t = (r-1)/2$ if r is odd.

3417 (*C. O. Williamson*). Construct a square so that each side shall pass through a given point.

3418 (*D. H. Dodge*). Prove that

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2 + n + p}{n^2 + n - p}.$$

3421* (*Otto Dunkel*). A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

3430* (*R. M. Sutton*). It is physically possible, given a large number of unit resistances, to make any resistance p/q between two points A and B, where p and q are integers. The result may be accomplished by connecting in parallel q groups of p resistances each, requiring pq resistances. However, it is usually possible to accomplish the same result by a fewer number of unit resistances. The problem is: Find the minimum number of unit resistances necessary to make a resistance p/q between two points A and B in an electric circuit, p and q being both integers.

3438 (*F. P. Matz*). Solve

$$\int_0^{dy/dx} \frac{\cos w \, dw}{16 + 9\sin^2 w} = \frac{1}{12}\tan^{-1} x.$$

3440 (*A. Pelletier*). A triangle is circumscribed about a circle. Prove that the three following lines are concurrent: (1) the line joining the points of contact of any two sides; (2) the line joining the points of intersection of these sides with the bisectors of the opposite angles; (3) the line joining the feet of the altitudes on these sides.

3445 (*Mannis Charosh*). If p is odd and greater than 1, prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \bmod p;$$
$$2^2 \cdot 4^2 \cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p.$$

3448 (*O. D. Kellogg*). Prove that whenever the infinite series with positive terms, $u_1+u_2+u_3+\cdots$, converges, the series

$$u_1/r_1 + u_2/r_2 + u_3/r_3 + \cdots$$

diverges, $r_n = u_n+u_{n+1}+u_{n+2}+\cdots$ being the remainder of the first series after $n-1$ terms.

3459* (*Norman Anning*). It is observed that $3003 = \binom{15}{5} = \binom{14}{6}$. Solve in positive integers the equation

$$\binom{x+1}{y} = \binom{x}{y+1}.$$

3468 (*C. A. Rupp*). Show that the determinant of n^2 elements in the upper left corner of the Pascal triangle

$$\begin{matrix} 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & \cdot & \cdot & \cdot & \\ 1 & 3 & \cdot & \cdot & \cdot & & \\ 1 & \cdot & \cdot & \cdot & & & \end{matrix}$$

has the value unity.

3506 (*E. B. Escott*). Solve the n simultaneous equations in n unknowns:

$$\begin{vmatrix} x_1 & x_2 \cdots x_{n-1} \\ x_n & x_1 \cdots x_{n-2} \\ \cdots & \cdots \\ x_3 & x_4 \cdots x_1 \end{vmatrix} = a_1; \quad \begin{vmatrix} x_2 & x_3 \cdots x_n \\ x_1 & x_2 \cdots x_{n-1} \\ \cdots & \cdots \\ x_4 & x_5 \cdots x_2 \end{vmatrix} = a_2; \quad \cdots$$

$$\begin{vmatrix} x_n & x_1 \cdots x_{n-2} \\ x_{n-1} & x_n \cdots x_{n-3} \\ \cdots & \cdots \\ x_2 & x_3 \cdots x_n \end{vmatrix} = a_n.$$

3514 (*J. P. Ballantine*). Let D_n denote a determinant of order n whose elements are all zeros and ones, and which has 2 ones and $n-2$ zeros in every row and column. Show that (a) for every n, $D_n = \pm 2^m$, where m and n are both even or both odd, or $D_n = 0$; (b) if $D_5 = 0$, then two rows are identical; (c) if two rows of D_n are identical, then two columns are alike, and conversely; (d) $3m \leqq n$;

and (e) property (b) does not hold except for D_2, D_3, and D_5.

3528 (*A. A. Bennett*). If a, b, c be complex numbers such that

$$|a| = |b| = |c| = r \neq 0,$$

then $|(ab+bc+ca)/(a+b+c)| = r$. Generalize.

3536 (*Martin Rosenman*). Consider fractions of the form 1/2, 1/3, 1/4, 1/5, $\cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7=41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807+\cdots$ give the desired result; in this series each denominator exceeds by 1 the product of all the preceding.

3539 (*R. E. Gaines*). A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends, will project over the edge of the table?

3547 (*Martin Rosenman*). Consider n points in a plane. Join these in any order to form a closed polygon. Repeat the operation on the n midpoints of the sides of the polygon thus formed, *etc.* Prove that the successive polygons converge to a point.

3552 (*J. M. Feld*). Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots 0 & A_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ e_{n1} & e_{n2} & e_{n3} & \cdots e_{n,n-1} & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) p_i, p_j, $\cdots$, are the distinct prime factors of n.

3565 (*Orrin Frink, Jr.*). Find the ellipse of least area circumscribing a given triangle.

3582 (*D. C. Duncan*). If α and β are positive integers and $\beta>2$ then $2^\alpha+1$ is never divisible by $2^\beta-1$.

3586* (*R. E. Gaines*). If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

3591* (*B. F. Kimball*). Let the nth difference of log x with difference interval

1 be denoted by $\Delta^n \log x$. Show that

$$\lim_{n\to\infty} n^x \log n \, \Delta^n \log x = \Gamma(x).$$

3594* (*H. T. R. Aude*). Find sets of integers for rational right triangles which, as the number increase, approach a 30°—60° right triangle.

3612 (*H. Grossman*). Prove that the ratio of the sum of the hth, $(h+k)$th, $(h+2k)$th, *etc.* coefficients to the sum of all the coefficients in the expansion of $(a+b)^n$ converges to the limit $1/k$, as n approaches ∞, where $h=1, 2, \cdots, k$; and k is a positive integer.

3621 (*A. S. Levens*). Extend the graphical method for the solution of real roots of a quadratic, as given in Dickson's *First Course in the Theory of Equations*, p. 29, to permit the reading of complex roots.

3625* (*R. E. Moritz*). Show that

$$\sum_{k=1}^{n} (-1)^{k+1} {}_nC_k \, k^n = (-1)^{n+1} n!$$

for all positive integral values of n.

3645* (*P. S. Dwyer*). Show that the value of the determinant formed by deleting the kth column from the array

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & {}_1C_1 & {}_2C_1 & {}_3C_1 & {}_4C_1 & \cdots & {}_nC_1 & {}_{n+1}C_1 \\ 0 & 0 & {}_2C_2 & {}_3C_2 & {}_4C_2 & \cdots & {}_nC_2 & {}_{n+1}C_2 \\ 0 & 0 & 0 & {}_3C_3 & {}_4C_3 & \cdots & {}_nC_3 & {}_{n+1}C_3 \\ 0 & 0 & 0 & 0 & {}_4C_4 & \cdots & {}_nC_4 & {}_{n+1}C_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & {}_nC_n & {}_{n+1}C_n \end{vmatrix}$$

is ${}_{n+1}C_{k-1}$.

3649 (*F. T. O'Doubler*). Solve the functional equation

$$f(xy) = [f(x)]^{y^\beta}[f(y)]^{x^\beta},$$

where β is a real constant and $f(x)$ is a real continuous and single-valued function of the real variable x.

3652* (*A. F. Stevenson*). The practice of certain cigarette manufacturers of supplying playing cards, poker hands, *etc.* with their cigarette packets, and of offering various articles in exchange for complete sets of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards,

and that these cards are distributed among the packets at random (the number of packets available being infinite), what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

3658* (*J. M. Feld*). The Simson line of a point P on the circumcircle of a triangle ABC is the tangent at the vertex of a parabola tangent to the sides of ABC and having its focus at P.

3666 (*Martin Rosenman*). Set up a one-to-one correspondence between the points in the open interval $0<x<1$ and the points in the closed interval $0\leqq x\leqq 1$.

3667 (*Raphael Robinson*). Show that $(-1)^{n-1}(n-1)2^{n-2}$ is the value of the n-rowed determinant for which $a_{ij}=|i-j|$.

3673 (*E. B. Escott*). Factor $x^8+98x^4y^4+y^8$ into two polynomial factors with integral coefficients.

3674* (*Garrett Birkhoff*). For any positive integer k show that

$$\phi_k = \frac{(2k-2)!}{k!(k-1)!} = \binom{2k-1}{k} \Big/ (2k-1)$$

is an integer; and prove the recurrence formula $\phi_n = \sum_{i=1}^{n-1}\phi_i\phi_{n-i}$.

3678 (*B. W. Jones*). Prove the following theorem: If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$, where n is a positive integer, $a_n\neq 0$, and all coefficients are real, has only real roots, then

1. Descartes' Rule of Signs gives exactly the number of positive and the number of negative roots of the equation.
2. $a_r=0$ $(n>r>0)$ implies $a_{r+1}\cdot a_{r-1}<0$.

3682* (*Mannis Charosh*). If a prime p has the form $p=4k+3$ and m is the number of quadratic non-residues less than $p/2$, prove that

(a) $$1\cdot 3\cdot 5\cdots(p-2)\equiv(-1)^{m+k} \bmod p,$$

(b) $$2\cdot 4\cdot 6\cdots(p-1)\equiv(-1)^{m+k+1} \bmod p.$$

3683 (*Raphael Robinson*). Show that the sum of the medians of a simplex of n dimensions is smaller than $2/n$ and greater than $(n+1)/n^2$ times the sum of the edges of the simplex, and that these are the best limits that can be given.

3687* (*Melvin Dresher*). If $S(i,j)$ denotes the sum of the divisors common to i and j, show that:

$$\begin{vmatrix} S(1,1) & S(1,2) & \cdots & S(1,n) \\ S(2,1) & S(2,2) & \cdots & S(2,n) \\ \cdot & \cdot & \cdots & \cdot \\ S(n,1) & S(n,2) & \cdots & S(n,n) \end{vmatrix} = n!.$$

3690 (*W. M. Whyburn*). Solve the functional equation

$$f(x)f(-x) = c^2 = [f(0)]^2,$$

subject to the single restriction that $f(x)$ be a single-valued, positive, real function of the real variable x.

3692 (*E. P. Starke*). Show that there are four distinct sets of integers which satisfy the equations

$$x_1 + x_2 + x_3 = 54, \qquad x_1^2 + x_2^2 + x_3^2 = 1406.$$

Develop a general method of attack for similar problems in which 54 and 1406 are replaced by a and b.

3696 (*J. B. Reynolds*). A dog directly opposite his master on the banks of a stream, flowing with uniform speed, swims at a still-water speed of two miles per hour heading directly towards his master at all times. The man notes that the dog does not stop drifting down stream until he is two-thirds across measured perpendicularly to the banks, and that it takes five minutes longer to make the trip than if the water had been still. How wide is the stream?

3700 (*J. H. M. Wedderburn*). Find a basis h_{pq} for matrices of order n such that each element of the basis is idempotent and $h_{pq}h_{rs} = k_{pqrs}h_{ps}$, where k_{pqrs} is a rational number.

3703 (*V. Thébault*). Prove that the integral part of the fourth root of the product of eight consecutive integers is equal to x^2+7x+6, where x is the smallest of the eight integers. This result may be used to show that the product of eight consecutive integers is never the fourth power of an integer.

3705* (*Raphael Robinson*). Show that when the quadratic form

$$\sum_1^n |i-j|\, x_i x_j, \qquad n > 1,$$

is reduced to the sum of squares by a real linear transformation, one of the terms will be positive, the other $n-1$, negative.

3707 (*Bernard Friedman*). If p is a prime of the form $3n+1$, it can be expressed as $p = A^2+27B^2$, where A and B are positive integers, if and only if 2 is a cubic residue of p.

3709* (*E. B. Escott*). Determine the values of A in the trinomial $x^{12}+Ax^6y^6+y^{12}$ so that it will have two polynomial factors of the sixth degree with rational coefficients.

3713* (*R. E. Gaines*). Determine the position of a normal chord of a conic which forms a segment of minimum area. Find the area of such a segment of an ellipse.

3716 (*J. D. Hill and H. J. Hamilton*). Let $E_1, E_2, \cdots, E_n, \cdots$ be an infinite sequence of measurable sets in the interval (a, b) such that $mE_n \geqq k > 0$ for $n = 1, 2, \cdots$. Does there necessarily exist some infinite sequence of indices, $1 \leqq r_1 < r_2 < \cdots < r_i < \cdots$ for which the measure of $E_{r_1} \cdot E_{r_2} \cdots E_{r_i} \cdots$ is greater than zero?

3718 (*Frank Morley*). Show that the ellipse through the points given by the complex numbers a, b, c and with center $(a+b+c)/3$ has semi-axes whose lengths are

$$|a + \omega^2 b + \omega c|/3 \pm |a + \omega b + \omega^2 c|/3,$$

where $\omega = (-1 + i\sqrt{3})/2$.

3719* (*Morgan Ward*). Prove that

$$\sum_{r=0}^{n} {}_nC_r/(x + r)(x + r + 1) \cdots (x + r + n) = 2^n/x(x + 2)(x + 4) \cdots (x + 2n).$$

3720 (*C. J. Coe*). In transforming coordinates from the rectangular system $OX_1Y_1Z_1$ to the congruent rectangular system $OX_2Y_2Z_2$ we have the following table of cosines:

	X_1	Y_1	Z_1
X_2	λ_1	μ_1	ν_1
Y_2	λ_2	μ_2	ν_2
Z_2	λ_3	μ_3	ν_3 ,

the determinant of the array having the value unity. Prove that

$$\lambda_1 + \mu_2 + \nu_3 \geqq -1.$$

3721 (*Albert Whiteman*). Prove by Fermat's method of infinite descent that an odd prime p of the form $3n+2$ has the quadratic non-residue -3.

3726 (*Mannis Charosh*). The vertices of a triangle inscribed in a given circle are the points of tangency of a triangle circumscribed about the circle. Prove that the product of the perpendiculars from any point on the circle to the sides of the inscribed triangle is equal to the product of the perpendiculars from the same point to the sides of the circumscribed triangle.

3731 (*Raphael Robinson*). In how many ways can a_1 1's, a_2 2's, $\cdots$, a_n n's be arranged, so that in reading from the beginning, none of the $(k+1)$'s are reached until at least one of the k's has been reached?

3734 (*A. A. Bennett*). A car with $n(n>2)$ passengers of different speeds of mental reaction passes through a tunnel and each passenger acquires unconsciously a smudge of soot upon his forehead. Suppose that each passenger

(1) laughs and continues to laugh as soon as and only so long as he sees a

smudge upon the forehead of a fellow passenger;

(2) can see the foreheads of all his fellows;

(3) reasons correctly;

(4) will clean his own forehead when and only when his reasoning forces him to conclude that he has a smudge;

(5) knows that (1), (2), (3), and (4) hold for each of his fellows.

Show that each passenger will eventually wipe his own forehead.

3737* (*J. P. Ballantine*). Derive the following formulas:

$$\text{(a)} \qquad \pi = \frac{10}{3} - 24\left\{\frac{1}{1\cdot2\cdot3\cdot4\cdot5} - \frac{1}{3\cdot4\cdot5\cdot6\cdot7} + \frac{1}{5\cdot6\cdot7\cdot8\cdot9} - \cdots\right\},$$

$$\text{(b)} \qquad \pi = 3.15 - 360\left\{\frac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8} - \frac{1}{4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10} + \cdots\right\},$$

$$\text{(c)} \qquad \log 2 = \frac{17}{24} - 12\left\{\frac{1}{2\cdot3\cdot4\cdot5\cdot6} - \frac{1}{4\cdot5\cdot6\cdot7\cdot8} + \cdots\right\},$$

$$\text{(d)} \qquad \pi = \frac{64}{21} + 96\left\{\frac{1}{1\cdot3\cdot5\cdot7\cdot9} - \frac{1}{3\cdot5\cdot7\cdot9\cdot11} + \cdots\right\}.$$

3739 (*Paul Erdös*). Given $n+1$ integers, $a_1, a_2, \cdots, a_{n+1}$, each less than or equal to $2n$, prove that at least one of them is divisible by some other of the set.

3740 (*Paul Erdös*). From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

3743 (*Norman Anning*). Two congruent coplanar parabolas have the same line as axis and open in the same direction. Tangents are drawn to the inner from any point of the outer. Prove that the area bounded by the tangents and the arc joining their points of contact is invariant.

3746 (*Paul Erdös*). Given a triangle ABC, with the sides $a>b>c$, and any point O in its interior. Let AO, BO, CO cut the opposite sides in P, Q, R. Prove that

$$OP + OQ + OR < a.$$

3747* (*Frank Irwin*). Find the single condition that all the roots of the secular equation

$$\begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix} = 0$$

should be equal, the a's being real and $a_{ji}=a_{ij}$; and hence determine the cases in which all the roots are equal.

3763 (*Paul Erdös*). Given any simple polygon P which is not convex, draw the smallest convex polygon P' which contains P. This convex polygon P' will contain the area P and certain additional areas. Reflect each of these additional areas with respect to the corresponding added side, thus obtaining a new polygon P_1. If P_1 is not convex, repeat the process, obtaining a polygon P_2. Prove that after a finite number of such steps a polygon P_n will be obtained which will be convex.

3764 (*J. S. Frame*). Prove that

$$\sinh^{n+1} nx + \cosh^{n+1} nx < \cosh^n (n+1)x$$

for all real positive values of x, and for all real values of n greater than unity. Show that the inequality is reversed when $0<n<1$ and $x>0$; whereas equality holds either if $n=0$ or 1, or if $x=0$.

3766* (*M. E. Levenson*). Evaluate

$$\int_0^\infty e^{-x} \log^2 x \, dx.$$

3771 (*Hansraj Gupta*). Prove that

$$61! + 1 \equiv 0 \bmod 71, \qquad 63! + 1 \equiv 0 \bmod 71.$$

Prove the general result of which these are particular cases.

3776 (*E. P. Starke*). Determine all triangles whose sides are relatively prime integers and such that one angle is double another.

3778 (*L. J. Adams*). Solve

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot$$

3780 (*J. M. Feld*). In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio $A_jD_i:D_iA_k=p_i:q_i$, where ijk is a cyclic permutation of 123. The transversals A_iD_i and A_jD_j intersect in P_k. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1} \Big/ \frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case.

3783 (*J. Barinaga*). Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix}$$

$$= 1!2!3! \cdots n!\left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}\right].$$

3784 (*J. M. Feld*). Prove that for any positive integer k the following determinant is zero.

$$\begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(2k+1)!} & \frac{1}{(2k+2)!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(2k)!} & \frac{1}{(2k+1)!} \\ 0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(2k-1)!} & \frac{1}{(2k)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2!} \end{vmatrix}.$$

3792 (*F. Morley*). A square is divided into n^2 unit squares, like a chessboard. Any two horizontal lines and any two vertical lines form a rectangle. We count a square as a rectangle. Each rectangle has a breadth b, less than or equal to its length. There is one rectangle of breadth n, namely the original square. Prove that there are 2^3 rectangles of breadth $n-1$, 3^3 of breadth $n-2$, $\cdots$, n^3 of breadth 1.

Deduce the formula

$$1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2.$$

3801 (*D. H. Lehmer*). Show that

$$\operatorname{arccot} 1 = \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 13 + \operatorname{arccot} 34 + \cdots,$$

where these integers constitute every other term of the Fibonacci series and satisfy the recurrence $u_{n+1} = 3u_n - u_{n-1}$.

3802 (*D. H. Lehmer*). Let 0, u_1, u_2, u_3, $\cdots$ be a sequence of numbers satisfying the recurrence $u_{n+1}=au_n+bu_{n-1}$, and consider the function

$$f(x) = \sum_{n=1}^{\infty} u_n x^n/n!.$$

Show that $f(x) = -e^{ax}f(-x)$.

3805* (*R. E. Gaines*). Determine a point P on $b^2x^2-a^2y^2=a^2b^2$ such that the tangent and normal lines at that point shall be normal and tangent respectively to $b^2x^2-a^2y^2=-a^2b^2$, and hence, that if the hyperbola and its conjugate be together considered as a single curve, $(b^2x^2-a^2y^2)^2-a^4b^4=0$, a rectangle may be drawn which is both an inscribed and a circumscribed figure.

3810 (*Oystein Ore*). One may define a new "multiplication" in the system of all positive real numbers by putting

$$[a, b] = a^b.$$

Determine all positive *rational* numbers for which this multiplication is:

I) *Commutative* $[a, b]=[b, a]$.

II) *Associative* $[a, [b, c]]=[[a, b], c]$.

III) *Right-hand or left-hand distributive*

$$[(a+b), c] = [a, c] + [b, c], \qquad [c, (a+b)] = [c, a] + [c, b].$$

3814 (*I. J. Schoenberg*). Let $f(t)$ be a one-valued and complex-valued function which satisfies the functional relation

$$e^{iks}f(t) = f(t+s) - f(s) \tag{1}$$

for all real values of t and s, where k is a real constant $\neq 0$. Prove that

$$f(t) = C(e^{ikt} - 1)/ik \qquad (C \text{ a constant}). \tag{2}$$

Remark. It should be noticed that for $k=0$, (1) and (2) reduce to (1′) $f(t+s)=f(t)+f(s)$ and (2′) $f(t)=Ct$ respectively. It is known that (1′) implies (2′) only if some additional assumption is made on $f(t)$, for instance, boundedness in the neighborhood of some point (see G. Hamel, *Mathematische Annalen*, vol. 60, 1905, pp. 459–462). No such assumption is needed if $k\neq 0$.

3815* (*P. Turán*). Given the function of a real variable x

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{4^n};$$

show that there is a positive constant c_1, independent of x, such that

$$|f(x)| < c_1 \log\log x, \qquad x > e. \tag{1}$$

Show also that there exists a sequence $x_1<x_2<\cdots\to\infty$, and a positive constant c_2, independent of x, such that

$$(2) \qquad |f(x_\nu)| > c_2 \log\log x, \qquad \nu = 1, 2, \cdots.$$

3816 (*E. Weiszfeld*). Given the function of the complex variable z

$$f(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|},$$

where a_i, $(i=1, 2, \cdots, n)$, denotes any given complex number; show that $|f(z)|$ takes on its maximum value in any domain not containing the points a_i at the boundary of the domain.

3820 (*Paul Erdös*). Let $a_1 < a_2 < \cdots < a_n < 2n$ be positive integers such that no one of them is divisible by any other member of the sequence; then $a_1 \geqq 2^k$, where k is defined by the inequalities $3^k < 2n < 3^{k+1}$. This estimate for a_1 is the best possible.

3824 (*F. A. Lewis*). Determine the roots of the characteristic equation of the matrix

$$V = (v_{rc}) = (\epsilon^{(r-1)(c-1)}), \quad \text{where } \epsilon = e^{2\pi i/n}.$$

3826 (*W. Macray*). Solve the system of partial differential equations,

$$U\frac{\partial W}{\partial x} + 2W\frac{\partial U}{\partial x} = 0, \qquad U\frac{\partial W}{\partial x} + 2W\frac{\partial V}{\partial y} = 0, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

3829 (*J. D. Hill*). Let C be a simple closed rectifiable plane curve and P an arbitrary point inside of C. (a) Show that there exist two points A and B on C such that P bisects the chord AB. (b) Does this property remain true if the curve is non-rectifiable?

3834 (*Paul Erdös*). Let $a_1 < a_2 < \cdots < a_n \leqq 2n$ be a sequence of positive integers. Then

$$\min\,[a_i, a_j] < 6([n/2] + 1),$$

where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j. This is the best possible estimate.

3835 (*Paul Erdös*). Let $a_1 < a_2 < \cdots < a_n < 2n$ be a sequence of positive integers. Then

$$\max\,(a_i, a_j) > \frac{38n}{147} - c,$$

where c is independent of n, and (a_i, a_j) denotes the greatest common divisor of a_i and a_j. This is the best possible estimate.

3863* (*S. B. Townes*). The vertices of a simplex in n dimensions are O, P_1, $P_2, \cdots, P_n$. Let $OP_i = (a_{ii})^{1/2}$ and the cosine of the angle between OP_i and OP_j be $a_{ij}/(a_{ii}a_{jj})^{1/2}$, $(i, j = 1, 2, \cdots, n)$. Show that r_n, the radius of the circum-

scribed hypersphere, is given by

$$r_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ii}a_{jj}A_{ij}/4 \mid a_{ij} \mid ,$$

where A_{ij} is the cofactor of a_{ij} in the symmetric determinant $|a_{ij}|$.

3868 (*Arnold Dresden*). Prove that, if $0 \leqq \alpha \leqq x_1 \leqq x_2$ and n is a positive integer, then

$$x_2^{1/n} - x_1^{1/n} \leqq (x_2 - \alpha)^{1/n} - (x_1 - \alpha)^{1/n}.$$

3872 (*E. R. Ott*). One man has m coins and another has n. They match coins until one player has won all the coins. Find the average number of tosses required to end the game.

3879 (*Victor Thébault*). Show that it is possible to determine a plane section, limited by three faces of a tetrahedron which divides both the surface and the volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

3885 (*Victor Thébault*). The product of n consecutive positive integers, n being odd, is divisible by their sum, except in the case where, n being prime, the arithmetic mean of the n integers is divisible by n. Examine the case where n is even.

3893 (*Norman Anning*). From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible triangles is the integer nearest to $n^2/12$.

3903 (*Simon Mowshowitz*). Denote $[(n!)!]!$ by $n(!)^3$, *etc.*, $n(!)^0 = n$. Prove that for $k \geqq 2$,

$$\frac{n(!)^k}{(n!)^{[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!}}$$

is an integer.

3907 (*V. W. Graham*). Find all the roots of the equation

$$\frac{(x^2 - x + 1)^3}{x^2(x - 1)^2} = \frac{(a^2 - a + 1)^3}{a^2(a - 1)^2}.$$

3909 (*Béla Sz.-Nagy*). Let $P_1, P_2, \cdots, P_{n+2}$ be $n+2$ points in n-dimensional space, $n \geqq 2$, no three of the points on the same straight line. Let the symbol $[P_{i_1}P_{i_2} \cdots P_{i_s}]$ denote the least convex polyhedron containing in its interior the points indicated. Show that if $n = 2, 3$, then there exist always subscripts i, k, such that $[P_iP_k]$ is not an edge of $[P_1P_2 \cdots P_{n+2}]$. Show that if

$n>3$, then this statement does not remain true.

3914 (*W. B. Campbell*). A dog is tied to a rope of length L, which is fastened on the other side of a smooth topped fence of height h at a point a units from the top, $L>h+a$. Discuss the shape and dimensions of the region over which he can roam; and find its area.

3918* (*B. M. Stewart*). Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

3919 (*Richard Bellman*). Prove that

$$\begin{vmatrix} \dfrac{x}{1-x} & 1 & 0 & 0 & 0 \cdots & 0 \\ \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 2 & 0 & 0 \cdots & 0 \\ \dfrac{x^3}{1-x^3} & \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 3 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{x^r}{1-x^r} & \dfrac{x^{r-1}}{1-x^{r-1}} & \cdot & & \cdot \cdots\cdot & \dfrac{x}{1-x} \end{vmatrix} = \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

3923 (*R. E. Gaines*). It is known that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus. Show that the diameter d of the circle is given by $d\sin\alpha\sin\beta\sin\gamma=a$, where α,β,γ are the angles which the tangents make with the axis of the parabola, $y^2=4ax$.

3939* (*J. H. Curtiss*). Define the function $f(x)$ by the relations

$$\begin{aligned} f(x) &= x\sin(1/x), && x>0, \\ &= 0, && x=0. \end{aligned}$$

Show that $|f(x_1)-f(x_0)|/|x_1-x_0|^\alpha$ is bounded for $0\leqq x_0\leqq 1$, $0\leqq x_1\leqq 1$, if and only if $\alpha\leqq 1/2$.

3942 (*H. E. Tester*). A man is standing at the junction of two perpendicular crossroads, and his dog, at a distance a from the junction along one of the roads, is watching him. At a given instant the man starts to walk with speed v along the other road, and the dog to run directly towards his master with speed $2v$. Determine the curve of pursuit.

3954* (*Oystein Ore*). From three elements a, b, c in given order one can form two products, namely $(ab)c$ and $a(bc)$ when the associative law is not assumed. Similarly four elements a, b, c, d give $N_4=5$ products; $[(ab)c]d$, $[a(bc)]d$, $a[(bc)d]$, $a[b(cd)]$, $(ab)(cd)$. Find the general expression for the number N_i of products with i factors.

3957* (*Otto Dunkel*). Given a triangle ABC with angles $A<B<C$, show that there are precisely one, two, three straight line segments which bisect both its perimeter and area according as

$$1-\frac{\sin A}{\sin B} \gtreqless 2\tan^2(A/2)\tan^2(B/2),$$

where we may replace B by C. If $B=C$, there are one, two, three such segments, according as $A \lesseqgtr A_0$, where $\sin(A_0/2)=\sqrt{2}-1$.

3965* (*H. S. Wall*). Show that for a real or complex x, $|x|\leqq 1$,

$$\frac{|x|}{1+|x|} \leqq |\log(1+x)| \leqq \frac{|x|(1+|x|)}{|1+x|}.$$

3967 (*V. Thébault*). For a given triangle ABC a second triangle $A'B'C'$ is formed, where AA', BB', CC' are segments of altitudes and $AA'/BC=BB'/CA=CC'/AB=k$. (1) Show that the two triangles have the same centroid. (2) Examine the variation of the area of $A'B'C'$. (3) For what value of k do the two triangles have the same angle of Brocard? (4) If $k=\pm 1$, show that the centers of squares constructed exteriorly, or interiorly, on the sides of $A'B'C'$ are the vertices of ABC.

3980 (*Esther Szekeres*). The symmetric polynomials $y_1, y_2, \cdots, y_n$ in the variables $x_1, x_2, \cdots, x_n$ are of the degrees indicated by the subscripts, and are algebraically independent. If $f(x_1, x_2, \cdots, x_n)$ is any given polynomial symmetric in the x's, show that it can be expressed as a polynomial in the y's.

3990 (*V. Thébault*). Let A', B', C' be the centers of squares $BCA_1'A_2'$, $CAB_1'B_2'$, $ABC_1'C_2'$ constructed interiorly on the sides of triangle ABC with the centroid G and the angle V of Brocard. If $\cot V=7/4$, show that: (1) The centers A'', B'', C'' of the squares constructed interiorly on the sides of $A'B'C'$ lie on a straight line through G. (2) The angle V' of Brocard of $A'B'C'$ is such that $\cot V'=2$. (3) The straight lines joining A, B, C respectively to the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are parallel. (4) The distance of the circumcenter from the orthocenter of the orthic triangle is equal to one fourth of the perimeter of the last triangle.

3996 (*E. H. Clarke*). Sum the series

$$\sum_{n=1}^{\infty} [(n-1)k]!/(nk)!,$$

where k is any integer greater than unity.

3999* (*G. B. Van Schaack*). Let $f(x)$ be a polynomial of degree n with distinct real roots x_i, $(i=1, 2, \cdots, n)$. Let λ_i be the reciprocal of the slope of the curve $y=f(x)$ at $x=x_i$. Let ρ_j, $(j=1, 2, \cdots, n-1)$, be the algebraic radius of curvature of the curve at the critical point of the curve which lies between x_j and x_{j+1}. (a) Show that if $n>1$, then $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. (b) Show that if $n>2$, then $\rho_1+\rho_2+\cdots+\rho_{n-1}=0$.

4002 (*F. A. Lewis*). Give an interpretation to the function that results from the Euler ϕ-function when the minus signs are changed to plus, namely $f(n)=n(1+1/p_1)(1+1/p_2)\cdots(1+1/p_k)$.

4009 (*J. H. M. Wedderburn*). If the roots of $x^n-c_1x^{n-1}+\cdots+(-1)^nc_n$ are the variables $x_1, x_2, \cdots, x_n$, find the Jacobian of $c_n, c_{n-1}, \cdots, c_1$ with respect to $x_1, x_2, \cdots, x_n$.

4012* (*V. Thébault*). Find a number of n digits $N_0=a_1a_2\cdots a_n$, $(a_1\neq 0)$. such that if we transpose the first k digits from left to right, $(k=1, 2, 3, \cdots, n-1)$, the $n-1$ numbers thus obtained $N_1=a_2a_3\cdots a_na_1$, $N_2=a_3a_4\cdots a_na_1a_2$, $\cdots$, $N_{n-1}=a_na_1a_2\cdots a_{n-1}$ are each multiples of N_0.

4014* (*P. Erdös*). Show that, if S_1 and S_2 are two squares contained in the unit square so that they have no point in common, the sum of their sides is less than unity.

It is very likely true that, if we have k^2+1 squares contained in the unit square so that no two of them have a point in common, the sum of their sides is less than k.

4019 (*Robin Robinson*). Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

4021 (*Orrin Frink, Jr.*). The differential operator D^2+1 may be factored in many ways; for example it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x\cdot D)(\cos x\cdot D+\sin x)$, or $(\sin x\cdot D+2\cos x)(D\csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D+a)^2+b^2=r^{-1}\left[D+a-b\tan(bx+c)-r'/r\right]\cdot r\left[D+a+b\tan(bx+c)\right],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

4023 (*J. A. Greenwood*). Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

4042 (*Henry Scheffé*). Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

4045 (*A. M. Glicksman*). Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right).$$

4046* (*Otto Dunkel*). Show that γ, Euler's constant, is given by

$$\gamma = 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots\right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}\cdot$$

4047 (*T. R. Running*). Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, and area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

4050 (*Arnold Dresden*). If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot (a_k - a_i) = \sin \frac{n\pi}{2}\cdot$$

4051 (*Arnold Dresden*). If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \cot a_j \prod_{i=1, i\neq j}^{n} \cot (a_j - a_i) + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin \frac{(n+1)\pi}{2}.$$

4053* (*E. P. Starke*). Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

4054* (*V. Thébault*). Find the base less than 100 for which the number 2101 is a perfect square.

4065* (*P. Erdös*). (1) Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Show that the n points lie on a straight line.

(2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

4066 (*Richard Bellman*). Prove that

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{e}{x} - \frac{e}{1!(x+1)} + \frac{e}{2!(x+2)} - \cdots \right] \frac{dx}{\Gamma(x)}.$$

4067 (*H. S. Wall*). Prove that, if $0<x<1$,

$$\prod_{n=1}^{\infty} (1 - x^{2n-1}) = 1 \Big/ \left[1 + \sum_{k=1}^{\infty} x^{k(k+1)/2}/(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)\right].$$

4070 (*P. Erdös*). Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho + r \leqq m$.

Correction. The proposer intended to exclude obtuse-angled triangles.

4072 (*Richard Bellman*). Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^6)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

4083 (*P. Erdös*). Let $a_1 < a_2 < \cdots < a_x \leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x \leqq \pi(n)$,

where $\pi(n)$ denotes the number of primes not exceeding n.

4085 (*V. Thébault*). Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

4086 (*P. Erdös*). Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon and let O be any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation:

$$\pi\left(1 - \frac{1}{2n+1}\right) \leqq A_iOA_j \leqq \pi.$$

4091 (*Morgan Ward*). Given the three series

$$z - \frac{z^5}{2\cdot4\cdot5} + \frac{z^9}{2\cdot4\cdot6\cdot8\cdot9} - \frac{z^{13}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot13} + \cdots,$$

$$\frac{z^3}{2\cdot3} - \frac{z^7}{2\cdot4\cdot6\cdot7} + \frac{z^{11}}{2\cdot4\cdot6\cdot8\cdot10\cdot11} - \frac{z^{15}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot15} + \cdots,$$

$$\frac{z^2}{1\cdot2} - \frac{z^6}{1\cdot3\cdot5\cdot6} + \frac{z^{10}}{1\cdot3\cdot5\cdot7\cdot9\cdot10} - \frac{z^{14}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot14} + \cdots,$$

prove that the sum of the squares of the first two series is double the third series.

4093 (*B. M. Stewart*). A store conducted the following game in which the player paid 10¢, and if he won received \$1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had fourteen or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the first throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

4101 (*R. C. Buck*). Show that

$$(n) = \begin{vmatrix} (1,1)^\lambda & (1,2)^\lambda \cdots (1,n)^\lambda \\ (2,1)^\lambda & (2,2)^\lambda \cdots (2,n)^\lambda \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (n,1)^\lambda & (n,2)^\lambda \cdots (n,n)^\lambda \end{vmatrix}$$

$$= (n!)^\lambda\left(1 - \frac{1}{2^\lambda}\right)^{[n/2]}\left(1 - \frac{1}{3^\lambda}\right)^{[n/3]}\left(1 - \frac{1}{5^\lambda}\right)^{[n/5]}\cdots,$$

where (i, j) means the greatest common divisor of the integers i, j.

4104 (*E. T. Bell*). Two symmetric functions, $M(x_1, \cdots, x_n)$, $S(x_1, \cdots, x_n)$, of n non-negative integers $x_1, \cdots, x_n$ are defined as follows

$$M(x_1, \cdots, x_n) \equiv M'(x_1) \cdots M'(x_n),$$

in which $M'(x) = 1, -1, 0$, according as $x=0$, $x=1$, $x>1$; if $S_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $x_1, \cdots, x_n$,

$$S(x_1, \cdots, x_n) \equiv 1 + \sum_{j=1}^{n} jS_j(x_1, \cdots, x_n).$$

Prove that $\sum M(x_1-b_1, \cdots, x_n-b_n)S(b_1, \cdots, b_n)$ equals the number of positive integers in the set $x_1, \cdots, x_n$; and equals unity if $x_1=x_2= \cdots =x_n=0$, where the summation refers to all integers b_i such that $0 \leqq b_i \leqq x_i$, $(i=1, 2, \cdots, n)$.

4108* (*G. Pólya*). Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2=11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2=7$. Prove that

$$(1) \qquad {}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1+x)(2+x)\cdots(n-1+x),$$

$$(2) \qquad {}_nQ_1x + {}_nQ_2x(x-1) + \cdots + {}_nQ_nx(x-1)\cdots(x-n+1) = x^n.$$

4118* (*Otto Dunkel*). Show that

$$\sum_{t=0}^{n}(-1)^{n+t}\frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3)\cdots n}{6!8}[15n^3+30n^2+5n-2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

4124 (*T. W. Anderson, Jr.*). Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries in the ith row be r_i, $i=1, 2, \cdots, n$, and the sum of the entries in the jth column be c_j, $j=1, 2, \cdots, n$. For specified r_i and c_j, positive integers or zero, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j,$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

4132 (*T. H. Matthews*). If an aircraft travels at a constant air speed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

4133 (*A. L. Putnam*). Let a, b, and c be integers with $b\neq 0$, and let d and f be the respective greatest common divisors of a and b and of c and b. Then if

$$a \not\equiv \pm d \pmod{b} \quad \text{and} \quad c \not\equiv \pm f \pmod{b},$$

there is an infinite number of integers k for which the equation

$$ax + bxy + cy = k$$

has no solution in integers.

4136 (*H. S. M. Coxeter*). The equation

$$\cos \pi x + \cos \pi y + \cos \pi z = 0, \qquad 0 \leqq x \leqq \tfrac{1}{2} \leqq y \leqq z \leqq 1,$$

has the trivial solutions $y=\frac{1}{2}$, $z=1-x$, and $y=\frac{2}{3}-x$, $z=\frac{2}{3}+x$. It has also the non-trivial solution $x=\frac{1}{5}$, $y=\frac{3}{5}$, $z=\frac{2}{3}$. Prove that it has no other rational solution.

4137 (*P. Erdös*). Given an integer $x\leqq n^2/4$ which has no prime factor greater than n, show that $n!\equiv 0 \bmod x$.

4138* (*G. Pólya*). Given two positive integers p and q define

$$S_{p,q} = \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2p}-\frac{1}{3}-\frac{1}{5}-\cdots-\frac{1}{2q+1}+\frac{1}{2p+2}+\cdots$$
$$+\frac{1}{4p}-\frac{1}{2q+3}-\cdots-\frac{1}{4q+1}+\cdots,$$

$$P_{p,q} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots$$
$$\times\left(1-\frac{1}{2q+1}\right)\left(1+\frac{1}{2p+2}\right)\cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots = S_{1,1} = 1 - \log 2,$$

and the product by rearranging correspondingly

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots = P_{1,1}.$$

Show directly that $P_{p,q}=(p/q)^{1/2}$ and hence derive the well known result

$$S_{p,q} - S_{1,1} = \log\,(p/q)^{1/2}.$$

4142 (*G. Pólya*). Find a sequence of real numbers $a_1, a_2, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges $\cdots$. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l=1, 2, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

4143 (*Paul Erdös*). Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n = 3$.

4152* (*W. J. Taylor*). Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\dfrac{N+x}{2}\right)!\left(\dfrac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

4161* (*R. E. Gaines*). Along a straight road a miles long are n persons. What is the probability that no two persons are less than b miles apart?

4171 (*Tibor Rado*). Let $\mathbf{x}_n$, $n=0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n| > \delta$ for all n, where δ is a fixed positive constant, and absolute value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n| + |\mathbf{x}_0| - |\mathbf{x}_n + \mathbf{x}_0| \to 0 \qquad \text{as } n \to \infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n - \mathbf{x}_0| \to 0$ as $n \to \infty$.

4172 (*R. P. Agnew*). Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

4173 (*Herbert Robbins*). Let $a < b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a \leqq t \leqq b$. By the law of the

mean for integrals, for every $p>0$ there will exist a unique number x_p, $a \leqq x_p \leqq b$, such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty} x_p$.

4174 (*Irving Kaplansky*). Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2=x$. Show that a ring in which $x^2=\pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

4176 (*H. S. M. Coxeter*). Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers.

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

4185 (*B. M. Stewart*). In an unexplored region known as Wild Basin, a hunter found himself lost. But he had on hand a compass, and there were visible on two distant peaks fire ranger stations, A and B, whose bearings from his own cabin, O, he knew. From one observation point, C, the hunter took bearings on A and B; walking to another observation point, D, nearby, he took bearings on A, B, and C. Somehow he felt these seven bearings ought to enable him to find the direction homeward, that is, the bearing from D to O.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

4186 (*Fritz Herzog*). It is known from the theory of probability that for x fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_rx^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of n and r with $n\geqq 1$, $0\leqq r\leqq n$ and for all values of x with $0<x<1$

$${}_nC_rx^r(1-x)^{n-r} < 1/[2enx(1-x)]^{1/2},$$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

4190 (*Norman Anning*). If a, b, c, R are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2+y^2+z^2 = R^2$$
$$ax+by+cz = 0.$$

4198 (*C. D. Olds*). Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where t is real, n is a positive integer, and $w(t)=(t-1)(t-2)\cdots(t-n+1)$.

4202 (*Vladimir Karapetoff*). In a certain game of chance, consecutive numbers from 1 to n are written on a table. The same number of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that k of the n discs covered the correct numbers.

4203 (*N. J. Fine*). If one is allowed n weighings on a beam balance, what is the maximum number A_n of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light?

4212 (*H. F. Sandham*). Evaluate

$$\int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2}.$$

4216 (*Herbert Robbins*). Write $\{x_n\}\vec{c}\alpha$ if $\lim_{n\to\infty}(x_1+\cdots+x_n)/n=\alpha$. A function $f(x)$ is said to be C.c. (Cesaro continuous) at $x=\alpha$ if $\{x_n\}\vec{c}\alpha$ implies $f(x_n)\vec{c}f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is C.c. at every value of x, and that if $f(x)$ is C.c. at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

4220 (*Paul Erdös*). Let m be a positive integer with no prime factors greater than n and $m\leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

4222 (*J. H. Butchart*). If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$ respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

4229 (*Paul Erdös*). Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by A the region where $|f(z)|\leqq 1$ and by B the region where $|g(z)|\leqq 1$. Prove that A cannot properly contain B.

4235 (*Irving Kaplansky and D. C. Lewis*). Show that the determinant

$$\begin{vmatrix} (x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\ (x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ (x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1) \end{vmatrix}$$

is a constant times $(x-1)^{n^2}$.

4236 (*H. D. Grossman*). Write down two 1's, then a 2 between them, then a 3 between any two numbers whose sum is 3, then a 4 between any two numbers whose sum is 4, and so forth. Prove that the number of n's written down is $\phi(n)$, ($n>1$, where ϕ is Euler's totient).

4239 (*H. F. Sandham*). $AXBZ$ is a jointed rhombus connected with a fixed point O by two equal rods OA, OB. $OCZD$ is a jointed rhombus and YC, YD are equal rods. (Two Peaucellier cells, as it were, "cross joined.") Prove that as Y describes a circle, X describes a conic.

4240 (*Victor Thébault*). Determine the relations which must connect N, B, B', in order that the number N may be written with the same three digits in the system of numeration of base B as in the system of base B'. Having given B, find B' and N. Apply the results when $B=10$.

4249 (*W. B. Campbell*). A body is projected from a point O in a plane making an angle A with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle B with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for t_n, the time consumed in the nth flight, and for x_n, the coordinate of the point of impact at the end of the nth flight. Will it ever strike O again, and will any of its flights be vertical? What is the maximum x_n?

4250 (*Richard Bellman*). If

$$s_n=\sum_{k=1}^{n}a_k,\qquad \sigma_n=\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)a_k,$$

and

$$\sum_{n=1}^{\infty}|s_n-\sigma_n|^k<\infty,$$

for any $k>0$, prove that $\sum_{n=1}^{\infty}a_n$ is convergent.

4252 (*Paul Erdös*). It is well known that $(2n)!/n!(n+1)!$ is always an integer. Prove that for every k there are infinitely many n's such that $(2n)!/n!(n+k)!$ is an integer.

4254 (*Paul Erdös*). We have seven points in the plane. Prove that we can always select three which do not form an isosceles triangle. For six points this does not necessarily hold. (If A, B, C are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

4255 (*G. Pólya*). A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{(n-1)g}{1+(n-1)g} x_{n-1} + \frac{1}{1+(n-1)g} x_{n-2},$$

where g is a given positive quantity. Find an expression for the limit of x_n as $n \to \infty$.

4258 (*H. F. Sandham*). Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm\ 34 \cdot 42 \cdot 23 \pm 41 \cdot 13 \cdot 34 \pm 12 \cdot 24 \cdot 41 \pm 23 \cdot 31 \cdot 12 = 0,$$

where rs denotes the distance between the rth and sth points, and three of the signs differ from the fourth.

4259 (*Richard Bellman*). If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1+x^{n_k}} = x \prod_{k=1}^{\infty} (1+x^{n_k}), \qquad |x| < 1,$$

show that, except perhaps for order, $n_k = 2^k$.

4263 (*Howard Eves and Paul Halmos*). Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \cdots, a_k\}$. Since $E = E(0) + E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1 = 0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i = 1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1} = 0$ or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property that for any value of k it is true that $E(a_1, \cdots, a_k)$ is uncountable. Write E^* for the union of all $E(a_1, \cdots, a_k)$ for $k = 1, 2, 3, \cdots$; then E^* is a subset (in fact an uncountable subset) of E.

For certain positive integers k it is true that both $E(a_1, \cdots, a_k, 0)$ and $E(a_1, \cdots, a_k, 1)$ are uncountable: in fact this must happen for an infinite number of k's. (Otherwise, for a sufficiently large k, $E(a_1, \cdots, a_k)$ would not be uncountable, contrary to its construction.) Let $k_1, k_2, k_3, \cdots$ be the integers for which this is true, and write, for any $\{x_1, x_2, x_3, \cdots\}$ in E^*, $y_n = x_{k_n+1}$; then $\{y_1, y_2, \cdots\}$ is an infinite sequence of 0's and 1's. From the way in which the k_n are defined it follows that every possible sequence of 0's and 1's occurs as a y sequence, and that consequently the sequences $\{x_1, x_2, \cdots\}$ in E^* correspond (in possibly a many to one manner) to a set (*viz.* the set of all y sequences) having the power of the continuum. It follows that the cardinal number of E^* (and hence of E) cannot be less, and since E is a subset of X it cannot be greater. In other words it has been proved that every uncountable subset of a set having the power of the continuum has also the power of the continuum.

4264 (*G. Pólya*). Given $a>0$, $b>0$, and given that $f(x)$ is a non-linear function such that $f(0)=0$, $f(a)=b$, and that

$$f(x) \geqq 0, \qquad f''(x) \geqq 0, \qquad 0 \leqq x \leqq a,$$

give an analytic proof that

$$2\pi \int_0^a f(x)[1 + (f'(x))^2]^{1/2} dx < \pi b(a^2 + b^2)^{1/2}.$$

(The inequality becomes intuitive when both sides are interpreted as areas of curved surfaces.)

4267 (*C. F. Pinzka*). Let p be a prime greater than 3, and let r/ps be the sum of the harmonic series, $1+1/2+1/3+\cdots$, to p terms. Prove that p^3 divides $r-s$.

4268 (*Paul Erdös*). Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of a's up to n, then the upper density is defined as $\overline{\lim}\, f(n)/n$ as $n \to \infty$.) Then for suitable t the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r} \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many t with this property.

4270 (*S. H. Gould*). Let b be a fixed positive integer, $m=1, 2, \cdots$, and q irrational with $0<q<1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of b successive gaps contains exactly one multiple of $1+b/q$.

4273 (*I. S. Cohen*). Prove that for any positive odd integer n, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find the explicit expressions in the case $n=3$.

4278 (*Peter Ungar*). Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k \leqq \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

4281 (*M. S. Knebelman*). Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

4283 (*E. P. Starke*). The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of C, $f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

4286 (*H. F. Sandham*). Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\, dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

4287 (*C. R. Phelps*). Show that for any given integer $k > 1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

4288 (*J. W. Campbell*). Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth? He used the exclusion method of solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

4290 (*P. T. Bateman*). The function $-\log |2 \sin \frac{1}{2}x|$ has the Fourier series

$$\cos x + \tfrac{1}{2} \cos 2x + \tfrac{1}{3} \cos 3x + \cdots.$$

Prove that no partial sum of the series is ever less than -1.

4293 (*H. F. Sandham*). Evaluate

$$\sum_{n=1}^{\infty} \frac{\left(\frac{3-\sqrt{5}}{2}\right)^n}{n^3}.$$

4295 (*Irving Kaplansky*). Show that any group with more than two elements admits an automorphism other than the identity.

4296 (*H. S. Wall*). It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \cdot_{\cdot_{\cdot}}}}$$

converges, then its value is the numerically larger root of the equation $x^2 - px + q = 0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

4299 (*R. C. Lyness*). If the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares.

4300 (*Leo Moser*). Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1 \leqq p \leqq q \leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

4302 (*Joseph Rosenbaum*). Prove that if x and y have no common factor then every odd factor of

$$x^{2^n} + y^{2^n},$$

where n is a positive integer, is of the form $2^{n+1}m + 1$.

4303 (*G. T. Williams*). If

$$\theta_n = \int_0^1 x(x-1)\cdots(x-n+1)dx, \qquad \phi_n = \int_0^1 x(x+1)\cdots(x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

4305 (*H. F. Sandham*). Prove that

$$1 + \left(\frac{1+\frac{1}{2}}{2}\right)^2 + \left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2 + \left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2 + \cdots = \frac{17\pi^4}{360}.$$

4310 (*Paul Erdös*). Let $f(z) = z^n + \cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)| \leqq 1$. Prove that

there always exists a z_0 in A_f with $|f'(z_0)| \geqq n$. Equality occurs only for z^n.

4311 (*V. L. Klee, Jr.*). If k and x are positive integers, let $f_k^1(x) = k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j = 2, 3, \cdots$, let $f_k^j(x) = f_k^{j-1}[f_k^1(x)]$. Show that for $k \leqq 3$, the sequence $f_k^1(x), f_k^2(x), \cdots$, is eventually constant, while for $k \geqq 4$, the sequence is eventually monotonically increasing.

4312 (*R. C. Lyness*). In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of sixpenny savings stamps. When I asked Morris how many stamps the winner got, he said that I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number; that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

4314 (*N. J. Fine*). Prove the identity:

$$1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^2(1-x^2)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots$$
$$= \frac{1 - x + x^3 - x^6 + x^{10} - \cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

4315 (*Albert Wilansky*). Consider the Clairaut differential equation $y = px + f(p)$, where $p = y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

4317 (*Leo Moser*). Let G be an Abelian group and A a subset of order n, such that $a \in A$ implies $a^{-1} \in G - A$. Consider the n^2 (not necessarily distinct) elements of G of the form $a_i a_j$, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

4318 (*H. F. Sandham*). Show that

$$\frac{1}{(\sinh \pi)^2} + \frac{1}{(2 \sinh 2\pi)^2} + \frac{1}{(3 \sinh 3\pi)^2} + \cdots = \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) - \frac{11\pi^2}{180}.$$

4319 (*Paul Erdös*). If p is a prime greater than 3 and if $n = (2^{2p} - 1)/3$, show that $2^n - 2$ is divisible by n.

4321 (*Paul Erdös*). Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of integers such that $\lim_{k\to\infty} n_k/n_1n_2 \cdots n_{k-1} = \infty$. Show that $\sum_{i=1}^{\infty} 1/n_i$ is irrational.

4322 (*Pedro A. Piza*). If p is a prime, prove that

$$\binom{n}{p} \equiv [n/p] \pmod{p},$$

where $\binom{n}{p}$ and $[n/p]$ have their usual meanings.

Prove also that when $[n/p]$ contains p^s as a factor, then $\binom{n}{p}$ is divisible by p^s.

4325 (*Orrin Frink*). Show that on every simple closed plane curve there are four points which are the vertices of a square.

4328 (*Victor Thébault*). Given a triangle ABC whose altitudes are AA', BB', CC'. Prove that the Euler lines of the triangles $AB'C'$, $A'BC'$, $A'B'C$ are concurrent on the nine-point circle at a point P which is such that one of the distances PA', PB', PC' equals the sum of the other two.

4329 (*A. W. Goodman*). Let θ be an irrational number, $a = e^{i\theta\pi}$. Prove that $f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$ has the unit circle as a natural boundary.

4330 (*Paul Erdös*). Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

4332 (*Paul Erdös*). Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that from the sequence $a_i + a_j$, $i = 1, 2, \cdots, j = 1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

4336 (*Orrin Frink*). Find the arc of fixed length $l > 2a$, lying below the x-axis and joining the points $(-a, 0)$ and $(a, 0)$, which includes between itself and the x-axis an area of lowest possible center of gravity. This will be the form actually assumed by a weightless flexible arc supported at its ends if it is holding water.

4337 (*R. M. Redheffer*). If the numbers R_{nk} are defined by

$$\frac{1 - z^2}{\sin \pi z} \prod_{k=2}^{n} \sin(\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk} z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

4341 (*D. H. Browne*). The sequence $\{a\} = 3, 7, 47, 2207, 4870847, \cdots$, used for the determination of primality of the Mersenne numbers, is usually defined by $a_{n+1} = a_n^2 - 2$. Show that it may also be defined by $a_k = f_{2^k+1}/f_{2^k}$, where the f's are the Fibonacci numbers, $1, 1, 2, 3, 5, 8, \cdots$.

4342. (*R. J. Walker*). A spherical planet whose density at any point P is a

function only of the distance of P from the center of the planet has the following property. If a straight frictionless tunnel is bored between two points on the planet's surface the time required for an object to slide from one of these points to the other is independent of the positions of the points. Prove that the planet has constant density.

4345 (*Irving Kaplansky*). An element x in a ring is said to be right quasi-regular if there exists an element y with $x+y+xy=0$. It is evident that in a division ring, every element except -1 is right quasi-regular. Prove the converse: if every element in a ring A is right quasi-regular, with exactly one exception, then A is a division ring.

4346 (*N. S. Mendelsohn*). Prove that

$$n - 1 = \sum_{r=1}^{\infty}\left[\frac{n + 2^{r-1} - 1}{2^r}\right],$$

for any positive integer n. The brackets denote, as usual, the greatest integer function.

4348 (*D. A. Darling*). This problem was brought from Poland by Professor H. Steinhaus. It appears that Professor Banach was accustomed to carrying a box of matches in each of two coat pockets. To light his pipe, he would take a match from either box at random. The boxes contained originally n matches each. Banach's question is: when first a box is opened and found empty, what is the expected number of matches left in the other box?

4349 (*H. F. Sandham*). Prove that

$$\frac{2}{1}\bigg/\frac{5}{4}\bigg/\frac{8}{7}\bigg/\frac{11}{10}\cdots = \sqrt{3}.$$

4351 (*Albert Wilansky*). Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

4352 (*Paul Erdös*). Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1 < a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

4353 (*H. F. Sandham*). Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

4358 (*Paul Erdös*). Let $a_1 < a_2 < \cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i,$$

have the same property. (If the a's are chosen to be the first k primes, then we have a well known result due to Dickson.)

4360 (*Free Jamison*). Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely.

4361 (*Melvin Dresher*). If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

4363 (*Paul Erdös*). Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive upper density (*i.e.*, $\underline{\lim}\ a_k/k < \infty$). Then there exists an infinite subsequence such that no element divides another. In fact, there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k} = \infty$ and no a_{i_k} divides any other.

4365 (*Paul Erdös*). Let $a_1 < a_2 < a_3 < \cdots < a_k \leqq n$ be such that the least common multiple of any two a's exceeds n. Prove that

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2.$$

4366 (*Joseph Rosenbaum*). Determine the condition which two concentric spheres must satisfy in order that a tetrahedron can be simultaneously inscribed in one and circumscribed about the other. Give a construction for the tetrahedron.

4371 (*Albert Wilansky*). Define $\theta(a, h)$ as the largest number θ satisfying

$$\text{(i) } 0 < \theta < 1, \qquad \text{(ii) } f(a + h) = f(a) + hf'(a + \theta h),$$

where $f(x) = x^2 \sin (1/x)$ for $x \neq 0$, $f(0) = 0$.

Now set $\lambda(h) = [h \cdot \theta(0, h)]^{-1}$. Prove that as h tends to zero, $\lambda(h)$ tends to infinity in a step function manner; specifically, given $\epsilon > 0$, there is a number $H(\epsilon)$ such that for every h with $|h| < H$ there is an integer $n(h)$ such that $|\lambda(h) - (n + \frac{1}{2})\pi| < \epsilon$.

4372 (*Ky Fan*). For what real values of x does the sequence $f_n(x) = \sin 7^n \pi x$ converge and what is the limit?

4375 (*N. J. Fine*). Let $((x)) = x - [x] - \frac{1}{2}$. Prove that the sums

$$\sum_{n=1}^{m} ((2^n x + \tfrac{1}{2}))$$

are uniformly bounded.

4381 (*R. J. Walker*). The inverse-square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

4382 (*E. P. Starke*). Let $f_1(x)$ be Riemann integrable in the interval $0 \leqq x \leqq M$ and let

$$f_{n+1}(x) = \int_0^x f_n(x)dx, \qquad n = 1, 2, \cdots.$$

Show that

$$\phi(x) = \sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of $f_1(x)$, and find a simple expression for $\phi(x)$.

4385 (*Peter Ungar*). The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of $2^n + n - 1$ digits has the same property as the case $n = 3$ cited at the outset.

4387 (*Paul Bateman*). If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m) = \sum_{d\mid(n,m)} d^r \sigma_r(nm/d^2),$$

where d runs through all common divisors of n and m.

4388 (*Paul Erdös and W. H. Fuchs*). Let

$$f(z) = \prod_{i=1}^{n} (z - z_i), \qquad |z_i| \leqq 1.$$

Consider the set $|f(z)| \leqq 1$. Prove that it consists of at most $n-1$ components.

4389 (*F. J. Dyson*). Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N,$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

4391 (*Paul Bateman*). Given a fixed integer k and a complex-valued function $f(n)$ defined on the positive integers and such that $f(n_1) = f(n_2)$ for $n_1 \equiv n_2 \pmod{k}$, $|f(n)| \leqq 1$ for all n, $f(n) = 0$ for $(n, k) > 1$, and $\sum_{n=1}^{k} f(n) = 0$. Show that

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n} \right| < \log k.$$

4392 (*Paul Erdös*). Let $f(z)$ be analytic for $|z| \leqq 1$. Let z_0 be the point $(|z_0| = 1)$ where $|f(z)|$ assumes its maximum on the unit circle. Prove that $f'(z_0) \neq 0$.

4394 (*H. F. Sandham*). Evaluate

$$\int_0^\infty \frac{\log x}{e^x + 1} dx.$$

4399 (*Ky Fan*). Let $f(n)$ and $g(n)$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1) = 1$.

(2) $g(n) = na - 1 - f(n)$, a being an integer > 4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1), f(2), \cdots, f(n); g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that $f(n) = [\alpha n]$, $g(n) = [\beta n]$. The brackets denote the greatest integer function.

4400 (*C. D. Olds*). Every positive root of the equation $\tan x = x$ can be expressed as follows: $x = (p + \frac{1}{2})\pi - \theta$ where p is an arbitrary integer $\geqq 0$ and

$$\theta = C_0\xi + C_1\xi^3 + C_2\xi^5 + \cdots, \quad \xi = 1/(p + \tfrac{1}{2})\pi.$$

The coefficients C_0, C_1, C_2, $\cdots$ are positive rational numbers. Show that, for n large,

$$C_n = 12^{-1/6}\Gamma(1/3)\pi^{-2/3} \cdot (\pi/2)^{2n+1}(2n+1)^{-4/3}(1 + \omega_n),$$

where $\omega_n \to 0$ when $n \to \infty$, so that the above series remains convergent even for $p = 0$.

4402 (*Carl Cohen*). Let the series of functions $\mathrm{dl}^{(n)}(x)$ be defined as

$$\mathrm{dl}^{(n)}(x) = \int_0^x \mathrm{dl}^{(n-1)}(t)\,dt/t, \tag{1}$$

$$\mathrm{dl}^{(1)}(x) = -\int_0^x \log(1-t)\,dt/t. \tag{2}$$

Show that for $|x| < \pi/2$,

$$\tan x = \sum_{n=1}^{\infty} (2/\pi)^{2n}[\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1)]x^{2n-1}, \tag{i}$$

$$\sec x = 1 + i\sum_{n=1}^{\infty} (2/\pi)^{2n+1}[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)]x^{2n}. \tag{ii}$$

4413 (*Paul Erdös*). Let $a_1 = 2\cdot 3$, $a_2 = 3\cdot 5$, $a_3 = 5\cdot 7, \cdots, a_k = p_k\cdot p_{k+1}, \cdots$, where p_k is the kth prime. Denote by $f(x)$ the number of integers $\leqq n$ composed entirely of the a's (*i.e.*, the integers of the form $\prod a_i^{\alpha_i}$, $0 \leqq \alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}),$$

where $\frac{1}{2} < c < 1$.

4415 (*R. P. Boas, Jr. and W. K. Hayman*). Find all the values of α and β for which the series $\sum_{n=1}^{\infty} n^{\alpha} \sin n^{\beta}$ converges.

4416 (*D. J. Newman*). Let $a_1, a_2, \cdots$ be a sequence of integers such that $\sum_{n=1}^{\infty} 1/a_n$ diverges. Show that almost all integers have a factor in common with some a_n.

4418 (*A. C. Aitken*). Evaluate the determinant

$$a_n = \begin{vmatrix} b_1 & -1 & 0 \cdots & 0 \\ b_2 & b_1 & -2 \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ b_{n-1} & b_{n-2} & b_{n-3} \cdots & 1-n \\ b_n & b_{n-1} & b_{n-2} \cdots & b_1 \end{vmatrix}, \quad \text{where} \quad b_n = \frac{n^n}{n!}.$$

E4 (*W. F. Cheney, Jr.*) What is the simplest way to cut a wooden block 1 ft.$\times$1 ft.$\times$2 ft. into pieces which may be reassembled into a cube?

E24 (*R. K. Morley*). There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}.$$

Find the other two and confirm the statement that there are no others.

E36 (*B. H. Brown*). Show that the thirteenth of the month is more likely to be Friday than any one of the other days of the week.

E46 (*B. H. Brown*). show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{N}$$

is never an integer for any N.

E83 (*Morgan Ward*). Show that in any arithmetic progression of positive integers with common difference less than two thousand, at most ten consecutive integers can be primes.

E91 (*Morgan Ward*). Let d be the greatest common divisor of the two positive integers, a and b, with $a=a'd$ and $b=b'd$. Now if n is any integer greater than unity, show that (n^a+1) and (n^b-1) can not have any common factor greater than 2 as long as b' is odd.

E124 (*B. W. Jones*). Show that the volume generated by revolving a cube of edge a about one of its space diagonals is $\pi a^3/\sqrt{3}$.

E182 (*D. H. Lehmer*). Show that the infinite product

$$\left(1-\frac{i}{3}\right)^4\left(1-\frac{i}{17}\right)^4\left(1-\frac{i}{99}\right)^4\left(1-\frac{i}{577}\right)^4\cdots,$$

in which the successive denominators satisfy the recurrence $D_n=6D_{n-1}-D_{n-2}$, is purely imaginary.

E190 (*Fred Discepoli*). If a and n are positive integers greater than the positive integer b, then if $a^n+b^n=c^n$, c can never be an integer.

E198 (*J. E. Trevor*). A multiplication of a three-place number by a two-place number has the form

$$\begin{array}{r} p\,p\,p \\ \underline{p\,p} \\ p\,p\,p\,p \\ \underline{p\,p\,p\,p} \\ p\,p\,p\,p\,p \end{array}$$

The p's are all prime digits, different from unity. Determine their values and show that the solution is unique.

E230 (*Meyer Karlin*). Prove that the sum of the series:

$$S_n = C_0^n - C_1^{n-1} + C_2^{n-2} - C_3^{n-3} + \cdots$$

will be $+1$, 0 or -1 according as n is a positive integer of the forms $6m$ or $6m+1$, $6m-1$ or $6m+2$, $6m+3$ or $6m+4$, respectively.

E231 (*A. A. Bennett*). In a certain bank there were eleven distinct positions; namely, in decreasing rank, President, First Vice-President, Second Vice-President, Third Vice-President, Cashier, Teller, Assistant Teller, Bookkeeper, First Stenographer, Second Stenographer, and Janitor. These eleven positions are occupied by the following, here listed alphabetically, Mr. Adams, Mrs. Brown, Mr. Camp, Miss Dale, Mr. Evans, Mrs. Ford, Mr. Grant, Miss Hill, Mr. Jones, Mrs. Kane, Mr. Long. Concerning them the following facts only are known:

1. The Third Vice-President is the pampered grandson of the president, but is disliked by both Mrs. Brown and the Assistant Teller.
2. The Assistant Teller and the Second Stenographer shared equally in their father's estate.
3. The Second Vice-President and the Assistant Teller wear the same style of hats.
4. Mr. Grant told Miss Hill to send him a stenographer at once.
5. The President's nearest neighbors are Mrs. Kane, Mr. Grant and Mr. Long.
6. The First Vice-President and the Cashier live at the exclusive Bachelors' Club.
7. The Janitor has occupied the same garret room since boyhood.
8. Mr. Adams and the Second Stenographer are leaders in the social life of the younger unmarried set.
9. The Second Vice-President and the Bookkeeper were once engaged to be married to each other.
10. The fashionable Teller is son-in-law of the First Stenographer.
11. Mr. Jones regularly gives Mr. Evans his discarded clothing to wear, without the elderly Bookkeeper knowing about it.

Show how to match correctly the eleven names against the eleven positions occupied.

E275 (*J. A. Benner*). In a certain town it began snowing before noon and continued at a constant rate until dark. At noon a crew of men set out along the highway, clearing the snow from it as they went. They cleared two miles in the first two hours, but only one mile in the next two hours. If the crew clears equal volumes of snow in equal times, at what time did it begin to snow?

E276 (*W. F. Cheney, Jr.*) The inside dimensions of a rectangular box with a lid on it are three feet, four feet, and five feet. A post in the form of a right circular cylinder nine inches in diameter just fits diagonally in the box, touching all six inner faces. How long is the post? (Note that the axis of the post, if prolonged, would miss the corners of the box.)

E285 (*D. L. MacKay*). If in triangle ABC, $\sin^2 A+\sin^2 B+\sin^2 C=1$, prove that the circumcircle cuts the nine-point circle orthogonally.

E312 (*D. L. MacKay*). If the scalene triangle ABC has its external angle bisectors at B and C equal, show that $(s-a)/a$ is the geometric mean of $(s-b)/b$ and $(s-c)/c$.

E319 (*Joseph Rosenbaum*). An electric light bulb is connected to n switches in such a way that the light is lit only when each switch is closed. Each switch is controlled by a push button, successive depressions of which will alternately open and close that switch. The push buttons are not provided with the usual marks, "on" and "off".

It is required to find in what order the push buttons should be pressed so that the greatest possible number of pushes which may be required to turn on the light will be as small as possible. Generalize the solution to cover the case in which each switch closes only at every p-th push.

E328 (*C. A. Murray*). In the equation $y=x^3+px^2+qx$, show how to determine all pairs of integral values of p and q for which the equation $y=0$ will have distinct integral roots and the two bend points, integral coordinates.

E332 (*Joseph Rosenbaum*). Prove that $1^p+2^p-3^p+4^p-5^p-6^p+7^p+8^p- \cdots +(2^n-1)^p=0$ for every positive integer $p<n$, where the sign of a term of the form m^p is negative or positive according as m is or is not one of the 2^k integers following 2^{k+1}, where k is any integer.

E345 (*F. E. Wood*). Let $S=a+b+c$, and $T=ab+ac+bc$, where a, b and c are the sides of a triangle. Show that $3T\leqq S^2<4T$. What are the analogous inequalities for a tetrahedron?

E364 (*A. V. Richardson*). If $(1+x)^n/(1-x)^3=a_0+a_1x+a_2x^2+\cdots$, show that

$$a_0+a_1+a_2+\cdots+a_{n-1}=\frac{n}{3}(n+2)(n+7)\cdot 2^{n-4}.$$

E378 (*J. L. Brenner*). Find the number of integral values of B which make B^2+m a perfect square, for any given, fixed, integer m.

E380 (*W. F. Cheney, Jr.*). If the radius of a circle is any odd prime, p, there are just two different primitive Pythagorean triangles circumscriptible about that circle. Show that, for each such pair of triangles:

(A) their shortest sides differ by one;

(B) their hypotenuses exceed their corresponding longer legs by one and by two, respectively;

(C) the sum of their perimeters is six times a perfect square;

(D) as p increases without limit, the ratio of their least angles approaches the limit 2;

(E) as p increases without limit, the ratio of their areas approaches the limit 2; and finally,

(F) the smaller triangle can always be placed inside the larger, so as not to touch it.

E395 (*E. P. Starke*). In high school geometry texts and elsewhere one frequently meets the statement that the reason for the straightness of the crease in a folded piece of paper is that the intersection of two planes is a straight line. This is fallacious. What is the correct reason?

E399 (*Victor Thébault*). Prove that the product of the first n positive integers $(1\cdot 2 \cdots n)$ is divisible by their sum $(1+2+\cdots+n)$ if and only if $n+1$ is not an odd prime.

E400 (*H. S. M. Coxeter*). Show how to dissect a regular hexagon by straight cuts into the smallest possible number of pieces which can be reassembled to form an equilateral triangle (of the same area).

E402 (*Irving Kaplansky*). If n, r, and a are positive integers, the congruence $n^2\equiv n \pmod{10^a}$ obviously implies $n^r\equiv n \pmod{10^a}$. (When such a number n has only a digits, it is called an automorphic number.) For what values of r does $n^r\equiv n \pmod{10^a}$ imply $n^2\equiv n \pmod{10^a}$?

E413 (*H. T. R. Aude*). The graph of a cubic function $y=x^3+ax^2+bx+c$ crosses the x-axis at three distinct points, two of which are A and B. The lines AP and BQ are drawn tangent to the humps of the curve, the points of contact being P and Q. Show that the ratio of the distance AB to the horizontal distance from P to Q is constant.

E419 (*Victor Thébault*). In what direction must a billiard ball be hit in order to return to its starting point after a given even number of rebounds? Neglect spin of the ball.

E432 (*C. W. Trigg*). If a and b are the radii of two spheres, tangent to each other and to a plane, show that the radius x of the largest sphere which can pass between them is given by the formula $x^{-1/2}=a^{-1/2}+b^{-1/2}$.

E433 (*A. A. Bennett*). Two parallel vertical walls, separated by a distance of d feet, have level ground between them. Two ladders of length a and b feet, respectively $(a>b)$, abut each against a foot of one of these walls and lean against the other wall, crossing each other at a height of c feet above the ground. Show that a solution in integers is given by

$$ka = (su+tv)(s-t)(u+v), \qquad kb = (sv+tu)(s-t)(u+v),$$
$$kc = (su-tv)(sv-tu), \qquad kd = 2(stuv)^{1/2}(s-t)(u+v),$$

where s, t, u, v are any positive integers subject to the three conditions $u>v$, $sv>tu$, and $stuv$ is a perfect square, k being the greatest common divisor of the four right-hand members. What is the simplest particular solution in which a, b, c, d are all odd?

E435 (*David Segal*). Show that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

is a necessary and sufficient condition for p to be an odd prime.

E436 (*E. H. Johnson*). A tall rectangular piece of furniture, of length a and width b, is moved down a hallway of width c, and goes through a door whose width d barely allows its passage into an adjacent room. If we neglect the thickness of the wall, it is easily seen by the comparison of similar triangles that $d=ab/c$. If the wall has a thickness h, find the value of d in terms of a, b, c, and h.

E438 (*J. S. Frame*). If p is any odd prime, show that the decimal expansion of the fraction $1/p$ will repeat in $(p-1)/2$ digits or some factor thereof if and only if $p \equiv \pm 3^k \pmod{40}$.

E444 (*Harry Goheen*). Prove that there is no prime p such that $p^n+1=2^m$ if $n>1$, and that there is no prime p such that $p^n-1=2^m$ if $n>2$.

E456 (*Leopold Infeld*). What is the smallest popular vote by which a President can be elected in the U. S. A. under the present electoral system? Assumptions: N is the total popular vote, the popular vote in each state is proportional to the electoral vote (which you will have to look up); there are just two candidates.

E458 (*J. L. Brenner*). Prove that in any power of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

two elements in the main diagonal will be the same. Show that the same result holds for any matrix (a_{rs}) in which $a_{r1}=a_{2r}$, $a_{1r}=a_{r2}$, $(r>2)$, and $a_{11}=a_{22}$.

E460 (*Henry Scheffé*). Let $s(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$ be the sum of the first n terms of the harmonic series. A well-known expression for $s(n)$ which does not formally involve the sum of n terms is the integral

$$\int_0^1 \frac{u^n-1}{u-1}\,du.$$

It is desired to write $s(n)$ in the form $s(n)=f^{(n)}(0)$. Find an expression for $f(x)$ in terms of the integral of an elementary function.

E468 (*W. R. Ransom*). The Fibonacci numbers, defined by $f_1=f_2=1$, $f_{j+1}=f_{j-1}+f_j$, are known to yield a puzzle in which a square of side f_n is cut into four pieces which can apparently be rearranged to form a rectangle $f_{n-1}\times f_{n+1}$.

Show that the same four pieces can be rearranged to form a figure which appears to consist of two rectangles $f_{n-1}\times 2f_{n-2}$ connected by a rectangle $f_{n-4}\times f_{n-2}$, the error being again one unit of area.

E481 (*J. A. Todd*). Let

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

be two matrices of non-vanishing numbers, the elements of the second being the co-factors of the corresponding elements of the first. Prove that the relation

$$\begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} = 0 \quad \text{implies} \quad \begin{vmatrix} X_1^{-1} & Y_1^{-1} & Z_1^{-1} \\ X_2^{-1} & Y_2^{-1} & Z_2^{-1} \\ X_3^{-1} & Y_3^{-1} & Z_3^{-1} \end{vmatrix} = 0.$$

E491 (*E. T. Frankel*). Prove that $\sqrt[\sqrt{-1}]{-1}=23\frac{1}{7}$ approximately.

E544 (*E. P. Starke*). Show that it is possible to construct a tetrahedron such that the length of every edge, the area of every face, and the volume all are integers.

E546 (*W. E. Buker*). Show that $\frac{1}{5}x^5+\frac{1}{3}x^3+\frac{7}{15}x$ is an integer for every integral value of x.

E554 (*J. L. Woodbridge*). Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

E555 (*Howard Eves*). Consider a rectangular parallelopiped of dimensions $a\times b\times c$, made up of abc unit cubes. Imagine the edges of all these cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting network is of genus $p=2abc+bc+ca+ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

E560 (*S. H. Gould*). In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n<p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m=3$.

E564 (*Ivan Niven*). Let a, b, and n be any positive integers such that n divides a^n-b^n. Prove that n divides $(a^n-b^n)/(a-b)$.

E598 (*H. S. Wall*). Let $g_1, g_2, g_3, \cdots$ be any numbers such that $0<g_p<1$, $(1-g_p)g_{p+1}>\frac{1}{4}$, $(p=1, 2, 3, \cdots)$. Prove that as $p\to\infty$, $\lim g_p=\frac{1}{2}$.

E610 (*Howard Eves*). (a) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd. (b) What is the least area that a closed curve of constant diameter d may have? (Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction.)

E624 (*D. H. Browne*). Show that the integer nearest $n!/e$ is a multiple of $n-1$.

E628 (*W. C. Rufus*). Find the smallest positive integer, one half of which is a square, one third of which is a cube, and one fifth of which is a fifth power.

E670 (*C. D. Olds*). Sum the series $\sum_{r=1}^{2n-1} (-1)^{r-1}r/\binom{2n}{r}$.

E680 (*Gordon Pall*). Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

E681 (*W. B. Campbell*). From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

E687 (*Victor Thébault*). A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

E706 (*D. H. Browne*). For what values of n does $(a+b)^n$ yield only odd coefficients?

E710 (*D. H. Lehmer*). Find the inverse of the symmetric matrix of the nth order: $A=\|a_{ij}\|$, in which $a_{ij}=i/j$ for $i\leqq j$.

E711 (*H. S. M. Coxeter*). Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

E712 (*Donald Eves*). A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings?

E735 (*Paul Erdös*). Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

E736 (*Paul Erdös*). Let $a_1<a_2<\cdots<a_k\leqq n$, where $k>[(n+1)/2]$, be k positive integers. Then $a_i+a_j=a_r$ is solvable.

E737 (*V. L. Klee, Jr.*). Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a}\cos(b\log n)$ for all values of $a\leqq 1$, regardless of the value of b.

E740 (*Esther Szekeres*). Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

E744 (*Paul Erdös*). Let $a_1<a_2<\cdots<a_n\leqq 2n$ be n positive integers such that the least common multiple of any two is greater than $2n$. Then $a_1>[2n/3]$.

E750 (*Paul Erdös*). Find the number of intersections of the diagonals of a convex polygon of n sides.

E751 (*Alan Wayne*). Find the digits represented by the letters in the following addition, if no two different letters represent the same digit:

$$\begin{array}{r} F\,O\,R\,T\,Y \\ T\,E\,N \\ T\,E\,N \\ \hline S\,I\,X\,T\,Y \end{array}$$

E753 (*L. M. Kelly*). How can one convince a class in elementary analytics that if the inside of a race track is a non-circular ellipse, and the track is of constant width, then the outside is not an ellipse?

E756 (*G. Pólya*). Show that

$$\begin{vmatrix} a-x & 1 & 0 & 0 & \cdots & 0 \\ \binom{a}{2} & a-x & 1 & 0 & \cdots & 0 \\ \binom{a}{3} & \binom{a}{2} & a-x & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{a}{n} & \binom{a}{n-1} & \binom{a}{n-2} & \binom{a}{n-3} & \cdots & a-x \end{vmatrix}$$

$$=\binom{a+n-1}{n}-\binom{2a+n-2}{n-1}x+\binom{3a+n-3}{n-2}x^2-\cdots+(-1)^n x^n.$$

E774 (*Norman Anning*). Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one third of the area. Through a point four fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

E776 (*L. R. Ford*). "Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

E780 (*G. Pólya*). A lampshade has the shape of a frustum of a right circular cone. Its perimeter is P at the bottom, p at the top, and its slant height is s. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions P and $s+p(P-p)/8s$. You can even save paper for a flap to glue the ends together, except for the limiting case where $P=p$, when not a bit of paper is wasted.

E785 (*R. J. Walker*). Each of $n-1$ tanks, $T_1, \cdots, T_{n-1}$, holds V gallons of water, and an nth tank, T_n, hold V gallons of a salt solution containing M pounds of salt. Liquid is circulated at the rate of g gallons per minute from T_n to T_{n-1}, T_{n-1} to $T_{n-2}, \cdots, T_2$ to T_1, T_1 to T_n. How much salt is in T_n after t minutes?

E788 (*Leo Moser*). Consider a map on a spherical surface where the countries are determined by n great circles of which no three are concurrent. Show that if n is a multiple of four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

E791 (*G. W. Walker*). The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars", which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any

of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that his was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!". Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars". How much salary must the mathematician have been receiving?

E793 (*Joseph Rosenbaum*). With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

E806 (*Leo Moser*). Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce a solution to the original problem without *a priori* knowledge of what these integers are.

E812 (*Monte Dernham*). Find the shortest perimeter common to two different primitive Pythagorean triangles.

E813 (*C. W. Trigg*). Let S be the sum of the integer elements of a magic square of order three, and let D be the value of the square considered as a determinant. Show that D/S is an integer.

E817 (*E. V. Hofler*). If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

E819 (*H. F. Sandham*). If $S_n = 1/1 + 1/2 + \cdots + 1/n$, prove that $\gamma < S_p + S_q - S_{pq} \leqq 1$, where γ is Euler's constant.

E824 (*E. P. Starke*). We modify the harmonic series by taking the first term positive, the next two negative, the next three positive, *etc.* Show that this modified series is convergent.

E827 (*Leo Moser*). Show that the reciprocal of every integer greater than 1 is the sum of a finite number of consecutive terms of the infinite series $\sum_{j=1}^{\infty} 1/j(j+1)$.

E832 (*V. E. Dietrich*). If a circle has a center with at least one irrational coordinate, then there are at most two points on the circle with rational coordinates.

E834 (*Don Walter*). Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence, $1, 1, 2, 3, 5, \cdots, x, y, x+y, \cdots$, and the determinant is of order $n-1$.

E846 (*H. J. Hamilton*). The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the principal part of the infinitesimal."

Show that this is not definitive and give a valid definition.

E853 (*C. S. Ogilvy*). If $y_1=x, y_2=x^{y_1}, \cdots, y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

E854 (*Jerome C. R. Li*). Show that $\pi=\sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)!$.

E860 (*Leo Moser*). Show that if all the faces of a polyhedron have central symmetry then it can be dissected by a finite number of plane cuts and the pieces fitted together to form a solid cube.

E879 (*Josef Langr*). Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2 meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear (3) $A_2, A_3, B_3, B_1, C_1, C_2$ lie on a conic.

E880 (*Peter Ungar*). Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

E888 (*H. D. Grossman*). Show how to cut a hole in a cube through which another cube of equal size can pass.

E919 (*Leo Moser*). Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi ibmn/a}.$$

E920 (*T. G. Room*). If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I+S)^{-1}(S+T)(I+ST)^{-1}(I+S) = (I-S)(I+TS)^{-1}(S+T)(I-S)^{-1}.$$

E924 (*D. J. Newman*). Find $\lim_{n\to\infty} n \sin(2\pi en!)$.

E929 (*H. S. Shapiro*). Given three non-concurrent straight lines l_1, l_2, l_3 in the plane. Let T_i denote reflection in l_i and set $T=T_1T_2T_3$. Show that T^2 is a translation.

E930 (*Gordon Raisbeck*). If $\prod_{i=1}^{n}(x+r_i)\equiv\sum_{j=0}^{n} a_jx^{n-j}$, show that

$$\sum_{i=1}^{n} \tan^{-1} r_i = \tan^{-1}\frac{a_1-a_3+a_5-\cdots}{a_0-a_2+a_4-\cdots}$$

and

$$\sum_{i=1}^{n} \tanh^{-1} r_i = \tanh^{-1}\frac{a_1+a_3+a_5+\cdots}{a_0+a_2+a_4+\cdots}.$$

E931 (*H. D. Larsen*). Ten balls numbered from 0 to 9 inclusive are placed in an urn. Five of the balls are then drawn at random (without replacement) and arranged in a row. What is the probability that the number thus formed is divisible by 396?

E937 (*I. N. Herstein*). If p is a prime and $n \geqq p$, then

$$n! \sum_{pi+j=n} 1/p^i i! j! \equiv 0, \bmod p.$$

E943 (*S. H. Gould*). By analogy with the motion of the planets, it seems natural to assume for any central motion that, at least if the particle never passes through the center, (a) the distance of the particle from the center is a maximum only when the velocity of the particle is a minimum, (b) the velocity is a maximum only when the distance is a minimum. Prove (a) and give a counter-example of (b).

E945 (*Leo Moser*). If all the faces of a convex polyhedron have central symmetry show that there are at least eight vertices where exactly three edges meet. (The cube has exactly eight such vertices.)

A CLASSIFICATION OF MONTHLY PROBLEMS
(1918–1950)

Note. All problems proposed in the MONTHLY between the years 1918 and 1950, inclusive, appear in the following classification. The classification is first divided into the three major divisions of *algebra*, *geometry*, and *analysis*; each of these major divisions is then further divided into subdivisions corresponding roughly to areas of instruction, arranged in general order of advancement. Most of the subdivisions are further broken down into appropriate topics, alphabetically arranged within the subdivisions. Topics containing four or fewer proposals, and any residue of unclassified problems in a subdivision, appear under the topic heading *miscellaneous* near the end of the subdivision. The topic for a given problem was determined principally by the mathematical tools used in the published solution of the problem; many problems accordingly appear under more than one topic. The numbers of the four hundred best problems are printed in italics. At the end of each of the three major divisions is a list of the unsolved problems in that division. (These are the problems proposed between 1918 and 1950, inclusive, for which solutions are still lacking. There are other problems proposed in the same period that could also be classified as unsolved but the statements of these problems have been followed up by either a partial solution, further comment, or an Editorial Note.)

ALGEBRA

Elementary algebra

Indeterminate equations—2790, E57, E106, E115, E135, E158, E163, E249

Verbal problems—2858, 3379, E135, E147, E163, E211, E222, E234, E249, E273, E366, E368, E408, E581, E641, E671, E872

Miscellaneous (*equations; maxima and minima; simultaneous equations; etc.*)—2662, E39; *2871*, 4180; 3050, E234, E686; 3651

College Algebra

Factoring—2881, *3673*, 3864, E162, E343, E614, E690, E716

Identities—2800, *3418*, 3545, 4062, E26, E152, E355, E787

Inequalities—*3207*, *3703*, E19, E186, *E190*

Permutations and combinations—3561, *3731*, 3761, 4277, E28, E97, E213, E272, E306, E360, E579, E589, E599, E650, E678, E772, E777, E801, E907

Progressions—2760, 2766, 2896, 3626, E11, E15, E48, E82, *E83*, E132, E186, E317, E324, E576, E584, E655, E658, E696, E726, E745, E768, E876, E877, E926

Simultaneous equations—2797, 2880, 3360, E235, E594, *E832*

Miscellaneous (*equations; rate problems; etc.*)—2724, E616, E736, E842; 2974, 3218, E466; *3528*, *4312*, E61, E390, *E456*, *E546*

Mathematics of finance—2744, 2857, 3113, 3306, E158, E169

Simultaneous equations—2833, 2835, 3239, 3259, *3506*

Solutions—3523, *3621*, 3730, 3894, *3907*, E223, E228

Miscellaneous (bounds on roots; quartics; roots of unity; etc.)—3162, E743, E884; 3036, *3173*, E588; 2839, 2864, 3672, E518; 3135, 3311, 3391, 3631, *4136*

Number theory

Arithmetico-geometrical—2661, 2671, 2738, 2741, 3386, 3733, *3776*, 3944, 3955, *4047*, 4205, E121, E283, E308, E322, E331, E352, E374, E379, E410, E418, E480, E515, E525, *E544*, E595, E620, E695, E747

Congruences—3125, 3198, 3208, 3403, *3445*, 3472, 3488, *3682*, *3707*, *3721*, *3771*, 3933, 4121, 4297, *4319*, *4322*, *E402*, E425, *E435*, E446, E462, E464, E654, E792, E896, E911, *E937*

Digit problems—2906, *4054*, 4055, 4100, 4103, 4107, 4114, 4123, 4135, 4140, 4144, 4157, 4164, 4184, 4217, 4227, 4230, 4237, *4240*, 4244, 4269, *4281*, 4327, 4355, *4385*, E13, E*24*, E42, E63, E69, E89, E94, E116, E118, E127, E136, E148, E159, E172, E176, E180, E187, E193, E195, E204, E207, E218, E232, E237, E240, E242, E246, E250, E261, E271, E278, E282, E284, E291, E299, E377, E389, E393, E404, E578, E582, E602, E608, E622, E638, E658, E718, E802, E856, E942

Diophantine equations—2700, 2729, 2751, *2784*, 2790, 2927, 3166, 3182, *3231*, *3251*, 3439, 3449, *3459*, 3467, 3584, 3677, *3692*, 3735, 3854, 3855, 3865, 3917, *4047*, 4123, *4190*, 4284, *4299*, 4401, E51, E55, E143, E151, E210, E227, E253, E313, E326, E353, E357, E371, E430, *E433*, E494, E509, E596, E640, E668, E682, E685, E695, E702, E755, *E806*, E816, E823, E901, E917, E926, E941, E942

Divisibility—3049, 3085, 3117, 3194, 3203, 3530, 3561, *3582*, 3638, *3687*, *3739*, 3761, *3885*, *3903*, *4083*, *4137*, *4143*, *4220*, 4226, *4252*, 4257, *4267*, *4281*, *4302*, *4330*, *4332*, 4339, *4352*, *4358*, *4363*, *4387*, *4416*, E28, *E46*, E77, *E83*, *E91*, E95, E123, E294, E369, *E399*, *E546*, *E560*, *E564*, *E628*, E655, E684, E692, E716, *E744*, E745, E768, E779, E808, E822, E903, *E919*, E932

Euler ϕ function—3058, 3122, *4002*, *4068*, 4121, 4221, *4236*, *4311*, 4327

Fibonacci numbers—2809, *4341*, *E468*, *E834*, E852

Partitions—*4067*, 4073, 4246, 4304, 4340, 4390, E883

Pythagorean numbers and triangles—2755, *3594*, 3901, *4047*, 4110, 4205, E18, E67, E73, E265, E283, E324, E327, E362, *E380*, E472, *E812*, E828

Representations—2841, *4133*, *4287*, *4413*, E759, E782, E861, E881

Reversed numbers and palindromic numbers—3116, E97, E237, E242, E278, E336, E427

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† For explanations of asterisk, see p. 79.

3910, 3951, 3997, 4003, 4022, 4097, 4139, 4207, 4261, 4289, 4326, 4350, E518, E534, E590, E604, E825

GEOMETRY

Elementary plane geometry

Circles—2679, 2814, 2823, 2900, 2901, 3021, 3074, 4253, E59, E107, E142, E171, E203, E409, E470, E571

Maxima and minima—3250, 3567, 3576, 3769, *E880*

Quadrilaterals—3064, 3080, 3359, 3791, E11, E38, E117, E129, E206, E334, E344, E346, E359, E391, E486, E643, *E740*, E767, E851, E870

Triangles—2660, 2735, 2736, 2765, 2834, 2993, 3326, 3341, *3726*, *3746*, *4254*, E52, E64, E68, E137, E139, E144, E150, E166, E179, E220, E281, E301, E305, E307, *E345*, E398, E497, E545, E626, E878

Miscellaneous (*loci; numerical problems; regular polygons; similar triangles; etc.*)—3479, E56; 3246, 3297; 3322, *4086*, E786; 3325, E214; E31, E455, *E735*

Elementary solid geometry

Polyhedra—E341, *E345*, *E711*, E859, *E888*

Miscellaneous (*volumes; etc.*)—3146, 3256; 2982, 3263, 3373, 3375, E79, E388, *E395*, *E432*, E689, E748

College plane geometry

Circles—3153, 3455, 3477, 3525, 3583, 3622, 3727, 3753, 3758, 3769, 3793, 3795, 3871, 3886, 3925, 4038, 4056, 4059, 4088, 4092, 4145, 4313, 4323, E93, E113, E238, E248, E262, E499, E526, E571, E591, E597, E705, E728, E886

Euler line—3228, 3956, 4057, *4328*, E102, E259, E547

Harmonic points, poles and polars—3213, 3338, 3436, 3751, 3828

Polygons—3867, 3886, 3970, 4008, 4025

Quadrilaterals and quadrangles—2723, 3295, 3371, 3609, 3738, 3788, 3890, 4194, 4323, E268, E420, E574, E619

Simson line—2776, 3461, *3658*, 4165, 4194, 4309, E535, E561

Triangles—2720, 2961, 3258, 3267, 3273, 3321, 3328, 3344, 3387, 3390, 3396, *3440*, 3508, 3525, 3628, 3634, 3640, 3653, 3675, 3751, 3753, 3758, 3772, *3780*, 3793, 3817, 3828, 3832, 3838, 3848, 3857, 3871, 3874, 3882, 3883, 3889, 3891, 3896, 3902, 3904, 3921, 3925, 3929, 3945, 3960, 3988, *3990*, *4019*, 4025, 4026, 4030, 4038, 4056, 4057, 4060, *4070*, 4074, 4076, 4089, 4092, 4102, 4119, 4134, 4145, 4169, 4179, 4182, 4196, 4210, 4260, 4274, 4294, 4301, 4344, 4386, 4401, E75, E122, E131, E236, E269, E350, E361, E367, E374, E383, E407, E411, E417, E459, E467, E469, E487, E521, E527, E537, E573, E613, E699, E720, E730, E742, E752, E803, E820, E849, *E879*, E905, E915

Miscellaneous (*concurrency and collinearity; isogonal conjugates; circles of Apollonius; etc.*)—2715, *4065*, E113, E831; 3797, 4038; 3234, 3973; 3937, *4325*, E323, E372, *E929*

3795, 3797, 3850, 3896, *3990*, 4059, 4154, 4170, 4193, 4200, 4398, E350
Miscellaneous (areas; cubics; intersections; points and lines; poles and polars; etc.)—3225, E65, E669; 2929, 3195, E904; 3533, 4273; 3338, 3823, *E774;* 4167, E666; 2839, 3291, 3359, 4343, *E419*, E649, E866

Solid analytic geometry
Assorted coordinate systems—3253, 4018, 4223, 4224, 4298, E443, E549, E800
Barycentric coordinates—3491, 3723, 3888, 3974, 3993, 4080, 4192, 4243, E370, E647, E885
Conicoids—2673, 2918, 3422, 3869, 4094, E175, E804, E843
Loci—2893, 2967, *3040*, 3041, 3224, 3311, 3407, *3586*, 3646, 3670, E74, E114, E200, E219, E260, E277, E314, E513, E868
Spheres—2897, 3174, 3544, 3941, 3953, 4015, 4064, E40, E553, E902
Tetrahedra—*3296*, 3560, 3639, 3644, 3689, 3759, 3840, 3849, 3986, 4015, 4248, E247, E252, E517
Miscellaneous (computations; surfaces; volumes; etc.)—3351, *E276;* 2992, 3773, E298; 3010, E145; 3370, 3619, 3627, 3992, *E481*, E820

Higher plane curves
Cardioid—2861, 3102, 3197, 3519, 3585, 3589, 3744, 3745, E569
Cubics—2824, 2928, 2978, 3444, *E413*, E501
Cycloids, epicycloids, hypocycloids—2691, 2863, 3221, 3235, 3533, 4115, 4181, E202, E600, E653, E731
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Miscellaneous (caustic curves; cissoid; limaçon; quartics; spirals; strophoid; etc.)—2966, 3642; 3105, 3197; 2865, E471; 2977, 3160, 3884, E465; 3339, E41, E639, E784; 3483, 4026, E339, E479; 2917, 2923, 3100, 3110, 3302, 3566, 3813, 3962, *3999*, 4099, E243, E303, E392, *E753*, E934

Synthetic treatment of conics and conicoids—2904, 3215, 3230, 3237, 3247, 3288, 3356, 3384, 3415, 3496, 3822, 3861, 3869, 3898, *4019*, *4085*, 4112, E384, E447, E739, E797

Euclidean constructions
Approximate constructions—2932, 3291, E6, E475, E697
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Impossible constructions—2706, 2963, 3059, 3324, 3686, E574
Polygons—2745, 3128, 3134, 3192, 3219, E480, E529, E627, E713
Quadrilaterals—2667, 2682, 3069, 3319, 3368, *3417*, 3443, 3489, 3878, E8, E197, E279, E381, E707
Space constructions—2895, 2962, *2994*, 3768, 3798, E266, E421, E637, E693, E709, E749, E778, E865
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ANALYSIS

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Miscellaneous—2746, 2771, 3216, *3816*, 4395, E873

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Kinematics—2685, 2779, 2862, 2951, 3072, 3214, *3345*, 3420, E224
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Statics—2665, 2670, 2732, 3139, 3175, 3280, 3411, 3433, 3557, 3930, 4030, *4381*, E536, E875

Inequalities
Geometrical—3572, *3746*, 3848, 3949, *4014*, 4036, *4070*, 4082, E174, *E345*, E417, E689, E741, E895
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Complex variables
Analytic functions—2851, 3257, 3516, 3985, 4275, *4283*, 4308, *4329*, *4337*, *4392*, *4400*
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Real variables
Set theory—*3661*, *3763*, *3829*, 4058, *4263*, *4268*, *4351*, 4405, E906
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Some special topics
Functional equations—*2759*, *2845*, 2851, 2959, 3620, *3649*, *3690*, *3814*
Recurrence relations—2686, 3005, 3400, 3588, *3674*, *3801*, *3802*, 3833, *3872*, 3940, *4255*, *E182*, *E319*, E510, E566, E694, *E853*, E874
Miscellaneous (calculus of variations; infinite products; hyperbolic functions; Stirling numbers)3252, 3566; *4349*, *E182*, E912; 3677, *3764*, 4191; E510, E520

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THE PRACTICE OF MATHEMATICS

R. E. GASKELL, Boeing Airplane Company

The place of the mathematician in our technology. Our society today benefits from the labors of several groups of dedicated individuals who are said to practice in certain fields. Physicians practice medicine, attorneys practice law, and we all know men who practice accounting, engineering or in other fields. We all recognize, too, that the most useful practicing physician may not be the one who knows the most about medicine, just as the most useful attorney may not be the one with the most profound knowledge of the law. Medical and legal training for these professions must of course be sound and thorough, yet it must be complemented with understanding and practical wisdom in many other fields.

Today we have an analogous situation in the field of mathematics. We have had professional mathematicians for some time, and now we find a growing number of practicing mathematicians. Because of the contributions of such pioneers as von Karman and Fry, the demand for mathematicians has increased tremendously, but to follow these pioneers faithfully, we must carefully control the product that we label "mathematician" if he is to meet the public professionally. To conform with the high standards set for practice in other professions, and for the practice of mathematics by those who were earliest in the field, we need mathematicians with more than a good solid grounding in mathematics. In addition, they need a sermonful of the attributes of the physician and attorney, plus considerable basic knowledge of engineering, physics and other sciences. As Dr. Fry explained some years ago, the mathematician in industry should be capable of doing a good engineering job if he turned his hand to it [1].* Today we find mathematicians assisting businessmen, production men, scientists, and many others in government and industry, so Fry's statement, though still true, should be broadly interpreted.

However, the consumer, whether he is an engineer, scientist, or businessman, will call upon the services of a mathematician only if and when he thinks he needs one. Too often he asks for help in an awkward situation that could have been avoided, or he asks for voluminous computations, with a "never mind what it's for or what method we're using." Of course, today's broader need for an understanding of mathematics means that businessmen, engineers and scientists must be far better trained in mathematics than was the custom fifteen years ago. Certainly more problems are solved, more sophisticated mathematics is used by these laymen—to borrow a much-borrowed word—than ever before. We must expect to see more mathematics of the do-it-yourself variety, and no one can really complain as long as the layman does not get beyond his depth. And since it is the layman who is first to judge the depth, the practice of mathematics is materially affected by what the public thinks mathematics and mathematicians are for.

* Numbers in brackets refer to the bibliography at the end of the article.

Fortunately, considerable mathematics does find its way to market, some of it under the name of Operations Research, but even so there is widespread ignorance as to what a mathematician is and what he can do. Not only is this ignorance widespread, but in a certain few individuals it is an intense, aggressive and contagious type of ignorance. Much credit is given to mathematicians in some quarters. Aerodynamicists will usually agree with von Karman when he points to the development of aerodynamic science as an "example of cooperation between 'men of mathematics' and creative engineers. Mathematical theories from the happy hunting grounds of pure mathematicians were found suitable to describe the airflow produced by aircraft with such excellent accuracy that they could be applied directly to airplane design" [**2**]. But we also read statements such as "There is great danger that the publicity given to these [computing] machines will develop an exaggerated idea of the value and place of mathematics in engineering design" [**3**].

We can only conclude that the position of the mathematician in our present day technology has not been uniformly established. Perhaps it can never be. Nevertheless, many people today are faced with the question, "Should the mathematical aspects of this study be carried out by an engineer or scientist with a flair for mathematics, or by a mathematician with a flair for the material under study?" In idler days this question would be interpreted as a bid for jurisdiction, but with work as abundant as it is, the question can only bear upon the efficiency of the utilization of our scientific manpower.

Popular notions about mathematics. It may be helpful to draw attention to the public attitude toward mathematics. Mathematicians must blame themselves for the widely held view that mathematics was all discovered years ago, and that it cannot possibly change—and therefore cannot grow. Our schoolchildren hear about Euclid and Pythagoras, our undergraduates hear about Apollonius, Descartes and Newton, and our mathematics majors may work up an acquaintance with the names of mathematicians who lived and worked in the nineteenth century—Cauchy, Fourier, Gauss, Laplace and others. Textbooks and classroom decorations hold the line in this respect, since custom seems to require that a mathematician spend a century in the grave before public display of his portrait is permitted.

It is common to find individuals who associate mathematical ability with winning at bridge (keeping score, anyway). Others place a mystic aura about the subject, as the executive who said that "Mathematics are quicker than the human eye" [**4**]. Some people fail to distinguish between accounting and mathematics; usually these folks just have bad consciences at income tax time and blame it on their poor arithmetic. The rash of cartoons depicting wheezing computers working unsuccessfully on income tax forms indicates that, in the public mind, computers have consciences, too.

Many other people, including most engineers and some scientists, equate mathematics with some of the more tangible tools of the mathematician, such

as formulas, graphs, charts, tables and computing machines. In this category we find those who place blind faith in the formula, those who delight in the various paradoxes concocted through use of extrapolation or sometimes through a sly division by zero.

The tremendous publicity given high-speed computing machines in the past five or six years has convinced many people that these machines are the very soul of mathematics. Well-known mathematicians have tripped on this one, too. One prominent mathematician was so completely overwhelmed that in reviewing a book on Laplace transformations he asked why the author bothered with Laplace transforms—why didn't he use a high-speed computer! It is quite true that the usefulness of mathematics has been increased enormously by the availability of high-speed computation, but it is a mistake to go further and assume that high-speed computers have replaced or will replace mathematics or mathematicians.

We mathematicians have our own ideas of what mathematics is. There are many definitions, of course. Mathematics has been called the queen of the sciences, and it also has been referred to as the handmaiden of the sciences—which led J. D. Williams, of RAND, to remark that while there is some doubt about the social position of mathematics, at least its sex is definitely known [**5**].

Alfred North Whitehead claimed for mathematics the distinction of being the most original creation of the human spirit. This led him to his definition, "Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about" [**6**]. And Bertrand Russell, Whitehead's colleague, figured he might as well agree with that, only he put it in plain English, defining mathematics as that science in which we do not know what we are talking about, nor whether what we say is true or not. This definition should enjoy almost universal acceptance. The mathematician accepts it because, in all seriousness, there is an element of truth in it, and by the time he explains its weaknesses, his listener will agree that it must be precisely correct. The non-mathematician accepts this definition readily—almost eagerly—because it explains so much of what he has heard and read.

In the face of these widely held and freely publicized notions, is it any wonder that a mathematician is so often called upon at the tail end of a decision and asked to "use this procedure to crank out a number on your little machine"?

The practice of mathematics. Many mathematicians in abstract branches of the subject must of necessity emphasize mathematical beauty and form. While it is true that their work finds applications, usually in another field of mathematics, we should compare their work to that of biological, pharmacological and medical researchers, who are among the leaders in the advancement of medical science, but who do not practice medicine. Louis Pasteur is the most notable example that comes to mind. We must agree, certainly, that this type of mathematician leads in the development of mathematical science.

Another type of mathematician works from the mathematics to the applica-

tion. He develops a mathematical method first, and then seeks instances in which he can apply it. He is an applied mathematician, but we should not call him a mathematical practitioner, at least not a *general* mathematical practitioner. He is more closely analogous to the medical specialist, who develops a technique and then stands ready to take over those cases in which his technique will prove beneficial.

Your true mathematical practitioner *does not seek the problem to fit his mathematics—he seeks that mathematics to which the problem can best be fitted*, with due regard for secondary effects, the quality of solution desired, time available for the solution, and the facilities available. Don't think for a moment that he ignores beauty and form, usually an elegant form goes hand in hand with efficiency. Nevertheless, his thought is *problem-centered*, rather than *method-centered.*

At this point let us boldly suggest that *the practice of mathematics involves the use of available and assimilable mathematical procedures to assist in a design or a decision.* The words *available* and *assimilable* are important here. A definitely superior method may exist, but time may be too short to develop it, the personnel assigned to carry out the work may not be trained enough in mathematics to use it safely, or the computing machinery necessary may not be accessible.

There seem to be five general activities in the practice of mathematics. For successful work all five must be practiced, though not necessarily by the same person or groups, and not necessarily in connection with every problem. Those who practice mathematics must

(1) *Recognize, understand and analyze the problem situation.** The man responsible for the design or the decision discusses the problem situation with a mathematical analyst. Together they pick out the fundamental points, and they strip away or ignore any irrelevant aspects. Compromise and negotiation usually feature these discussion sessions, which may be carried on intermittently for a considerable period of time—weeks, months, and sometimes for more than a year. The aim of these discussions is to arrive at a compromise which will be sufficiently realistic to satisfy the designer, yet practical for the mathematician to handle.

This activity, which may be called negotiation, almost always requires considerable ability in other sciences, in economics, business or in engineering.

(2) *Formulate the problem situation.* Formulation requires the ability to solve "word problems." Those who feel they have this ability to state problems in mathematical terms will be interested in Rufus Oldenburger's book "Mathematical Engineering Analysis" [7], where problems of this kind are emphasized. For example, Oldenburger poses the problem: "The bottom of a metal plate is

* The term *problem situation* is perhaps somewhat better than *problem* when it comes to describing the work of the practicing mathematician, though we shall use both terms interchangeably.

being uniformly heated by hot gases; the top is exposed to the atmosphere." That's the problem. The student takes over from there. Perhaps Oldenburger's masterpiece is this: "A pile of powdered coal is burning." That's it!

Continued negotiation is almost always necessary in formulating a problem for solution. Both parties must keep in mind the time, money, personnel and machinery available for its study. There is no point in formulating a problem precisely if it cannot be handled in finite time with the tools at hand, except that the precise formulation may persist as a challenge to develop useful tools for it at some future time.

(3) *Solve the problem stated* in mathematical form, negotiating modifications of the form where they are necessary to make the study practical. The solution of the problem is essentially the manipulation of its mathematical statement into an alternative form which can be interpreted, or which can be efficiently used in computation.

Let us consider a hypothetical illustration here. Suppose your client wants to design a pendulum which has a period of one second, and he is willing to neglect such things as air resistance, friction and movement at the point of support, elastic effects of the string, wire or bob, *etc.* Then your differential equation is nonlinear,

$$L \frac{d^2\theta}{dt^2} + g \sin \theta = 0.$$

If you have only a short time to get the solution, or if you do not have tables of elliptic integrals handy, you would negotiate with your client to replace $\sin \theta$ by θ, explaining that for small θ there isn't much error. Then

$$L \frac{d^2\theta}{dt^2} + g\theta = 0,$$

and this equation is easily solved:

$$\theta = A \sin t\sqrt{g/L} + B \cos t\sqrt{g/L}.$$

Note that this solution is interchangeable with the original differential equation from which it was obtained—hence the equivalence of the terms "solve" and "manipulate."

The period of this motion is determined from the period of the sine or cosine function, namely $T = 2\pi\sqrt{L/g}$. This is the formula usually given for the period of a pendulum.

But if your client will not allow restriction to small angles, and has the time and money for the extra effort, you would solve the differential equation as it stands, thus the period turns out to be

$$T = 4\sqrt{L/g}K\left(\sin \frac{\alpha}{2}\right),$$

where K is the complete elliptic integral of the first kind, and α is the initial deflection.

(4) *Compute cases of interest to the client.* When the analytical solution cannot be used directly to give the desired assistance, and this is almost always the case, then we must compute a sufficient number of instances to enable us to interpret the solution. Before the development of high-speed computing machinery, computation was a long and tedious chore, something that was too often skimped.

Today there is a fast developing branch of mathematics that we might call computational mathematics, and a growing number of specialists in this field. Those who practice mathematics with this as their special field might be called *computer engineers*—certainly the term programmer is too narrow. This is a term suggested by Professor Tukey. In his words, "Computational mathematics recognizes one interpolating polynomial through a given set of points, and notes in passing that values of this polynomial can be found by [several different methods]. Computation engineering is concerned with the advantages and disadvantages of the various methods, and with the choice between them in particular situations" [**8**].

Of course the computer engineer must not stop there. His field of study must encompass the characteristics and special abilities of the various commercially available machines, the principles of the design of computing machines, and all developments in the use of computing machinery.

There is a tendency to use the high speed and immense size of these computers with little or no regard for efficiency. The computer engineer must combat this tendency. To him a computing machine is not a brain, but a supersonic moron. It must be told what to do every step of the way, but once it is told, it doesn't easily forget, and it is extremely obedient. The computer engineer must watch for spurious results or methods, and call attention of the client or a mathematical analyst to the difficulty. Very few investments lead to more embarrassment than computing voluminous data from the wrong set of formulas. We should see redder faces when an expensive computation is proved useless by a day or two of pencil and paper mathematics.

(5) *Interpret the results.* This activity is similar to problem recognition and understanding, and will almost always be performed by the same individual. In fact, often the groundwork for interpretation will be laid during the negotiation activity.

These five activities of the practicing mathematician are given grossly unequal emphasis in our training of mathematicians today. Most of our course work is geared toward Manipulation, with Computation coming more recently into the picture because of the vast number of computer engineers required today in business and industry. Some attempts at Formulation are included in our curricula, especially in applied mathematics courses but more often in nonmathematical courses. As for Negotiation and Interpretation, the almost com-

plete absence of anything resembling them from our mathematical curricula definitely handicaps newly trained mathematicians in their contacts with their clients.

Getting down to cases. A classical example of the practice of mathematics was the discovery of the planet Neptune by Leverrier [9]. The problem situation arose as observations of the erratic behavior of Uranus were recorded. Negotiation with astronomers led Leverrier to the hypothesis that another planet was causing the disturbance, and ruled out possible gravitational anomalies, ether resistance or disturbance by a comet. The mathematical model required was formulated and then manipulated until it was in a form suitable for computation. After the computations were completed, Leverrier interpreted the results to the German astronomer, Galle, as coordinates of a point in the sky where he would find the new planet. Galle found it within one degree of the predicted position.

Other examples of the practice of mathematics may be interesting. These examples are selected for their ease of explanation, not for their mathematical sophistication nor for their economic importance. No single example, or group of examples, could fairly picture the activity of the practicing mathematician.

Most of us have equated mathematics with what we refer to in this article as Manipulation, although many of us have now come to accept Computation as a legitimate mathematical occupation. The next illustration shows that Interpretation and Formulation may be all that is necessary in some situations.

A client was using a strange method for determining the buckling load of a stepped column, that is, one made up of several different uniform segments. It seems that there were two different ways to apply this method, but the client couldn't make the two answers jibe. Not only that, but neither answer was at all reasonable, and even averaging the two doubtful answers failed to improve them, strangely enough. First the mathematician wanted to know where the client found his method. He referred to an article in a reputable journal, and it turned out that as far as could be determined he was following the directions supplied in the article correctly. Careful study disclosed, however, that the article contained a set of three equations in only two unknowns, one of the alleged unknowns being a well-masked combination of the other two. Following this lead, a fundamental error was soon found hidden on the first page of the article. It was only necessary to point out the difficulty to the client, who was quite familiar with the manipulation required to get the corrected result that he needed.

A detective story writer could give some nice lilting title like "The Case of the Repressed Redundancy", or the "Case of the Curious Client", to this problem situation. It is quite fortunate that the redundant equations were incompatible, leading the curious client to suspect murder. Not all the errors in our voluminous printed literature lie so near the surface. But it is tempting for an engineer or other potential client to run down to the formula he wants and pay no

attention to where it came from. It seems that much valuable time is saved that way.

The next case is also a very simple one. The client complained that formulas for friction coefficients were not giving expected results in a certain case. He had a "dolly" for transporting aircraft sub-assemblies, which was equipped with V-grooved casters, and which was to be pulled along angle-iron "tracks". The client had to find the force necessary to pull the dolly along the track. At first he assumed rolling friction, and found a result which was absurdly low, but sliding friction gave a result far too high. After looking the problem over, the mathematician convinced himself and the client that rotation of the caster took place about some instantaneous axis which wasn't yet known, and whose determination was a part of the problem. Above this instantaneous axis the friction force opposed the forward motion of the caster, but below the instantaneous axis the friction force was actually in the direction of the motion. Points below the instantaneous axis move in the same fashion as points on the flange of a locomotive wheel—they actually move backward part of the time. Force and moment balance equations readily disclosed the force required and the position of the instantaneous axis.

Here again Manipulation and Computation played a minor role, but in this case conviction or Negotiation was an important part of the job. The client will not use the results obtained for him if he is not convinced of their validity. And he must be willingly convinced—not merely beaten with the force of authority. Both the mathematician and his client must agree that the problem as stated will lead to a conclusion in which some faith can be placed. And the more earth-shaking the conclusion, the more abiding the faith must be.

These problems are very simple ones, although they do look simpler after they are completed than they do before. There is a multitude of other problems —much more complex, more challenging. How should the throat of a supersonic wind tunnel be designed so that shock waves will not be generated by the walls of the tunnel itself? What temperatures are generated during machining? What is the best design for a shipping container? And there are many others.

How many clerks should be stationed behind the counter of a tool crib—taking into account the cost of clerks' idle time as well as the cost of mechanics' waiting time in the lines at the counter [10]? There are routines that can be used to solve linear programming problems on high-speed digital computing equipment. Can a much less expensive analog computer be used to solve such problems, and are there any other advantages in the use of analog equipment [11]? What change in performance of a given airplane can be expected if different engines are used?

To go on to even broader fields, consider that we human beings have had centuries of experience in detecting signals in the presence of noise. We can see a tomato worm on a tomato plant in spite of his protective coloring, we may hear a trout splashing in a stream even above the noise of the stream, we hear radio signals even when there is static. How can this human experience be built into

machinery, so that signals can be detected mechanically, electronically, mathematically, automatically—in the presence of noise? Progress continues in this important field [**12**].

Another very important problem facing us today is that of educating a computing machine. We must take these arithmetic strong-backs and teach them to communicate more intelligently with their masters, so that hordes of people aren't required to boss them. We have already taught them algebra [**13**], but this is just a beginning.

Diversity in the training of mathematicians. For the good of our technology, it is imperative that the position of mathematicians be established as basic. They should be used continuously and regularly, and not just to provide occasional flashes of genius when all else fails. Until this happy day dawns, the position of the practicing mathematician will be somewhat less than clear, and much of his best work—that in which his ideas and methods provoke an evolution in the designs and decisions of his clientèle—will be unrecognized and unrewarded.

We can speed the coming of that day if we develop more mathematicians whose thought is problem-centered rather than method-centered. We need mathematicians whose training is broad enough so that they can give sound advice in a wide variety of problem situations. Furthermore, we should encourage more diversity in the curricula of our mathematicians. I am in good company in making this suggestion [**8**], and the same conclusion, by reasoning from a different direction, has been much more forcefully drawn by Oakley [**14**].

As it is now, many of those who practice mathematics today are converted from other fields, such as physics, electrical and mechanical engineering, and chemistry. Perhaps conversions of this kind provide the most practicable way to get broadly trained mathematicians, since many courses in engineering and science are accessible only through long chains of prerequisites, some of them realistic and some not so realistic. On the other hand, we would do well to consider the needs of the practicing mathematician when we find a student who wants to build up a stronger-than-usual background in a cognate field.

References

1. T. C. Fry, Industrial mathematics, this MONTHLY, vol. 48, no. 6, part II, 1941, p. 5.
2. Th. von Karman, Aerodynamics, Cornell University, 1954, p. 2.
3. George F. Nordenholt, A graduate can measure a bottle, Product Engineering, April 1953, p. 135.
4. Anonymous, Business Week, June 30, 1951, p. 45.
5. J. D. Williams, The Compleat Strategyst, New York, 1954, p. 215.
6. A. N. Whitehead, Science in the Modern World, New York, 1931, p. 32.
7. Rufus Oldenburger, Mathematical Engineering Analysis, New York, 1950.
8. J. W. Tukey, Mathematical consultants, computational mathematics, and mathematical engineering, this MONTHLY, vol. 62, 1955, p. 566.
9. Louis Figuier, Principales Découvertes Scientifiques Modernes, vol. IV, Paris, 1858, pp. 291–319. This contemporary account of the discovery is the one which we follow.
10. Georges Brigham, On a congestion problem in an aircraft factory, J. Operations Res. Soc. Amer., vol. 3, 1955, p. 412.

11. C. M. Ablow and Georges Brigham, An analog solution of programming problems, J. Operations Res. Soc. Amer., vol. 3, 1955, p. 388.

12. W. M. Stone, On the statistical theory of detection of a randomly modulated carrier, J. Appl. Phys., vol. 24, 1953, p. 935.

13. Mandalay Grems and R. E. Porter, A truly automatic computing system, Proc. Western Joint Computer Conference, Feb. 1956.

14. C. O. Oakley, Engineering for mathematicians, J. Engineering Education, vol. 47, 1956, p. 292.

ON HIGHER ORDERED DIFFERENTIATION

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1. Introduction. Courant in [1] gives Leibniz' rule for finding the Nth derivative of the product of two functions, and then remarks that no such easily remembered law has been found for the repeated differentiation of the compound function $Y=F[U(X)]$. Rather complicated formulae for the Nth derivative of a compound function have been found by Schlömilch [2], Faa' di Bruno [3], and McKiernan [4]. In this paper, however, a simple and easily remembered law for repeated differentiation is given which is valid not only for a compound function, but also for the more general composite function $Y=F[U_1(X), \cdots, \cdots, U_Q(X)]$. Partially to illustrate the power of this law, but more particularly because the problem is of some interest in its own right, a formula is derived for writing the power series representation of a meromorphic function of finite genre in terms of its Weierstrass product representation.

In each of the results given in this paper, a certain matrix of simple structure plays a prominent role. It is shown that the characteristic polynomial of this matrix has interesting properties, especially with regard to differentiation processes.

It may be worthy of note that in the formula for the repeated differentiation of a composite function, the elements of the matrix are operational quantities similar in form to the familiar chain rule for first ordered differentiation. Thus, the formula given here may be regarded as a chain rule for higher ordered differentiation.

2. The matrix Δ_N. The symbol Δ_N denotes the Nth ordered matrix

$$\begin{bmatrix} D_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ D_2 & D_1 & -2 & 0 & 0 & \cdots & 0 \\ D_3 & D_2 & D_1 & -3 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & \cdot & \cdot & 1-N \\ D_N & \cdots & \cdots & \cdots & D_3 & D_2 & D_1 \end{bmatrix}.$$

The elements $D_1, \cdots, D_N$ will be assigned meanings appropriate to the particular topics under discussion. In some cases these elements will be operational in nature, while in others they will be purely algebraic. The symbol $|\Delta_N|$ denotes the determinant of Δ_N. The zero'th ordered determinant $|\Delta_0|$ is defined as unity.

3. The *N*th derivative of a composite function. Let the function $Y = F(U_1, \cdots, U_Q)$ be continuous together with its mixed partial derivatives of the first N orders with respect to $U_1, U_2, \cdots, U_Q$. Let $U_1(X), \cdots, U_Q(X)$ be continuous together with their first N derivatives with respect to X. It is then possible to state the following theorem.

THEOREM 1. *The Nth derivative of Y with respect to X is given by*

$$\frac{d^N Y}{dX^N} = |\Delta_N| F(U_1, \cdots, U_Q),$$

where the elements $D_1, \cdots, D_N$ of Δ_N are defined by the relation

$$D_i = \frac{1}{(i-1)!} \sum_{h=1}^{Q} \frac{d^i U_h}{dX^i} \frac{\partial}{\partial U_h}.$$

Proof. For $N=1$, the theorem is merely an operational statement of the familiar chain rule for first ordered differentiation. In order to complete a proof by induction, it will now be assumed that the theorem is true for $N=1, \cdots, K$, and it will be shown that it must then be true also for $N=K+1$. Upon expanding $|\Delta_K|$ by cofactors of its last row, one obtains the recurrence relation

$$|\Delta_K| = \sum_{i=1}^{K} \frac{(K-1)!}{(K-i)!} D_i |\Delta_{K-i}|.$$

Then, substituting the operational definitions for the elements D_i gives

$$|\Delta_K| = \sum_{i=1}^{K} \sum_{h=1}^{Q} \frac{(K-1)!}{(K-i)!(i-1)!} \frac{d^i U_h}{dX^i} \frac{\partial}{\partial U_h} |\Delta_{K-i}|.$$

If both sides of this relation are allowed to operate upon Y, the result is

$$\frac{d^K Y}{dX^K} = \sum_{i=1}^{K} \sum_{h=1}^{Q} \frac{(K-1)!}{(K-i)!(i-1)!} \frac{d^i U_h}{dX^i} \frac{\partial}{\partial U_h} \frac{d^{K-i} Y}{dX^{K-i}}.$$

Then, differentiation with respect to X yields the expression

$$\frac{d^{K+1} Y}{dX^{K+1}} = \sum_{i=1}^{K} \sum_{h=1}^{Q} \frac{(K-1)!}{(K-i)!(i-1)!} \left[\frac{d^{i+1} U_h}{dX^{i+1}} \frac{\partial}{\partial U_h} \frac{d^{K-i} Y}{dX^{K-i}} + \frac{d^i U_h}{dX^i} \frac{\partial}{\partial U_h} \frac{d^{K-i+1} Y}{dX^{K-i+1}} \right].$$

This result may be written in the alternative form

$$\frac{d^{K+1} Y}{dX^{K+1}} = \sum_{i=1}^{K} \frac{(K-1)!}{(K-i)!} (i D_{i+1} | \Delta_{K-i} | + D_i | \Delta_{K-i+1} |) Y.$$

Then, upon collecting like terms, one obtains

$$\frac{d^{K+1} Y}{dX^{K+1}} = \sum_{i=1}^{K+1} \frac{K!}{(K+1-i)!} D_i | \Delta_{K+1-i} | \; Y.$$

The expression operating upon Y is precisely $|\Delta_{K+1}|$ expanded by cofactors of its last row. Therefore, $d^{k+1} Y/dX^{k+1} = |\Delta_{K+1}| \; Y$, and hence, by the postulate of mathematical induction, $d^N Y/dX^N = |\Delta_N| \; Y$ for all positive integral values of N.

COROLLARY. *Let* $Y = \epsilon^{U(X)}$ *where* $U(X)$ *is continuous together with its first* N *derivatives. Then*

$$\frac{d^N Y}{dX^N} = | \Delta_N | \, \epsilon^{U(X)},$$

where the elements $D_1, \cdots, D_N$ *of* Δ_N *are defined by the relation*

$$D_i = \frac{1}{(i-1)!} \frac{d^i U}{dX^i}.$$

Proof. The corollary follows directly from Theorem 1. Since each of the derivatives of ϵ^U with respect to U is precisely ϵ^U itself, one may expand $|\Delta_N|$, allow the expansion to operate upon ϵ^U, and then place the result again in the form of a determinant by merely omitting the partial differentiation symbols in the relation defining $D_1, \cdots, D_N$.

4. Meromorphic functions of finite genre. A function which is analytic, except for a finite number of poles within any finite region, at every finite point of the complex plane is said to be a meromorphic function. Let $F(Z)$ be a meromorphic function whose zeros (excluding any which may lie at the origin) are at $a_1, a_2, \cdots$. Let the zero at a_i be of order m_i. Poles are regarded as zeros of negative order. Let the zero at the origin be of order m_0. It then follows by the Weierstrass factor theorem [**5**]. that $F(Z)$ may be represented at every point for which it is defined by the following infinite product:

$$F(Z) = \epsilon^{G(Z)} Z^{m_0} \prod_{i=1}^{\infty} \left[\left(1 - \frac{Z}{a_i}\right) \epsilon^{H_i(Z)} \right]^{m_i},$$

where

$$H_i(Z) = \sum_{n=1}^{k_i} \frac{1}{n} \left(\frac{Z}{a_i}\right)^n,$$

$G(Z)$ is analytic throughout the entire finite plane, and the k_i are chosen so as to ensure the convergence of the product.

It is shown in [6] that if K is the smallest nonnegative integer such that the summation $\sum_{i=1}^{\infty} m_i/a_i^{K+1}$ is absolutely convergent, then it is sufficient to choose $k_i = K$ for all values of i. If with this choice for k_i the function $G(Z)$ is of (finite) degree K', then the function $F(Z)$ is said to be of finite genre, the genre P being the larger of the two nonnegative integers K and K'. In other words the genre of the function $F(Z)$ is the highest (necessary) power of Z in any exponential function within or without the product sign. If $F(Z)$ is of genre P, therefore, it may be represented by the Weierstrass product given above, except that the functions $G(Z)$ and $H_i(Z)$ will now be polynomials of (formal) degree P. That is,

$$G(Z) = \sum_{n=0}^{P} C_n Z^n, \quad H_i(Z) = \sum_{n=1}^{P} \frac{1}{n} \left(\frac{Z}{a_i}\right)^n.$$

5. The series expansion of a function of finite genre. Let $F(Z)$ be the function of finite genre described above. Let its representation by a power series be

$$F(Z) = Z^{m_0} \sum_{n=0}^{\infty} A_n Z^n.$$

It is then possible to state the following theorem.

THEOREM 2.

$$A_N = \frac{1}{N!} \mid \Delta_N \mid \epsilon^{C_0},$$

where the elements $D_1, \cdots, D_N$ *of* Δ_N *are defined by*

$$D_K = KC_K, \qquad K \leqq P; \qquad D_K = -\sum_{i=1}^{\infty} \frac{m_i}{a_i^K}, \qquad K > P.$$

Proof. Since $F(Z) = \epsilon^{\log F(Z)}$, one may write at once from the corollary to Theorem 1,

$$\frac{d^N F(Z)}{dZ^N} = \mid \Delta_N \mid \epsilon^{\log F(Z)}, \qquad \text{where} \qquad D_K = \frac{1}{(K-1)!} \frac{d^K \log F(Z)}{dZ^K}.$$

But the logarithm of the Weierstrass product representation of $F(Z)$ is given by

$$\log F(Z) = \sum_{n=0}^{P} C_n Z^n + \sum_{i=1}^{\infty} m_i \left[\log\left(1 - \frac{Z}{a_i}\right) + \sum_{n=1}^{P} \frac{1}{n}\left(\frac{Z}{a_i}\right)^n \right].$$

(Here, it is temporarily assumed that $m_0 = 0$.) Upon differentiating K times with respect to Z, one obtains

$$\frac{d^K \log F(Z)}{dZ^K} = \sum_{n=K}^{P} \frac{n!}{(n-K)!} C_n Z^{n-K} + \sum_{i=1}^{\infty} m_i \left[-\frac{(K-1)!}{(a_i - Z)^K} + \sum_{n=K}^{P} \frac{(n-1)!}{(n-K)!} \frac{Z^{n-K}}{a_i^n} \right].$$

Evaluating this expression for $Z = 0$, and dividing by $(K-1)!$ gives the result

$$\frac{1}{(K-1)!} \left. \frac{d^K \log F(Z)}{dZ^K} \right|_{Z=0} = \begin{cases} -\sum_{i=1}^{\infty} \frac{m_i}{a_i^K}, & K > P \\ KC_K, & K \leqq P. \end{cases}$$

Now, $\epsilon^{\log F(Z)}$ evaluated for $Z = 0$ is simply ϵ^{C_0}, so

$$\left. \frac{d^N F(Z)}{dZ^N} \right|_{Z=0} = |\Delta_N| \epsilon^{C_0}.$$

It follows at once from Taylor's theorem that the coefficient of the Nth term in the series expansion of $F(Z)$ is given by

$$A_N = \frac{1}{N!} |\Delta_N| \epsilon^{C_0},$$

where

$$D_K = -\sum_{i=1}^{\infty} \frac{m_i}{a_i^K}, \quad K > P; \qquad D_K = KC_K, \quad K \leqq P.$$

Now it is obvious that if m_0 is not zero, then each term in the series expansion is merely multiplied by Z^{m_0}. Thus, the coefficient of the Nth term is unchanged, although the power of the term will now be $N + m_0$.

It should be noted that in the proof given here, it is not necessary to assume that the multiplicities m_i and m_0 are integers, or even that they are real, although these assumptions are satisfied if the function $F(Z)$ is meromorphic. Thus, the theorem given here is valid not only for meromorphic functions, but also for any other functions which may be expressed in the *form* of a Weierstrass product for a meromorphic function of finite genre. If, however, multiplicities are present which are not real integers, then it must be recognized that the resulting series will represent the principal branch of the function.

6. The characteristic polynomial of the matrix Δ_N. Let the elements of Δ_N be complex numbers, and let $P_N(X)$ denote the characteristic polynomial of Δ_N. It is then possible to state the following theorem.

THEOREM 3.

$$\frac{dP_N(X)}{dX} = NP_{N-1}(X).$$

Proof. The characteristic polynomials associated with Δ_1 and Δ_2 are

$$P_1(X) = X - D_1,$$

$$P_2(X) = \begin{vmatrix} X - D_1 & 1 \\ -D_2 & X - D_1 \end{vmatrix} = X^2 - 2D_1X + D_1^2 + D_2.$$

The derivative of $P_2(X)$ is $2P_1(X)$. Thus, the theorem is seen to be true for $N=2$. It is also formally true for $N=1$, since $P_0(X)$ is unity by definition. In order to complete a proof by induction, it will now be shown that if the theorem is true for $N=1, \cdots, K-1$, then it is also true for $N=K$.

In order to obtain the characteristic polynomial associated with the matrix Δ_K, it is only necessary to substitute for D_i the quantity $(\delta_i^1 X - D_i)$, and for the elements $-1, -2, \cdots, 1-K$, their absolute values, and to expand the result as a determinant. Here, the symbol δ_i^1 denotes the so-called Kronecker delta, which by definition assumes the value unity for $i=1$ and the value zero for $i\neq 1$. Thus, expanding the determinant by cofactors of its last row gives

$$P_K(X) = \sum_{i=1}^{K} \frac{(K-1)!}{(K-i)!}(-1)^{i-1}(\delta_i^1 X - D_i)P_{K-i}(X).$$

Then, upon differentiating with respect to X, one obtains

$$P'(X) = \sum_{i=1}^{K} \frac{(K-1)!}{(K-i)!}(-1)^{i-1}[\delta_i^1 P_{K-i}(X) + (\delta_i^1 X - D_i)P'_{K-i}(X)].$$

But, by assumption $P'_N(X) = NP_{N-1}(X)$, for $N=1, 2, \cdots, K-1$, so

$$P'_K(X) = \sum_{i=1}^{K-1} \frac{(K-1)!}{(K-i)!}(-1)^{i-1}[\delta_i^1 P_{K-i}(X) + (\delta_i^1 X - D_i)(K-i)P_{K-i-1}(X)].$$

Then, because of the definition of the Kronecker delta,

$$P_K(X) = P_{K-1}(X) + (K-1)\sum_{i=1}^{K-1} \frac{(K-2)!}{(K-1-i)!}(-1)^{i-1}(\delta_i^1 X - D_i)P_{K-i-1}(X)$$

But the summation of the right is precisely the characteristic polynomial of Δ_{K-1}. Thus, $P'_K(X) = KP_{K-1}(X)$. And, by the postulate of mathematical induction, $P'_N(X) = NP_{N-1}(X)$for all N.

COROLLARY. *Let the symbol C_K denote the coefficient of X^K in $P_N(X)$. It then follows that*

$$C_K = (-1)^{N-K}\frac{N!}{(N-K)!K!}\,|\Delta_{N-K}|.$$

Proof. From Theorem 3, one may write at once for $K \leqq N$,

$$\frac{d^K P_N(X)}{dX^K} = \frac{N!}{(N-K)!} P_{N-K}(X).$$

But, by definition

$$P_N(X) = \sum_{n=0}^{N} C_n X^n.$$

So it follows that

$$\frac{d^K P_N(X)}{dX^K} = \sum_{n=K}^{N} \frac{n!}{(n-K)!} C_n X^{n-K}.$$

Equating the two expressions for $d^K P_N(X)/dX^K$ and setting $X=0$ gives

$$K! C_K = \frac{N!}{(N-K)!} P_{N-K}(0).$$

But, $P_{N-K}(0)$ is simply the constant term in $P_{N-K}(X)$, so its value is $(-1)^{N-K}$ times $|\Delta_{N-K}|$. Thus, it follows that

$$K! C_K = \frac{N!}{(N-K)!} (-1)^{N-K} |\Delta_{N-K}|,$$

and finally that

$$C_K = (-1)^{N-K} \frac{N!}{(N-K)!K!} |\Delta_{N-K}|.$$

References

1. R. Courant, Differential and Integral Calculus, 2nd ed., New York, 1937, vol. I, p. 202.
2. O. Schlömilch, Compendium der Höheren Analysis, Brunswick, 1879, vol. 2, p. 4.
3. F. Faa' di Bruno, Annales de Tortolini, 1855 and Théorie des Formes Binaires.
4. M. McKiernan, On the *N*th derivative of composite functions, this MONTHLY, vol. 63, 1956, pp. 331–333.
5. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge, 1935, p. 137.
6. E. Borel, Leçons sur les Fonctions Entières, Paris, 1900, p. 24.

ON CERTAIN MATRIX EQUATIONS

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This paper exhibits some properties of matrix solutions for certain matrix equations, where the elements in the coefficient matrices range over the field of complex numbers. In particular, the first section gives the number of parameters along with a minimum and maximum rank of the matrix solution for $AYB=C$. The second section gives the minimum number of parameters in the solution for matrix equations which are of a more general type, and the third section is devoted to remarks.

Throughout, capital letters denote matrices, while lower case letters denote scalars. The symbol A^I means the inverse of the matrix A, when A is nonsingular, and without confusion, the symbol 0 is used to designate the matrix with every element zero or the scalar of value zero as the case may be. The collection of symbols $A(m, n, a)$ indicates that the matrix A has m rows, n columns and is of rank a.

1. Concerning the matrix equation $AYB=C$. The following theorem establishes the minimum rank of the matrix solution Y of

$$AYB = C, \tag{1}$$

where $A(m, n, a)$, $0<a\leqq\min(m, n)$, $B(k, t, b)$, $0<b\leqq\min(k, t)$, $C(m, t, c)$, $0\leqq c\leqq\min(m, t)$, while Y is an unknown n by k matrix.

THEOREM 1. *If $c\leqq a$, $c\leqq b$, then the rank of the matrix solution Y of* (1) *is not less than c, providing necessary and sufficient conditions* [1] *are satisfied for the existence of a solution.*

Proof. The rank of the product AYB does not exceed the rank of its factors [**2**, p. 42], but this rank is c. Hence the rank of Y is not less than c.

A well known result [**2**, p. **11**] is that if A is a matrix as described above, there exist nonsingular matrices L_A, R_A such that $L_AAR_A=E_a$, where E_a is a matrix whose elements e_{ij} have the following values: $e_{ij}=1$, $(i=j=1, \cdots, a)$, $e_{ij}=0$, $i\neq j$, $(i=1, \cdots, m)$, $(j=1, \cdots, n)$.

THEOREM 2. *If $c\leqq a$, $c\leqq b$ and the last $(m-a)$ rows and $(t-b)$ columns of L_ACR_B consist of zero elements, then there exists an $(nk-ab)$-parameter family of solutions for Y and of these, there is at least a p^*-parameter family of solutions each of maximum rank r, where p^* and r are given below.*

Proof. Equation (1) is rewritten as

$$E_aZE_b = D, \tag{2}$$

where $Z=R_A^IYL_B^I$, $D=L_ACR_B$. It is observed that with the hypotheses of this theorem, equation (2) satisfies necessary and sufficient conditions [1] for a matrix solution to exist, hence a matrix solution Y of (1) exists. The product

E_aZE_b is an m by t matrix of the form

$$\left(\begin{array}{c|c} \begin{matrix} z_{11} & \cdots & z_{1b} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ z_{a1} & \cdots & z_{ab} \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right),$$

and by hypothesis

$$D = L_ACR_B = \left(\begin{array}{c|c} \begin{matrix} d_{11} & \cdots & d_{1b} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ d_{a1} & \cdots & d_{ab} \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right).$$

Hence ab of the elements in Z are uniquely determined while $(nk-ab)$ of its elements are parameters. Further the matrix solution Y of (1) contains these parameters, since $Y=R_AZL_B$.

To show that, of these solutions, there is at least a p^*-parameter family of solutions each of maximum rank r, first transform Z, by use of elementary transformations, to an equivalent matrix X—in rank—so that the nonvanishing determinant of order c appears in the first c rows and c columns of X. Assign the following nonzero values to the parametric elements x_{pq} of X in each of the following cases:

I. If $n \leqq k$

(A) and $n=a$, either

1. $a>c$ and $k>b$, let $x_{c+i,b+i}=\lambda_i$, $(i=1, \cdots, p^*)$, where $p^*=\min(a-c, k-b)$, otherwise $x_{pq}=0$. In this case $r=c+p^*$.
2. or $a=c$, $k>b$, let $x_{pq}=0$. In this case $p^*=0$, $r=n$. Note this case does not exist when $n=k$.
3. or $a\geqq c$, $k=b$. In this case there are no parameters, hence $p^*=0$, $r=c$.

(B) or $n>a$, and either

1. $c+n-a\geqq b$, and
   (a) $a>c$, let $x_{a+i,c+i}=\lambda_i$, $(i=1, \cdots, n-a)$, $x_{c+j,c+n-a+j}=\lambda_j$, $(j=1, \cdots, a-c)$, otherwise $x_{pq}=0$. In this case $p^*=n-c$, $r=n$.
   (b) or $a=c$, let $x_{a+i,c+i}=\lambda_i$, $(i=1, \cdots, n-a)$, otherwise $x_{pq}=0$. In this case $p^*=n-a$ and $r=n$.
2. or $c+n-a<b$, and
   (a) $a>c$ and $k>b$, let $x_{a+i,c+i}=\lambda_i$, $(i=1, \cdots, n-a)$, $x_{b+j,c+j}=\lambda_j$, $(j=1, \cdots, s)$, where $s=\min(a-c, k-b)$, otherwise $x_{pq}=0$. In

this case $p^* = n - a + s$ and $r = c + p^*$.

(b) or $a = c$ and $k \geqq b$ or $a \geqq c$ and $k = b$, let $x_{a+i,c+i} = \lambda_i$, $(i = 1, \cdots, n-a)$, otherwise $x_{pq} = 0$. In this case $p^* = n - a$ and $r = n$ or $r = c + p^*$ respectively.

II. Or $n \geqq k$. This is treated as Case I by considering the transpose of (1), namely, $B^T Y^T A^T = C^T$, and various cases follow in a manner entirely similar to that above. Since R_A and L_B are nonsingular matrices and $Y = R_A Z L_B$, the matrix Y is of the same rank as Z.

It is worth noting that the hypothesis on the matrix $L_A C R_B$ is a necessary condition for the systems to be consistent.

Certain cases of these results are essential in the study of differential systems with interface conditions as indicated in [**3**].

2. Certain general matrix equations. Consider the matrix equation

$$\sum_{h=1}^{\alpha} A_h Y_h B_h = C, \tag{3}$$

where $A_h(m, n_h, a_h)$, $0 < a_h \leqq \min(m, n_h)$, $B_h(k_h, t, b_h)$, $0 < b_h \leqq \min(k_h, t)$, $C(m, t, c)$, $0 \leqq c \leqq \min(m, t)$, while Y_h are unknown n_h by k_h matrices. Denote with "a" and "b" the ranks of the matrices such that $a_h b_h \leqq ab$, $(h = 1, \cdots, \alpha)$. Also "condition LR" means $L^I_{A_1} = L^I_{A_2} = \cdots = L^I_{A_\alpha}$, $R^I_{B_1} = R^I_{B_2} = \cdots = R^I_{B_\alpha}$.

THEOREM 3. *If in* (3) *condition LR holds, $C \neq 0$, the last $(m-a)$ rows and $(t-b)$ columns of $L_{A_h} C R_{B_h}$ consist of zero elements, there is at least an $(n_h k_h - a_h b_h)$-parameter family of solutions for each Y_h.*

Proof. Equation (3) is rewritten as

$$\sum_{h=1}^{\alpha} E_{a_h} Z_h E_{b_h} = D, \tag{4}$$

where $Z_h = R^I_{A_h} Y_h L^I_{B_h}$, $D = L_{A_h} C R_{B_h}$. With close observation, it is seen equation (4) satisfies necessary and sufficient conditions [1] for solutions, Z_h, to exist under the hypotheses of this theorem. In each matrix Z_h, at least $(n_h k_h - a_h b_h)$ of the elements cannot be determined. Since $Y_h = R_{A_h} Z_h L_{B_h}$, then Y_h contains these parameters.

THEOREM 4. *If in* (3) *condition LR holds, $C = 0$, $ab < \sum_{h=1}^{\alpha} n_h k_h$, there is at least an $(n_h k_h - a_h b_h)$-parameter family of solutions for Y_h not all zero.*

In case $Y = Y_h$, $n = n_h$, $k = k_h$, $(h = 1, \cdots, \alpha)$, then (3) becomes

$$\sum_{h=1}^{\alpha} A_h Y B_h = C. \tag{5}$$

Under the same hypotheses as in Theorem 2 and in a manner entirely similar to that above follows

THEOREM 5. *There is at least an $(nk-ab)$-parameter family of solutions.*

THEOREM 6. *If in* (5), *condition LR holds,* $C=0$, $nk\neq ab$, *there is at least an $(nk-ab)$-parameter family of nonzero solutions.*

3. Remarks. In a recent and interesting note [4], the existence and number of solutions were determined for the matrix equation $AX=B$, where the elements in A and B belong to a finite field. The number of solutions is given in terms of the order of the field and the number of parameters in the solution. Further, the author gives the number of solutions for the matrix equation $UA+BV=C$, if solutions exist.

In my work presented above and following Hodges [4], the number of solutions of (1), (3), and (5) is unbounded. If one restricts the elements of the matrices in (1) to range over a finite field of order f, then the number of solutions is f^{nk-ab}.

References

1. W. E. Roth, Direct product matrices, Bull. Amer. Math. Soc., vol. 40, 1934, pp. 461–468.
2. C. C. MacDuffee, The Theory of Matrices, New York, 1946.
3. T. J. Pignani and W. M. Whyburn, Differential systems with interface and general boundary conditions, J. Elisha Mitchell Sci. Soc., vol. 72, 1956, pp. 1–14.
4. J. H. Hodges, The matrix equation $AX=B$ in a finite field, this MONTHLY, vol. 63, 1956, pp. 243–244.

THE DIRICHLET SERIES TRANSFORMATION*

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1. Introduction. In references [3] and [4], Tomlinson Fort introduced the Dirichlet series transformation as a tool for solving difference equations. We define the transformation by the relationship,

$$\mathfrak{D}\{F(t)\} = f(s) = \sum_{t=0}^{\infty} a^{-st}F(t), \qquad a > 1. \tag{1}$$

$F(t)$ is assumed to be a function of a discrete variable t, where t takes on nonnegative integral values. Deviating from Fort's definition, s is assumed complex.

If $F(t)=O(k^t)$ for $t>t_0$, then (1) is absolutely convergent when $a^{\mathrm{Re}(s)}>k$, (k, a positive constant).

2. Inversion. Inversion of (1) can be accomplished by differentiation or contour integration if s is a complex variable. It can be shown that $F(t)$ is given by the theorem.

* Excerpt from a thesis written under the direction of Professor J. R. Britton, University of Colorado.

INVERSION THEOREM. *Assume that* $f(s)=\sum_{t=0}^{\infty} a^{-st}F(t)$, $a>1$. (a) *If* $f(s)$ *is such that* $f\{-(\operatorname{Ln} w)/(\operatorname{Ln} a)\}$ *is single-valued and analytic inside some circle* C *with center* $w=0$, *then*

$$F(t) = \lim_{s\to\infty} \frac{1}{t!}\frac{d^t}{dw^t} f\left(\frac{-\operatorname{Ln} w}{\operatorname{Ln} a}\right)\Bigg]_{w=a^{-s}}$$

(b) *If* $f(s)$ *is such that* $f\{(\operatorname{Ln} z)/(\operatorname{Ln} a)\}$ *is single-valued and analytic outside some circle* C *with center* $z=0$, *then*

$$F(t) = \frac{1}{2\pi i}\int_C z^{t-1}f\{(\operatorname{Ln} z)/(\operatorname{Ln} a)\}dz = \sum R_n,$$

where R_n *are the residues at the singularities,* Z_n, *of the integrand inside* C. (*We use* Ln *to stand for the principal value of the logarithm.*)

In addition to the inversion theorem, the use of a table of transforms relating $f(s)$ and $F(t)$ is desirable for the problem of inversion. If this latter procedure is used, it is imperative that for a given $f(s)$ there be associated a function $F(t)$ which is unique. Uniqueness can be justified, if $F(t)$ is of exponential order, by a process described in [**2**, pp. 166–167].

3. Derivatives of transforms. If $F(t)=O(k_0^t)$ and $a^{\operatorname{Re}(s)}\geqq k$ where $k>k_0$, then $|a^{-st}|\,|F(t)|<K(k_0k^{-1})^t$. K and k_0 are positive constants. The function $(k_0k^{-1})^t$ is independent of s and its sum converges over the interval $(0,\infty)$. Therefore, (1) is uniformly convergent with respect to s. The Cauchy-Riemann equations are satisfied with $a^{-st}F(t)$. Hence termwise differentiation of (1) is permissible. Therefore,

$$(d/ds)f(s) = \sum_{t=0}^{\infty} a^{-st}[-tF(t)]\operatorname{Ln}^{\neg} a.$$

If $t^{n-1}F(t)$ satisfies the conditions imposed on $F(t)$, then

$$(d^n/ds^n)f(s) = \sum_{t=0}^{\infty} a^{-st}[(-t)^nF(t)](\operatorname{Ln} a)^n,$$

or

$$\mathfrak{D}\{t^nF(t)\} = (-1)^n/(\operatorname{Ln} a)^n\cdot(d^n/ds^n)f(s).$$

4. Translation of the transform $f(s)$. Let $F(t)$ in (1) be of exponential order.

$$f(s+\sigma) = \sum_{t=0}^{\infty} a^{-(s+\sigma)t}F(t), \text{ where } \sigma \text{ is complex.}$$

$$f(s+\sigma) = \sum_{t=0}^{\infty} a^{-st}a^{-\sigma t}F(t) = \mathfrak{D}\{a^{-\sigma t}F(t)\}, \text{ when } a^{\operatorname{Re}(s)+\operatorname{Re}(\sigma)} > k.$$

5. Convolution. Assume $F(t)$ and $G(t)$ are two functions, each of exponential order. Then $\mathfrak{D}\{F(t)\}$ and $\mathfrak{D}\{G(t)\}$ converge absolutely. The Cauchy product of the two series,

$$f(s)g(s) = \sum_{t=0}^{\infty} a^{-st}F(t) \sum_{t=0}^{\infty} a^{-st}G(t),$$

may be expressed as

$$f(s)g(s) = \sum_{t=0}^{\infty} a^{-st}\left[\sum_{\tau=0}^{t} F(\tau)G(t-\tau) \right],$$

or

$$f(s)g(s) = \sum_{t=0}^{\infty} a^{-st}\left[\sum_{\tau=0}^{t} G(\tau)F(t-\tau) \right].$$

Therefore, we may express

$$f(s)g(s) = \mathfrak{D}\{F(t) * G(t)\} = \mathfrak{D}\{G(t) * F(t)\}, \tag{2}$$

where the combination of $F(t)$ and $G(t)$ appearing inside the sum is called the convolution of these functions. It is understood that

$$F(t) * G(t) = \sum_{\tau=0}^{t} F(\tau)G(t-\tau).$$

With $G(t)=1$, in (2), it follows that

$$\frac{a^s}{a^s - 1} f(s) = \mathfrak{D}\{1 * F(t)\} = \mathfrak{D}\left\{ \sum_{\tau=0}^{t} F(\tau) \right\}. \tag{3}$$

If the process of (3) is continued n times,

$$\frac{a^{ns}}{(a^s - 1)^n} f(s) = \mathfrak{D}\left\{ \sum_{\tau=0}^{t} \sum_{\sigma=0}^{\tau} \cdots \sum_{\alpha=0}^{\beta} F(\alpha) \right\}, \tag{4}$$

where n indicates the number of summations.

6. Applications. The transform method, used by Fort [3] for the solution of a difference equation, can be applied to systems of linear difference equations with constant coefficients. This method is useful for the solution of certain summation equations also. Consider the equation,

$$\sum_{\tau=0}^{t} \tau^{(k)} F(t-\tau) = (t+n)^{(m)}, \qquad 1 \leqq n \leqq m - k. \tag{5}$$

(The notation $t^{(q)}$ represents the factorial polynomial $t(t-1) \cdots (t-q+1)$.) The transformed equation of (5) is

$$\frac{k!a^s}{(a^s-1)^{k+1}}f(s) = \frac{m!a^{(n+1)s}}{(a^s-1)^{m+1}},$$

or

$$f(s) = \frac{m!a^{ns}}{k!(a^s-1)^{m-k}}.$$

Since it can be shown that

$$\mathfrak{D}\{(t+p)^{(q)}\} = \frac{q!a^{(p+1)s}}{(a^s-1)^{q+1}}, \text{ if } p \leqq q,$$

the solution of the equation follows immediately.

$$F(t) = \frac{m!}{k!(m-k-1)!}(t+n-1)^{(m-k-1)}.$$

The summation of finite series may be accomplished by employing results suggested by the convolution theorem. We observe that (3) and (4) may be written in the form,

$$\sum_{\tau=0}^{t} F(\tau) = \mathfrak{D}^{-1}\left\{\frac{a^s}{a^s-1}f(s)\right\}, \tag{6}$$

and

$$\sum_{\tau=0}^{t}\sum_{\sigma=0}^{\tau}\cdots\sum_{\alpha=0}^{\beta} F(\alpha) = \mathfrak{D}^{-1}\left\{\frac{a^{ns}}{(a^s-1)^n}f(s)\right\}. \tag{7}$$

($\mathfrak{D}^{-1}\{f(s)\}$ is the inversion of $f(s)$ and is equal to $F(t)$.)

The cosine series, $1+\cos g+\cos 2g+\cdots+\cos tg$, may be summed by formula (6).

$$\begin{aligned}\sum_{\tau=0}^{t}\cos g\tau &= \mathfrak{D}^{-1}\left\{\frac{a^s}{a^s-1}\,\frac{a^s}{2}\left[\frac{1}{a^s-e^{ig}}+\frac{1}{a^s-e^{-ig}}\right]\right\}\\ &= \mathfrak{D}^{-1}\left\{\frac{a^{2s}}{2(a^s-1)(a^s-e^{ig})}+\frac{a^{2s}}{2(a^s-1)(a^s-e^{-ig})}\right\}.\end{aligned}$$

Since it can be shown that

$$\mathfrak{D}\left\{\frac{b^{t+1}-1}{b-1}\right\} = \frac{a^{2s}}{(a^s-1)(a^s-b)},$$

then

$$\mathfrak{D}^{-1}\left\{\frac{a^{2s}}{2(a^s-1)(a^s-e^{ig})}+\frac{a^{2s}}{2(a^s-1)(a^s-e^{-ig})}\right\}$$

$$= (1/2)\left[\frac{e^{ig(t+1)} - 1}{e^{ig} - 1} + \frac{e^{-ig(t+1)} - 1}{e^{-ig} - 1}\right]$$

$$= (1/2)\left[\frac{e^{igt} + e^{-igt} - e^{ig(t+1)} - e^{-ig(t+1)} - e^{ig} - e^{-ig} + 2}{2 - e^{ig} - e^{-ig}}\right]$$

$$= \frac{\cos gt - \cos g(t+1) - \cos g + 1}{2(1 - \cos g)}$$

$$= \frac{\cos gt - \cos g(t+1)}{2(1 - \cos g)} + \frac{1}{2}$$

$$= \frac{\sin g(t + 1/2)}{2 \sin (g/2)} + \frac{1}{2}\cdot$$

Therefore,

$$\sum_{\tau=0}^{t} \cos g\tau = \frac{\sin g(t + 1/2)}{2 \sin (g/2)} + \frac{1}{2}\cdot$$

The sum,

$$1^2 + 1^2 + 2^2 + 1^2 + 2^2 + 3^2 + \cdots + 1^2 + 2^2 + 3^2 + \cdots + t^2,$$

may be determined by (7).

$$\sum_{\tau=0}^{t}\sum_{\sigma=0}^{\tau} \sigma^2 = \mathfrak{D}^{-1}\left\{\frac{a^{2s}}{(a^s - 1)^2} \quad \frac{a^{2s} + a^s}{(a^s - 1)^3}\right\} = \mathfrak{D}^{-1}\left\{\frac{a^{4s} + a^{3s}}{(a^s - 1)^5}\right\}.$$

$$\sum_{\tau=0}^{t}\sum_{\sigma=0}^{\tau} \sigma^2 = \frac{(t+3)^{(4)}}{4!} + \frac{(t+2)^{(4)}}{4!} = \frac{(t+1)(t+2)^{(3)}}{12}.$$

7. Table of transforms. A short list of transforms is given below. See Fort [**3**] for additional transformations.

	$F(t)$	$f(s)$
1	t^n	$\dfrac{(-1)^n}{(\mathrm{Ln}\ a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^s}{a^s - 1}\right)$
2	$\dbinom{t+n}{m}$	$\dfrac{a^{(n+1)s}}{(a^s - 1)^{m+1}} - \sum_{j=1}^{n-m} a^{js}\dbinom{n-j}{m}$, if $m < n$, $\dfrac{a^{(n+1)s}}{(a^s - 1)^{m+1}}$, if $n \leqq m$

3	$\sum_{\tau=0}^{t-1} F(\tau)$	$\dfrac{1}{a^s - 1} f(s)$
4	$t^n b^{ht}$	$\dfrac{(-1)^n}{(\text{Ln } a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^s}{a^s - b^h}\right)$
5	$t^n b^{ht} \sin gt$	$\dfrac{(-1)^n}{(\text{Ln } a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^s b^h \sin g}{a^{2s} - 2a^s b^h \cos g + b^{2h}}\right)$
6	$t^n b^{ht} \cos gt$	$\dfrac{(-1)^n}{(\text{Ln } a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^{2s} - a^s b^h \cos g}{a^{2s} - 2a^s b^h \cos g + b^{2h}}\right)$
7	$t^n b^{ht} \sinh gt$	$\dfrac{(-1)^n}{(\text{Ln } a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^s b^h \sinh g}{a^{2s} - 2a^s b^h \cosh g + b^{2h}}\right)$
8	$t^n b^{ht} \cosh gt$	$\dfrac{(-1)^n}{(\text{Ln } a)^n} \dfrac{d^n}{ds^n}\left(\dfrac{a^{2s} - a^s b^h \cosh g}{a^{2s} - 2a^s b^h \cosh g + b^{2h}}\right)$
9	$t^{(n)} b^{ht} \sinh gt$	$n! a^s b^{nh}\left[\dfrac{V \cosh ng + U \sinh ng}{(a^{2s} - 2a^s b^h \cosh g + b^{2h})^{n+1}}\right]^*$
10	$t^{(n)} b^{ht} \cosh gt$	$n! a^s b^{nh}\left[\dfrac{U \cosh ng + V \sinh ng}{(a^{2s} - 2a^s b^h \cosh g + b^{2h})^{n+1}}\right]^*$

* $U+V=(a^s-b^h \cosh g+b^h \sinh g)^{n+1}$, $U-V=(a^s-b^h \cosh g-b^h \sinh g)^{n+1}$.

References

1. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford, 1935.

2. Tomlinson Fort, Infinite Series, Oxford, 1930.

3. ———, Linear difference equations and the Dirichlet series transform, this MONTHLY, vol. 62, 1955, pp. 641–645.

4. ———, Solving difference equations by means of the Dirichlet series transform, Bull. Amer. Math. Soc., vol. 61, 1955, pp. 62–63.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE FUNDAMENTAL THEOREM OF ALGEBRA*

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We prove anew the theorem of Gauss, that a nonconstant polynomial has at least one root. The discussion is based on the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x, \tag{1}$$

and makes no reference to geometry or topology. The novelty is thought to consist partly in the derivation of (1), but chiefly in the discussion of Remark 4. The latter yields a new proof for the minimum-modulus theorem of function theory.

As background we assume a few elementary facts of differential calculus such as Leibniz' rule and the properties of absolute value. These do not require the completeness of the real-number system, hence can be proved in all rigor in the beginning calculus class. It is necessary to know also that mixed partial derivatives satisfy $u_{xy} = u_{yx}$. Although this is a rather deep theorem in general, it is trivial for polynomials. (Verify the equation directly for $u = a_{pq}x^p y^q$; it then follows by linearity for a sum of such terms.)

A second theorem of some complexity is that

$$g_x = g_y = 0, \qquad g_{xx} \geqq 0, \qquad g_{yy} \geqq 0$$

at a minimum of $g(x, y)$. This too is trivial for polynomials. (Without loss of generality let the polynomial $g(x, y)$ have a minimum of value 0 at $x = y = 0$. Then

$$g(x, 0) = bx + cx^2 + \cdots + kx^n = x(b + cx + \cdots + kx^{n-1}),$$

where $b, c, \cdots, k$ are constant. The assumption that $b \neq 0$ shows immediately that $x = 0$ is not a minimum. When $b = 0$ the same procedure shows that $c \geqq 0$).

The only other "deep" theorem we need is that a continuous function in the closed circle attains a minimum in the closed circle. This is not trivial even for polynomials; in fact the fundamental theorem of algebra is deducible therefrom.

1. Algebraic properties of the Cauchy-Riemann equations. We say that a complex function $w(x, y)$ satisfies the Cauchy-Riemann equations if its real and imaginary parts do so; that is, if $w(x, y) = u(x, y) + iv(x, y)$ where u and v are real and satisfy (1).

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Remark 1. If w and W are complex functions which satisfy the Cauchy-Riemann equations, then so are $w+W$ and wW.

If $w=u+iv$, $W=U+iV$, then $w+W=(u+U)+i(v+V)$. Now the equations (1) for w and W yield

$$(u+U)_x = u_x + U_x = v_y + V_y = (v+V)_y$$

which is the first equation for $w+W$. The second follows similarly.

Again, we have $wW=uU-vV+i(uV+vU)$. By Leibniz' rule,

$$(uU - vV)_x = uU_x + u_xU - vV_x - v_xV,$$
$$(uV + vU)_y = uV_y + Vu_y + vU_y + Uv_y.$$

That these expressions are equal follows by comparing coefficients of u, U, v, V, with due regard to (1). We now have the first equation for wW; proof of the second is similar.

Remark 2. Let u and v be real and imaginary parts of a polynomial, so that

$$u + iv = a_nz^n + \cdots + a_1z + a_0,$$

where $z=x+iy$ and the a's are constant. Then the polynomials u and v satisfy (1).

The reader may verify that $x+iy$ satisfies the Cauchy-Riemann equations; that is, $u=x$, $v=y$ satisfy (1). Hence, by Remark 1, the same is true of $(x+iy)(x+iy)=(x+iy)^2$ and of $(x+iy)(x+iy)^2=(x+iy)^3$, and so on. Hence, $(x+iy)^m$ satisfies the Cauchy-Riemann equations for each integer m.

Again, a constant $a+ib$ is clearly a solution; that is, $u=a$, $v=b$ satisfies (1). Hence, by Remark 1, $a_m(x+iy)^m$ satisfies the Cauchy-Riemann equations; and, again by Remark 1, a sum of such terms does so.

2. A minimum modulus theorem. In the following remarks, u and v are solutions of (1) as given by Remark 2.

Remark 3. If $2f=u^2+v^2$, and if $m=u_x^2+u_y^2$, then

$$f_x^2 + f_y^2 = m(u^2 + v^2), \tag{2}$$
$$f_{xx} + f_{yy} = 2m. \tag{3}$$

For proof, we use (1) to obtain three expressions for f_x and f_y:

$$f_x = uu_x + vv_x = uu_x - vu_y = uv_y - vu_y,$$
$$f_y = uu_y + vv_y = uu_y + vu_x = -uv_x + vu_x.$$

The second of these expressions gives

$$f_x^2 + f_y^2 = (uu_x - vu_y)^2 + (uu_y + vu_x)^2 = m(u^2 + v^2)$$

as desired. The third of these expressions gives

$$f_{xx} = uv_{yx} + u_xv_y - vu_{yx} - v_xu_y,$$
$$f_{yy} = - uv_{xy} - u_yv_x + vu_{xy} + v_yu_x,$$

whence, by addition and by (1),

$$f_{xx} + f_{yy} = u_xv_y - u_yv_x - v_xu_y + v_yu_x$$
$$= u_x^2 + u_y^2 + u_y^2 + u_x^2 = 2m.$$

Remark 4. With $f(x, y)$ as above, suppose $f(0, 0) = c$, and suppose $f(x, y) \geqq c+1$ on the circle $x^2+y^2=r^2$. Then $f=0$ at some point inside this circle.

For proof, let h be a small positive constant and set

$$g(x, y) = f(x, y) - hx^2. \tag{4}$$

At the origin and on the circle $x^2+y^2=r^2$ we have, respectively,

$$g(0, 0) = c, \tag{5}$$

$$g(x, y) \geqq c + 1 - hx^2 \geqq c + 1 - hr^2 > c, \tag{6}$$

provided h is sufficiently small.

The function $g(x, y)$ attains an absolute minimum in $x^2+y^2 \leqq r^2$ since g is continuous. Moreover, this minimum is not attained on the boundary, since (5) and (6) show that $g(0, 0)$ is smaller than any value attained on the boundary. There is, then, an interior minimum.

At the minimum point we have, by (4),

$$f_x - 2hx = g_x = 0, \qquad f_y = g_y = 0; \tag{7}$$

$$f_{xx} - 2h = g_{xx} \geqq 0, \qquad f_{yy} = g_{yy} \geqq 0. \tag{8}$$

Equations (7) combine with (2), and (8) with (3) to give, respectively,

$$m(u^2 + v^2) = f_x^2 + f_y^2 = 4h^2x^2, \tag{9}$$

$$2m = f_{xx} + f_{yy} \geqq 2h. \tag{10}$$

By division of (9) and (10), $(u^2+v^2)/2 \leqq 2hx^2 \leqq 2hr^2$. Since h is arbitrary this shows that u^2+v^2 cannot have a positive lower bound in the circle, whence we deduce $u^2+v^2=0$ at some point.

3. The fundamental theorem. Remark 4 contains the fundamental theorem of algebra as a special case. For, if

$$u + iv = a_nz^n + \cdots + a_1z + a_0, \tag{11}$$

where $z=x+iy$, $a_n \neq 0$, and $n \geqq 1$, then we have the Cauchy-Riemann equations

(1), by Remark 2. And we also have (trivially)

$$\lim \frac{u + iv}{a_n z^n} = 1 \qquad \text{as } |z| \to \infty.$$

Hence, for large $|z|$ we have $|u+iv|/|a_n z^n| > 1/2$ say, or, equivalently, $|u+iv| > (1/2)|a_n| |z|^n$. This shows that $u^2+v^2 \to \infty$ as $x^2+y^2 \to \infty$, and therefore that the hypothesis of Remark 4 is satisfied if r is large enough. The point (x, y) where u and v both vanish yields a complex number $z = x+iy$ where the polynomial vanishes, and the existence of a root is established.

NOTE ON THE DISTRIBUTION OF THE TOTATIVES

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If n is a positive integer, the $\phi(n)$ numbers $\leqq n$ and relatively prime to n are called the *totatives* of n.* We shall be concerned in this note with the distribution of the totatives of n in certain subintervals from 0 to n. In fact, the intervals are given by

$$(1) \qquad nq/k < m < n(q + 1)/k, \qquad q = 0, 1, \cdots, k - 1.$$

No totative can occur as the endpoint of such an interval if we make the natural assumption that $n > k$. We make this assumption throughout this note.

We denote by $\phi(k, q, n)$ the number of totatives of n in the interval (1). Then a natural measure of the distribution of the totatives of n in the intervals (1) is the function

$$(2) \qquad E(k, q, n) = \phi(n) - k\phi(k, q, n).$$

The totatives of n are said to be *uniformly distributed* with respect to k if $E(k, q, n) = 0$ for all q: then each of the intervals (1) contains $\phi(n)/k$ of the totatives of n. Lehmer* called *n exceptional* with respect to k if n is divisible by k^2 or by a prime of the form $kx+1$. He showed that the totatives of n are uniformly distributed with respect to k when n is exceptional with respect to k. Thus we have a sufficient condition for the uniform distribution of the totatives of n. It is not a necessary condition as can be seen from the fact that the totatives 1, 2, 4, 5, 7, 8 of 9 are uniformly distributed with respect to 6. In fact, we have the following result.

THEOREM. *If k is a prime, then the totatives of n are uniformly distributed with respect to k, if and only if n is exceptional with respect to k. Moreover, if k is not squarefree there is an integer $n > k$ such that the totatives of n are uniformly distributed with respect to k, but n is not exceptional with respect to k.*

We need only prove the necessity of the condition. From (2), $E(k, q, n) \equiv \phi(n) \pmod{k}$, and since $\phi(n)$ is multiplicative we have for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$,

* D. H. Lehmer, The distribution of totatives, Canad. J. Math., vol. 7, 1955, pp. 347–357.

$$E(k, q, n) \equiv \prod_{i=1}^{t} p_i^{\alpha_i-1}(p_i - 1) \pmod{k}.$$

Suppose $E(k, q, n)=0$. Then, since k is prime, either $p_i \equiv 1 \pmod{k}$ for some i, or $k=p_i$ and $\alpha_i>1$ for some i. In either case, n is exceptional with respect to k.

To prove the second part of the theorem we shall make use of the

LEMMA. *If k divides n, and n/k has the same prime divisors as n, then the totatives of n are uniformly distributed with respect to k.*

If $r<n/k$ is prime to n, so also is $r+qn/k$ prime to n for all q, and *vice versa*. Hence the number of totatives of n in the interval (1) is independent of q, so all the $E(k, q, n)$ are equal, and by (6) of Lehmer's paper* each $E(k, q, n)=0$.

Now suppose k is not squarefree and write $k=p^\alpha k_1$, where $\alpha>1$ and k_1 is prime to p. Let $n=p^{\alpha+1}k_1^2$. By the lemma the totatives of n are uniformly distributed with respect to k. Clearly k^2 does not divide n. If n has a prime divisor of the form $kx+1$, then so does k, which is impossible. Thus n is not exceptional with respect to k. This completes the proof of the theorem.

Remark. The case when k is squarefree and composite presents added difficulties. The example preceding the statement of the theorem shows that the first part of the theorem is not true for such k (see also Theorem 7 of Lehmer's paper*), and we have been unable to determine whether or not the second part is true.

We wish to thank the referee for his kind suggestions.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

PROBABILITY AND SUMS OF SERIES

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Many interesting problems in probability stem from the simple model of repeated Bernoulli trials, where the probability of success is the same at each trial. If the trials are independent, but the probability p_n of success at trial n is allowed to vary with n, the trials are called Poisson trials [1, p. 233]). When we consider an infinite sequence of Poisson trials we are led quite naturally to certain associated series (for example, generating functions, *cf.* Feller, Chapter 11). Here we consider the probability of eventually obtaining a success. This is given by

$$Pr \text{ (success on some trial)} = \sum_{n=1}^{\infty} f_n, \tag{1}$$

where f_n is the probability of *first* success at trial n,

$$f_1 = p_1, \qquad f_n = (1 - p_1)(1 - p_2) \cdots (1 - p_{n-1})p_n, \qquad n > 1. \tag{2}$$

In particular, we are interested in circumstances under which the probability of getting at least one success is one. The following theorem gives the answer. By specializing the p_n we also obtain the sums of certain infinite series in explicit form.

THEOREM. *Let* $\{p_n\}$ *be a sequence of real numbers,* $0 \leqq p_n < 1$. *Let* f_n *be given by* (2). *Then*

$$\sum_{n=1}^{\infty} f_n \leqq 1, \tag{3}$$

and equality holds in (3), *if and only if*

$$\prod_{n=1}^{\infty} (1 - p_n) \text{ diverges to zero,} \tag{4}$$

or (*equivalently*), *if and only if*

$$\sum_{n=1}^{\infty} p_n \text{ diverges to plus infinity.} \tag{5}$$

Proof. Let N be a positive integer. Let

$$P_N = \sum_1^N f_n, \qquad Q_N = \prod_1^N (1 - p_n). \tag{6}$$

Then P_N is the probability of at least one success in the first N trials, while Q_N is the probability of no successes in the same N trials. Hence

$$P_N + Q_N \equiv 1. \tag{7}$$

Equation (7) is an identity for all sequences $\{p_n\}$ and for all positive integers N. In case $0 \leqq p_n \leqq 1$, the sequence $\{Q_N\}$ is monotone nonincreasing and bounded below by zero. Hence Q_N tends to a limit as N increases indefinitely. Likewise P_N has a limit. We denote these limits by Q and P respectively and have from (7)

$$P + Q = 1, \tag{8}$$

identically for all sequences $\{p_n\}$ such that $0 \leqq p_n \leqq 1$, with

$$P = \sum_1^{\infty} f_n \leqq 1, \qquad Q = \prod_1^{\infty} (1 - p_n) \geqq 0. \tag{9}$$

From (8) it follows at once that $P = 1$ if and only if $Q = 0$. And $Q = 0$ if and only

if either (i) $p_n=1$ for some n, or (ii) $\sum p_n$ diverges to plus infinity [2]. The theorem is thus established.

Example 1. Let $p_n=1/(n+1)$ and obtain

$$\sum_1^\infty \frac{1}{n(n+1)} = 1.$$

Example 2. Let $p_n=n/(n+1)$ and obtain

$$\sum_1^\infty \frac{n}{(n+1)!} = 1.$$

Example 3. Let $p_n=n/(n+k)$, $k>0$, and obtain

$$\sum_{n=1}^\infty \frac{nk^{n-1}}{\Gamma(n+k+1)} = \frac{1}{\Gamma(k+1)}\,.$$

In particular, for $k=1/2$,

$$\sum_{n=1}^\infty n \Big/ \left(\prod_{k=1}^n (2k+1)\right) = 1/2.$$

Remarks. (a) The identity (7) may be of interest to algebra teachers looking for another easy exercise in mathematical induction. Feller's book provides many more. Their probabilistic interpretations may provide additional interest.

(b) The series in Example 2 above is clearly closely related to the series for e^x. In fact, if we look at the generating function for the sequence $\{f_n\}$ we have

$$F(x) \equiv \sum_1^\infty f_n x^n = \sum_1^\infty nx^n/(n+1)!,$$

and we soon recognize this as being x times the derivative of

$$G(x) = \sum_1^\infty x^n/(n+1)! = x^{-1}(e^x - 1 - x).$$

Hence

$$F(x) = x\frac{d}{dx}G(x) = e^x - x^{-1}(e^x - 1),$$

and the series in Example 2 is evaluated by taking $x=1$; $F(1)=1$. To obtain the expected number of trials until first success occurs, we would compute

$$F'(1) = \sum_{n=1}^\infty nf_n = e - 1.$$

(c) One of the Borel-Cantelli lemmas [1, p. 155] asserts: *If the events A_k are*

mutually independent, and if $\sum Pr(A_k)$ diverges, then with probability one infinitely many A_k occur.

We may take A_k to be the event, success at trial k, and apply this lemma and our theorem to obtain the following interesting corollary.

COROLLARY. *Let p_n be the probability of success on the nth trial in a sequence of Poisson trials. If $0 \leqq p_n < 1$, and the probability of obtaining at least one success is one, then the probability of obtaining infinitely many successes is also one.*

References

1. W. Feller, An Introduction to Probability and its Applications, New York, 1950.
2. J. M. Hyslop, Infinite Series, New York, 1942, p. 94.

THE GEOMETRY OF VARIATION OF PARAMETERS

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The usual first course in differential equations treats the method of variation of parameters for finding particular integrals of nth order linear nonhomogeneous differential equations somewhat formally. The method of Lagrange multipliers for extreme values of functions of several variables under constraints often suffers from the same kind of treatment. There is, of course, a geometric interpretation of both methods (see the treatment of Lagrange multipliers in Kaplan, *Advanced Calculus*, Cambridge, 1952). Since the variation of parameters method is, for most students, the first encounter with perturbation theory, it seems appropriate to provide at least the better students with some geometry to go with it. It is the purpose of this note to present a rather simple example together with suggestions as to what happens in the general case.

We consider the second order nonhomogeneous differential equation with initial conditions

$$\ddot{x} + x = \mu f(x, \dot{x}, t), \qquad x(0) = u, \qquad \dot{x}(0) = v.$$

The homogeneous equation is that of a harmonic oscillator with angular velocity unity, whose solution is $x(t) = u \cos t + v \sin t$. The variation of parameters method seeks to find $u(t)$ and $v(t)$ so that $u(t) \cos t + v(t) \sin t$ is a solution of the nonhomogeneous (perturbed) equation.

Changing (1) to the corresponding first order system, we have

$$(1) \qquad \begin{aligned} \dot{x} &= y, & x(0) &= u, \\ \dot{y} &= -x + f(x, y, t), & y(0) &= v, \end{aligned}$$

and the unperturbed system

$$(2) \qquad \begin{aligned} \dot{x} &= y, & x(0) &= u, \\ \dot{y} &= -x, & y(0) &= v. \end{aligned}$$

The solutions in the xy-(phase) plane of the conservative (homogeneous) system

(2) are given parametrically by

$$
(3) \qquad \begin{aligned} x &= u \cos t + v \sin t = \phi(u, v, t), \\ y &= -u \sin t + v \cos t = \psi(u, v, t). \end{aligned}
$$

Eliminating t, we have

$$
\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx} = -\frac{x}{y},
$$

so that the solutions are concentric circles. The representative point travels clockwise on one of these circles (determined by the initial values u and v) with angular velocity unity.

Equations (3), considered as the equations of a rotation through the angle t, transform the xy system of coordinates into a system of coordinates which rotates clockwise with angular velocity unity. In the new coordinate system, the point (u, v) is an equilibrium state for the system (2). Regarding (3), then, as a change of variables, and substituting into (1), we have

$$
(4) \qquad \frac{\partial \phi}{\partial u} \dot{u} + \frac{\partial \phi}{\partial v} \dot{v} + \frac{\partial \phi}{\partial t} = \psi(u, v, t),
$$

$$
(5) \qquad \frac{\partial \psi}{\partial u} \dot{u} + \frac{\partial \psi}{\partial v} \dot{v} + \frac{\partial \psi}{\partial t} = -\phi(u, v, t) + \mu f(\phi, \psi, t).
$$

On the other hand, regarding (3) as the solutions of (2), we have

$$
(6) \qquad \frac{d\phi}{dt} \stackrel{t}{\equiv} \psi(u, v, t),
$$

$$
(7) \qquad \frac{d\psi}{dt} \stackrel{t}{\equiv} -\phi(u, v, t),
$$

$$
(8) \qquad \phi(0) = u, \qquad \psi(0) = v.
$$

By virtue of (6), (4) becomes

$$
\frac{\partial \phi}{\partial u} \dot{u} + \frac{\partial \phi}{\partial v} \dot{v} = 0.
$$

This equation is arrived at in the usual treatment by imposing one of two allowable conditions on the functions $u(t)$ and $v(t)$, the other being the result of using (7) in (5). In the harmonic oscillator example, where, if u and v are regarded as unknown functions of t in (3), we have $x = -u \sin t + v \cos t + \dot{u} \cos t + \dot{v} \sin t$, and one of the conditions imposed on u and v is that $\dot{u} \cos t + \dot{v} \sin t = 0$. It is implied that any condition could be imposed at this point, but it is clear that in this geometric setting, the usual condition is the only reasonable one. Using (6) and (7) to simplify (4) and (5),

$$\frac{\partial\phi}{\partial u}\dot{u}+\frac{\partial\phi}{\partial v}\dot{v}=0,\quad \frac{\partial\psi}{\partial u}\dot{u}+\frac{\partial\psi}{\partial v}\dot{v}=\mu f(\phi,\psi,t). \tag{9}$$

Since, when $t=0$, the determinant of the coefficients (the Jacobian J of the transformation (3)) is unity, we can solve for u and v in some neighborhood of $t=0$:

$$\dot{u}=-\frac{\mu}{J}f(\phi,\psi,t)\frac{\partial\phi}{\partial v},\qquad \dot{v}=\frac{\mu}{J}f(\phi,\psi,t)\frac{\partial\phi}{\partial u}\cdot \tag{10}$$

In the harmonic oscillator problem, these become $\dot{u}=-\mu f(\phi,\ \psi,\ t)\sin t$, $\dot{v}=\mu f(\phi,\ \psi,\ t)\cos t$.

In any case, u and v will be linear in $\mu f(\phi,\psi,t)$, the perturbing function, so that for small values of the constant μ, the rates of change of u and v will be small, *i.e.*, u and v are nearly constant. The quantities $u(t)-u(0)$, $v(t)-v(0)$ are perturbations, measuring the change that the initial point of the solution of the homogeneous equation undergoes as a result of the perturbing function $\mu f(x,\dot{x},t)$.

If, for the harmonic oscillator, we have $\mu f(x,\dot{x},t)=\mu e^{-t}$, the perturbations are

$$\begin{aligned} u(t)-u(0)&=-\mu\left[\frac{e^{-t}}{2}(-\sin t-\cos t)+1/2\right],\\ v(t)-v(0)&=\mu\left[\frac{e^{-t}}{2}(-\cos t+\sin t)+1/2\right]. \end{aligned} \tag{11}$$

For large t, $u(t)\approx u(0)-\mu/2$, and $v(t)\approx v(0)+\mu/2$, so that the steady state solution of the perturbed equation in the phase plane is the circle whose center is at the origin, passing through $(u-\mu/2,\ v+\mu/2)$. It is interesting to note that there finally results no change in the orbit if the initial conditions and the parameter μ are such that $u-v=\mu/2$, no matter how large is μ. The perturbing function μe^{-t} produces no net change in the energy state of the oscillator in this case. However, in general, small perturbations are associated only with small values of μ and short intervals of time. The steady state solution may be greatly different from the unperturbed solution, since the perturbing function has had a long time in which to change the energy level of the system, even though this level changes very slowly when μ is small.

In the above example, the perturbations, viewed in the uv-plane, appear as the curves given parametrically by equations (11). That is, if one were riding along on the trajectory of the solution of the unperturbed equation, one would view only the effect of the perturbing function.

For the relation of the variation of parameters method to one-sided Green's functions, see Miller, *Engineering Mathematics*, New York, 1956.

References

1. A. S. Andronow and C. E. Chaikin, Theory of Oscillations, Princeton, 1949.
2. F. R. Moulton, Celestial Mechanics, New York, 1914.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1281. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Prove that no perfect square is 7 more than a perfect cube.

E 1282. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Tangents PQ, PR are drawn from a point P to a circle (O). Another tangent touches the circle at a point B on the minor arc QR and cuts PQ, PR at A, C. Show that QR is equal to the sum of the sides of the triangle of minimum perimeter that can be inscribed in triangle OCA.

E 1283. *Proposed by J. W. Andrushkiw, Seton Hall University*

If three consecutive coefficients of the real polynomial

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

are equal and different from zero, show that the zeros of $P(x)$ are not all real.

E 1284. *Proposed by Charles Fox, McGill University*

Let P be any point on a central conic C, and let the normal to C at P and the polar of P with respect to the director circle of C meet at Q. Show that PQ is equal to the radius of curvature of C at P. What form does this result take for a parabola?

E 1285. *Proposed by James Ax and Lawrence Shepp, Polytechnic Institute of Brooklyn*

Let A and B be two nth order determinants, and let C be a third determinant whose (i, j)th element is the determinant A with its ith column replaced by the jth column of B. Show that $C = A^{n-1}B$.

SOLUTIONS

Willie's Age

E 1251 [1957, 109]. *Proposed by J. S. Lew, Princeton University*

"Did your teacher give you that problem?" I asked. "It looks rather tedious." "No," said Willie, "I made it up. It's a polynomial with my age as a root.

That is, x stands for my age last birthday."

"Well, then," I remarked, "It shouldn't be so hard to work out—integer coefficients, integral root. Suppose I try $x=7 \cdots$. No, that gives 77."

"Do I look only seven years old?" demanded Willie.

"Well, let me try a larger integer $\cdots$. No, that gives 85, not zero."

"Oh, stop kidding!" said Willie, looking over my shoulder. "You know I'm older than that!"

How old is Willie?

Solution by D. C. B. Marsh, Colorado School of Mines. Willie's friend, dispensing with trial-and-error, might make use of the fact that $a-b$ is an integral divisor of $P(a)-P(b)$ when a, b are distinct integers and $P(x)$ is a polynomial with integral coefficients. Denoting the "larger integer" tried by I and Willie's age by A, we have $I-7$ divides 8, $A-7$ divides 77, $A-I$ divides 85, and $7<I<A$. It follows that I must be one of 8, 9, 11, 15 and A one of 14, 18, 84. To have $A-I$ divide 85, we see that the second integer tried was 9 and that Willie is 14 years old.

Also solved by Merrill Barnebey, H. F. Bennett, J. L. Botsford, C. K. Bradshaw, Julian Braun, D. A. Breault, W. B. Carver, J. E. Davis, Underwood Dudley, P. L. Duren, Hazel E. Evans, M. A. Feldstein, H. J. Fletcher, L. R. Ford, C. B. Germain, Cornelius Groenewoud, A. R. Hyde, I. M. Isaacs, Ray Jurgensen, Seymour Kass, Andrew Kraus, Joe Lipman, Wallace Manheimer, J. B. Muskat, C. S. Ogilvy, W. D. Peeples, Jr., C. F. Pinzka, R. I. Purry, L. A. Ringenberg, David Rosen, Azriel Rosenfeld, David Rothman, D. K. Stillinger, Chih-yi Wang, R. M. Warten, Alan Wayne, and the proposer. Late solution by L. D. Goldstone.

Carver, Ford, and Wayne pointed out that Willie's polynomial must be of the form

$$(x-7)(x-9)(x-14)Q(x)-3x^2+52x-140,$$

where $Q(x)$ is any polynomial with integral coefficients.

A Property of Positive Integral Powers of an Integer

E 1252 [1957, 109]. *Proposed by Jerome Rothstein, Signal Corps Engineering Laboratories, Fort Monmouth, N. J.*

Prove that the difference between two positive integral powers of the same integer is exactly divisible by six unless the integer gives the remainder two on division by three and one power is odd while the other is even. Show that in this exceptional situation the sum of the two powers is exactly divisible by six.

Solution by Andrew Kraus, University of Colorado. Let n, m be positive integers, and a an integer. In every case $a^n \pm a^m \equiv 0 \pmod 2$. If $a\equiv 0 \pmod 3$, then $a^n-a^m=(3k)^n-(3k)^m\equiv 0 \pmod 3 \equiv 0 \pmod 6$. If $a\equiv 1 \pmod 3$ and n, m are of like parity, then $a^n-a^m=(3k+1)^n-(3k+1)^m\equiv 0 \pmod 3 \equiv 0 \pmod 6$. If $a\equiv -1 \pmod 3$, and n, m are of unlike parity, then $a^n+a^m=(3k-1)^n+(3k-1)^m \equiv 0 \pmod 3 \equiv 0 \pmod 6$.

Also solved by J. L. Alperin, R. V. Andree, Leon Bankoff, H. F. Bennett, P. L. Chessin, J. E. Darraugh, A. G. Davis, Irma Esrig, Hazel E. Evans, Michael Goldberg, Cornelius Groenewoud, Emil Grosswald, J. H. Hodges, J. T. Humphrey, A. R. Hyde, I. M. Isaacs, J. B. Johnson, Ray Jurgensen, Sidney Kravitz, Aaron Lieberman, Joe Lipman, Marshall Luban, W. S. McCulley, Wallace Manheimer, D. C. B. Marsh, L. C. Marshall, G. E. Meador, David Muskat, J. B. Muskat,

C. S. Ogilvy, Hiram Paley, D. S. Passman, D. J. Persico, C. F. Pinzka, L. A. Ringenberg, Azriel Rosenfeld, Kenneth Ross, David Rothman, Frank Saunders, J. A. Schumaker, Lawrence Shepp, W. A. Soper, Jr., D. R. Sudborough, Eldon Vought, Chih-yi Wang, R. M. Warten, Maud Willey, L. K. Williams, and the proposer. Late solution by D. A. Breault.

Application of a Formula of Finite Differences

E 1253 [1957, 109]. *Proposed by T. R. Jenkins, University of Idaho, and by Nathaniel Macon and Abraham Spitzbart, Flight Propulsion Laboratory, General Electric Co.*

For n and p positive integers with $n>p$, and x and y arbitrary, show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + ky)^p = 0.$$

Find the value of the sum for $n=p$.

I. *Solution by Chih-yi Wang, University of Minnesota.* We have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + ky)^p = \frac{d^p}{dt^p}\left\{ \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{(x+ky)t} \right\}\Bigg|_{t=0}$$
$$= \frac{d^p}{dt^p} \left\{ e^{xt}(1 - e^{yt})^n \right\} \Big|_{t=0}.$$

For $n>p$, the pth derivative of the above expression involves all terms containing the factors $(1-e^{yt})^r$, $n-p\leqq r\leqq n$, whence the stated result follows. For $n=p$, the only nonvanishing term is $n!(-y)^n$.

II. *Solution by W. A. Al-Salam, Duke University.* Denote the sum by S_p. Expanding $(x+ky)^p$ and changing order of summation, we find

$$S_p = \sum_{s=0}^{p} \binom{p}{s} x^{p-s} y^s \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^s.$$

But the inside sum vanishes for $s<n$ and has the value $(-1)^n n!$ if $s=n$. Hence $S_p=0$ or $(-1)^n y^n n!$ according as $p<n$ or $p=n$.

III. *Solution by R. S. Pinkham, Princeton University.* From the calculus of finite differences

$$\Delta^n w^p = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (w + k)^p.$$

Setting $w=x/y$ and multiplying both sides by $(-1)^n y^p$, we find

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + ky)^p = \Delta^n w^p (-1)^n y^p.$$

But when $n\geqq p$, $\Delta^n w^p = \delta_p^n n!$, and we obtain the desired results.

Also solved by J. L. Alperin, J. L. Brown, Jr., J. E. Darraugh and F. D. Parker (jointly), R. V. Esperti, C. B. Germain, Cornelius Groenewoud, Emil Grosswald, R. H. Hou, J. B. Johnson, P. G.

Kirmser, M. S. Klamkin, Joseph Lehner, Aaron Lieberman, Joe Lipman, D. C. B. Marsh, Mary P. Moseley, C. F. Pinzka, Ronald Pyke, L. A. Ringenberg, J. B. Roberts, D. A Robinson, E. P. Rozycki, Paul Schaefer, F. C. Smith, E. P. Starke, Julius Vogel, David Zeitlin, and the proposers.

Pinzka pointed out that the desired conclusion follows from Problem 4183 [1947, 235], and that Problem 3625 [1934, 454] also deals with the limiting case. For the valuation of the inside sum in Solution II see, *e.g.*, Schwatt, *An Introduction to Operations with Series*, p. 101, or Chrystal, *Algebra*, Part II, 2nd ed., formulas (3) and (4) p. 207 (by setting $x=-n$ and introducing the factor $(-1)^n$). The fundamental formula of the calculus of finite differences used in Solution III may be found in Jordan, *Calculus of Finite Differences*, p. 132 *et seq.*, or Boole, *Calculus of Finite Differences*, 3rd ed., formula (3), p. 19.

Pencils of Conics Containing a Circle

E 1254 [1957, 109]. *Proposed by Robin Robinson, Dartmouth College*

Prove that if two conics intersect in four distinct points, these points are concyclic if and only if the axes of the two conics are parallel or perpendicular.

I. *Solution by L. R. Ford, Illinois Institute of Technology.* Let the conics be the noncircles

$$S_i = A_ix^2 + H_ixy + B_iy^2 + E_ix + F_iy + C_i = 0, \qquad i = 1, 2.$$

The general equation of a conic through their points of intersection is

$$S_1 + kS_2 = (A_1 + kA_2)x^2 + (H_1 + kH_2)xy + (B_1 + kB_2)y^2 + \cdots = 0.$$

This is made a circle by taking $A_1+kA_2=B_1+kB_2$, $H_1+kH_2=0$, which is possible if and only if $H_1/(A_1-B_1)=H_2/(A_2-B_2)$. This can be written $\tan 2\theta_1 = \tan 2\theta_2$, where θ_i is the angle between the x-axis and an axis of the conic. Hence, $2\theta_1=2\theta_2+n\pi$, or $\theta_1=\theta_2+n\pi/2$, as required.

II. *Solution by Roland Deaux, Faculté Polytechnique de Mons, Belgium.* By virtue of Desargues' theorem, conics passing through the vertices A, B, C, D of a proper quadrangle determine on the line at infinity pairs of points of an involution I. In order that A, B, C, D be concyclic, it is necessary and sufficient that the circular points U, V constitute a pair of I. The projection of I from any proper point P is an involution i which, because of the pair formed by the isotropic lines PU, PV, has two perpendicular double rays p_1, p_2, and these lines are thus the bisectors of the angles formed by any pair of conjugate lines belonging to i. The pencil of conics considered contains two parabolas whose axes are parallel to p_1, p_2, and these lines are parallel to the axes of any other conic of the pencil, whence the property.

Also solved by W. A. Al-Salam, N. A. Court, Michael Goldberg, A. R. Hyde, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy, O. J. Ramler, Sister M. Stephanie, Chih-yi Wang, Maud Willey, and an anonymous solver.

R. H. F. Denniston gave the references: Durell, *Algebraic Geometry*, London, 1955, p. 351; Robson, *An Introduction to Analytic Geometry*, vol. ii, Cambridge, 1947, p. 97; Walker, *Cartesian and Projective Geometry*, London, 1953, pp. 193 (analytic proof), 276 (projective proof). Al-Salam pointed out that the desired result follows immediately from the theorem, "The common chords of a conic and a circle taken in pairs are equally inclined to the axis of the conic," and he gave the reference, Askwith, *Analytic Geometry of the Conic Sections*, London, 1950, p. 218.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4753. *Proposed by J. K. Senior, University of Chicago*

Are there any integers larger than 31 which can be represented in more than one way by the form $(a^m-1)/(a-1)$, a and m integers and $m>2$.

4754. *Proposed by T. G. Room, University of Sydney, Australia*

Given the four quadrics

$$L \equiv x^2 + dyz - cyt + bzt = 0, \qquad M \equiv y^2 + dzx + cxt - azt = 0,$$
$$N \equiv z^2 + dxy - bxt + ayt = 0, \qquad K \equiv t^2 - ayz - bzx - cxy = 0.$$

Show that if $1+a^3+b^3+c^3+d^3+3abcd=0$ then they have six points in common, and otherwise, none.

4755. *Proposed by Chandler Davis, Columbia University*

In how many ways can the first n positive integers be arranged in alternately increasing and decreasing order? That is, how many permutations π: $\pi(1), \cdots, \pi(n)$ are there such that the quantities $(-1)^k\{\pi(k+1)-\pi(k)\}$, for $k=1, \cdots, n-1$, have all the same sign?

4756. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, x_2, \cdots, x_n)$, $q(x_1, x_2, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope R: $0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$, when $x_1x_2 \cdots x_n \neq 0$. Prove that there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0)=0$, such that throughout R

$$h[q(x_1, \cdots)] < p(x_1, \cdots).$$

4757. *Proposed by O. P. Aggarwal, University of Washington, Seattle*

Prove for every integer $n \geqq 0$, and for any positive c,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(c+k)^2} = \frac{\Gamma(c)\Gamma(n+1)}{\Gamma(c+n+1)} \sum_{k=0}^{n} \frac{1}{c+k}.$$

SOLUTIONS

Convergence of a Sequence

4703 [1956, 584]. *Proposed by R. E. Bellman, Rand Corporation, Santa Monica, California*

Show that the conditions (a) $u_n \geqq u_{n-1} - a_{n-1}$, $a_n \geqq 0$, $\sum_{n=0}^{\infty} a_n < \infty$, (b) $u_n \leqq M$ imply the convergence of the sequence $\{u_n\}$.

I. *Solution by E. M. Wright, King's College, Aberdeen, Scotland.* Put $A_n = \sum_{m=1}^{n} a_m \to A$ as $n \to \infty$. Then

$$u_{n-1} + A_{n-2} \leqq u_n + A_{n-1} < M + A,$$

and so $\{u_n + A_{n-1}\}$ is a convergent sequence. Hence, so is $\{u_n\}$.

II. *Solution by R. P. Agnew, Cornell University.* The hypothesis can be weakened to (a′) $u_n \geqq u_{n-1} - a_{n-1}$, (a″) $\sum a_n$ is convergent, and (b′) $\liminf u_n < +\infty$. Using (a′) gives $u_p - (a_p + a_{p+1} + \cdots + a_{q-1}) \leqq u_q$ when $0 < p < q$. For each fixed p, taking inferior limits as $q \to \infty$ gives

$$u_p - (a_p + a_{p+1} + \cdots) \leqq \liminf_{q\to\infty} u_q$$

where the right member is either finite or $+\infty$. Taking superior limits as $p \to \infty$ gives

$$\limsup_{p\to\infty} u_p \leqq \liminf_{q\to\infty} u_q.$$

This implies that either $\lim u_p$ exists or $\lim u_p = +\infty$. The hypothesis (b′) eliminates the latter possibility and gives the conclusion that the sequence $u_1, u_2, \cdots$ is convergent.

Also solved by P. M. Anselone, Joshua Barlaz, Ranko Bojanic, G. U. Brauer, A. Césaro, Ward Cheney, T. Y. Chow, R. M. Conkling, R. J. Driscoll, Harley Flanders, L. C. Graue, D. S. Greenstein, C. A. Grimm, A. S. Hendler, Peter Henrici, S. S. Holland, Jr., J. B. Johnston, J. B. Kelly, J. H. B. Kemperman, W. A. Kistler, W. S. Lawton, A. E. Livingston, Marshall Luban, R. M. Meisel, E. J. Miller, D. J. Newman, E. N. Nilson, F. D. Parker, B. E. Rhoades, Michael Skalsky, D. P. Squier and David Zeitlin, Chih-yi Wang, L. E. Ward and L. E. Ward, Jr., and the proposer.

Necessary and Sufficient Condition for a Prime

4704 [1956, 584]. *Proposed by L. E. Clarke, University College of the Gold Coast*

If m is a positive integer greater than 1, show that a necessary and sufficient condition that m be prime is that, for every integer n,

$$\binom{n}{m} \equiv [n/m] \pmod{m}.$$

Solution by E. M. Wright, King's College, Aberdeen, Scotland. Given $|x| < 1$, we have

$$\sum_{n=m}^{\infty}\binom{n}{m}x^{n-m} = (1-x)^{-m-1},$$

$$\sum_{n=m}^{\infty}[n/m]x^{n-m} = (1+x+x^2+\cdots+x^{m-1})(1+2x^m+3x^{2m}+\cdots)$$

$$= (1-x)^{-1}(1-x^m)^{-1}.$$

Hence the condition of the problem is equivalent to the truth of the identical congruence

$$(1) \qquad (1-x)^m \equiv 1-x^m \pmod{m}.$$

If m is a prime, this is well known (Hardy and Wright, *Theory of Numbers*, Theorems 75, 76) and almost trivial. If m is composite, let p be a prime divisor of m. Then (1) implies that $m\mid\binom{m}{p}$, and so $p\mid(m-1)(m-2)\cdots(m-p+1)$, which is impossible since $p\mid m$. Hence (1) is a necessary and sufficient condition that m be prime.

Also solved by D. M. Bloom, Leonard Carlitz, Leopold Flatto, A. S. Hendler, J. H. Hodges, J. H. B. Kemperman, D. C. B. Marsh, J. B. Muskat, C. D. Olds, Paul Schaefer, G. J. Smith, and the proposer.

Isomorphisms of the Additive Group

4705 [1956, 584]. *Proposed by Albert Wilansky, Lehigh University*

It is possible to construct an additive but discontinuous real function of one real variable. Is it possible to construct one with the property that $f(x)=f(y)$ implies $x=y$? (This would give an algebraic isomorphism of the additive group of the reals which is not continuous.)

Solution by G. E. Bredon, Harvard University. All possible additive functions are obtained by assigning arbitrary values to the elements of a Hamel basis over the rationals and extending the function by additivity to the reals. The isomorphisms of the additive group of the reals are obtained by mapping one Hamel basis in a one-to-one manner onto itself or some other basis, and all of these isomorphisms are discontinuous except the (trivial) ones of the form $x\to\alpha x$, α fixed.

Also solved by Ward Cheney, Harley Flanders, J. Horváth, H. H. Johnson, J. B. Johnston, J. B. Kelly, J. H. B. Kemperman, L. E. Ward, Jr., and the proposer.

Editorial Note. See Hamel's original discussion, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y)=f(x)+f(y)$, Math. Ann. vol. 50, 1905, pp. 459–462.

Periodic Functions Having Incommensurable Periods

4706 [1956, 584]. *Proposed by S. W. Golomb, California Institute of Technology*

If $f(x)$ and $g(x)$ are real periodic functions of the real variable x, having incommensurable periods, is it possible for $f(x)+g(x)$ to be periodic?

I. *Solution by R. H. F. Denniston, University College, Ibadan, Nigeria.* Let

a, b, c be real numbers, no two being commensurable. For real x, let

$$d(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0; \end{cases}$$

$$f(x) = \sum_{m,n-\infty}^{\infty} d(x - ma - nb) - \sum d(x - ma - nc);$$

$$g(x) = \sum d(x - mb - nc) - \sum d(x - ma - nb).$$

Then $f(x)$, $g(x)$, $f(x)+g(x)$ are periodic with periods a, b, c.

II. *Solution by Harley Flanders, University of California, Berkeley.* An affirmative answer is indicated by the following construction. Let a, b, c be real numbers which are linearly independent over the rationals and imbed these in a Hamel basis $\{a, b, c, \rho_i\}$ of the reals over the rationals. We define linear (over the rationals) real valued functions f and g by $f(a)=0$: $f(b)$, $f(c)$, $f(\rho_i)$ arbitrary; $g(b)=0$; $g(c)=-f(c)$; $g(a)$, $g(\rho_i)$ arbitrary. Then f has period a, g has period b, and $f+g$ has period c.

Also solved by G. E. Bredon, J. B. Kelly, J. H. B. Kemperman, Darald Rohb, and the proposer.

Magic Matrix

4707 [1956, 584]. *Proposed by Charles Fox, McGill University*

A magic matrix is one whose elements are the numbers of a magic square, *i.e.*, every row, column and diagonal has the same sum. (a) Show that a 3 by 3 magic matrix inverts into a magic matrix. (b) Can this result be extended to magic matrices of higher order?

Solution by F. D. Parker, Clarkson College of Technology, Potsdam, N. Y. Let A be a nonsingular magic matrix of any order, let $R(x)$ be a row matrix whose elements are all x, and $C(x)$ be a column matrix whose elements are all x, and let A' be the inverse of A. The condition that the rows of A all have the sum s is $AC(1)=C(s)$. Multiplying on the left by A' we have $C(1)=A'C(s)$, or the rows of A' have the same sum $1/s$. The corresponding analysis of the column sums goes from $R(1)A=R(s)$ to $R(1)=R(s)A'$, which shows that the columns of A' all have the same sum $1/s$.

It is not hard to show in a straightforward way that for $n=3$, if the diagonals of A also have the sum s, then the diagonals of A' have the same sum $1/s$. That this result does not necessarily extend to the fourth order is shown by the counterexample:

$$A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 2 & 1 & 5 & 0 \\ 3 & 0 & 2 & 3 \\ 1 & 4 & 0 & 3 \end{bmatrix}, \qquad A' = \frac{1}{64}\begin{bmatrix} 66 & -14 & 2 & -46 \\ 18 & 2 & -14 & 2 \\ -30 & 18 & 2 & 18 \\ -46 & 2 & 18 & 34 \end{bmatrix}.$$

Also solved by G. E. Bredon, D. C. B. Marsh, B. E. Mitchell, Chih-yi Wang, and the proposer.

Editorial Note. The fact that the rows of the inverse matrix have the same sum $1/s$ for all orders has been noted by A. Wilansky. See this MONTHLY, vol. 58, 1951, pp. 614–615 and vol. 63, 1956, pp. 652–653, where also the result is extended to certain infinite matrices.

Series Evaluation of Definite Integral

4708 [1956, 669]. *Proposed by H. A. Bender, University of Rhode Island*

Prove the following relationship:

$$\int_0^1 e^{c(1-x^2)}dx = \sum_{n=0}^{\infty} \frac{2^n c^n}{3\cdot 5 \cdots (2n+1)}.$$

I. *Solution by Margaret M. LaSalle, Southwestern Louisiana Institute.* Repeated integration by parts gives:

$$\int_0^1 e^{c(1-x^2)}dx = 1 + \frac{2c}{3} + \frac{4c^2}{3\cdot 5} + \cdots + \frac{2^n c^n}{3\cdot 5 \cdots (2n+1)} + \frac{2^{n+1}c^{n+1}}{3\cdot 5 \cdots (2n+1)} \int_0^1 x^{2n+2}e^{c(1-x^2)}dx.$$

Since the integral on the right hand side is bounded, and the coefficient of the integral approaches zero as n approaches infinity, the result follows.

II. *Solution by L. A. Ringenberg, Eastern Illinois State College.* Expand the integrand in a series of powers of $(1-x^2)$, substitute $\sin\theta$ for x, and integrate termwise using Wallis' formula:

$$\int_0^1 e^{c(1-x^2)}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{c^n(1-x^2)^n}{n!}dx = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{c^n \cos^{2n+1}\theta d\theta}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{c^n\cdot 2\cdot 4 \cdots 2n}{n!\cdot 1\cdot 3 \cdots (2n+1)} = \sum_{n=0}^{\infty} \frac{2^n c^n}{1\cdot 3 \cdots (2n+1)}.$$

III. *Solution by D. C. B. Marsh, Colorado School of Mines.* Define

$$S(x) = \sum_{n=0}^{\infty} \frac{2^n c^n x^{2n+1}}{3\cdot 5 \cdots (2n+1)}.$$

It is easily seen that $S(x)$ satisfies the linear differential equation

$$S'(x) - 2cxS(x) = 1.$$

Introduce the integrating factor e^{-cx^2} and integrate from $(x, S) = (0, 0)$ to $(1, S(1))$ to obtain the desired result in the form

$$e^{-c}S(1) = \int_0^1 e^{-cx^2}dx.$$

Also solved by D. M. Bloom, J. L. Boal and J. B. Muskat, Fred Brafman, Leonard Carlitz,

Tien Chi Chen, R. A. Costa, A. E. Danese, J. O. C. Ezeilo, G. B. Findley, Harley Flanders, J. S. Frame, Nathaniel Grossman, Emil Grosswald, R. R. Gutzman, A. B. Harper, Jr., Juris Hartmanis, Peter Henrici, Frank Herlihy, J. H. Hodges, A. F. Horne, A. R. Hyde, W. C. James, J. B. Johnston, Marshall Lubin, Y. L. Luke, R. T. Mahoney, R. M. Meisel, C. D. Olds, F. D. Parker, M. J. Pascual, Mary Payne, B. E. Rhoades, R. E. Shafer, Lawrence Shepp, F. C. Smith, J. R. Trollope, Chih-yi Wang, R. E. Williamson, and David Zeitlin.

Editorial Note. As pointed out by Lubin and Rhoades, above solutions I and III, virtually unchanged, yield

$$\int_0^1 x^r e^{c(1-x^a)} dx = \sum_{n=0}^{\infty} \frac{a^n c^n}{(a+r+1)(2a+r+1)\cdots(na+r+1)}, \quad r > -1, a > 0.$$

Isotomic Lines

4710 [1956, 669]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

I. *Solution by O. J. Ramler, Catholic University of America.* Using a system of conjugate coordinates we take the circle (O) as the unit circle, and on it points whose vector coordinates are T_1, T_2, T_3 as the inscribed triangle and t_1, t_2, t_3, t_4 as the inscribed complete quadrangle. Then the line t_2t_3 intersects the side T_2T_3 of the triangle in a point whose vector coordinate is

$$z = \frac{T_2T_3(t_2+t_3) - (T_2+T_3)t_2t_3}{T_2T_3 - t_2t_3}.$$

Similarly t_1t_4 meets side T_2T_3 where

$$z' = \frac{T_2T_3(t_4+t_1) - (T_2+T_3)t_1t_4}{T_2T_3 - t_1t_4}.$$

The hypothesis implies $T_3 - z = z' - T_2$ which becomes, upon making proper substitutions and simplifying

$$T_2^2T_3^2(T_2 + T_3 - s_1) + T_2T_3s_3 - (T_2+T_3)s_4 = 0,$$

where s_1, s_3, s_4 are elementary symmetric functions of t_1, t_2, t_3, t_4. The symmetry of this result establishes the proposed theorem.

II. *Solution by Roland Deaux, Faculté Polytechnique, Mons, Belgium.* Circle (O) may be replaced by any conic Γ. Let ABC and $PQRS$ be the triangle and the quadrangle inscribed in Γ. By virtue of Desargues' theorem, the four conics Γ, (PQ, RS), (PR, QS), (PS, QR) determine on each side of ABC four pairs of an involution. This involution, defined by Γ and the isotomic lines PQ, RS, has for double points the midpoint and the point at infinity of the side. Hence the property.

Also solved by J. W. Clawson, N. A. Court, R. Goormaghtigh, Josef Langr, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Fundamental Concepts of Higher Algebra. By A. Adrian Albert. University of Chicago Press, 1956. ix+165 pp. $6.50.

The classic book of L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory* (Teubner, 1901) has gained stature with the passing years. The treatment of linear groups was more complete than even the author probably realized, and the book has been, over the years, the major source of information on finite fields. It has inspired considerable research in the last half century, and furthermore, notations and points of view have changed. It is timely, therefore, to have an up-to-date exposition of the subject of finite fields using the slick proofs of modern algebra and embracing the latest results.

As the author states, the real reason for the existence of this book is the fifth and last chapter of 29 pages. The usual basic theorems on Galois fields are quickly developed. A proof is given of Gauss' theorem determining the integers m for which primitive roots modulo m exist. The Galois group of the finite field is determined. The familiar theorems of Serret and Dedekind on irreducible polynomials are given, and the concept of exponent to which an irreducible polynomial belongs is used to extend and clarify this theory. Now come many detailed results on the construction of irreducible polynomials. The book ends with a discussion of Dickson's method for computing primitive irreducible polynomials, and a list without proofs of eighteen known theorems.

In order to make the book essentially self-contained, the author has put in four preliminary chapters totaling 122 pages. These are entitled: I, Groups; II, Rings and Fields; III, Vector Spaces and Matrices; IV, Theory of Algebraic Extensions. In order to cover so much material in so short a space, the author has employed two devices. He has limited the theory to those topics which are necessary for his Chapter V, and he has in places drastically condensed the usual presentation. Thus Chapter III omits quadratic form theory, and all proofs of theorems on determinants.

It is thus evident that the title of the book is too comprehensive. In the reviewer's opinion the book is not well suited for use as a textbook with beginning graduate students. For those who have been over the material once, it would furnish an excellent review and summary of many parts of modern algebra. It is a scholarly piece of work, with some misprints, to be sure (but what book does not have some?), and is the last word on finite fields.

CYRUS COLTON MACDUFFEE
The University of Wisconsin

The Enjoyment of Mathematics. By Hans Rademacher and Otto Toeplitz. Princeton University, 1956, 204 pp. $4.50.

The first edition of *Von Zahlen und Figuren*, by Rademacher and Toeplitz, was published just twenty-seven years ago, but it is recognized today as a superlative piece of mathematical writing to which it is not inappropriate to attach the adjective "classical". Although the beauty of the German language may have escaped some English speaking people, the depth and subtle power of the mathematics were apparent to any serious reader.

The Enjoyment of Mathematics is a faithful translation into English of this classic to which the translator, Herbert Zuckerman, has added two fine chapters of his own. Because this is a translation of a well-known work we mention only a few of the topics discussed: The Sequence of Prime Numbers, Traversing Nets of Curves, Incommensurable Segments and Irrational Numbers, The Theory of Sets, Waring's Problem, Pythagorean Numbers and Fermat's Theorem, The Figure of Greatest Area with a Given Perimeter, Periodic Decimal Fractions, Curves of Constant Breadth. However a mere listing of the contents can give no hint of the simple but fundamentally deep character of the book.

Each chapter is a gem of mathematical exposition and the plan of attack is the same. The reader, who needs no more mathematical training than a good high school affords, is first introduced to some innocent-looking mathematical question. Next, with an absolute minimum amount of computation, he is carried deeper and deeper into the problem until the basic issues are made excitingly clear. Where final answers are still unknown (Fermat's Theorem, The Four-Color Problem, *etc.*), partial answers are obtained.

The Princeton University Press is to be congratulated for making this English edition available. It will not only stretch the imagination of the amateur, but it will also give pleasure to the sophisticated mathematician. It should prove extremely useful to teachers in quickening the mathematical spirit of students.

C. O. OAKLEY,
Haverford College

The Basic Concepts of Mathematics. Part I, *Algebra.* By Karl Menger, The Bookstore, Illinois Institute of Technology, 1957, v+91 pp.

In the words of the author, "*this book . . . is addressed to everyone desirous to clarify his ideas about some of the basic concepts and fundamental procedures of mathematics.*" The chapter headings indicate the procedures and concepts treated. They are: Numbers and Numerals, Facts and Formulas, Variables in General Statements, Unknowns in Equations, Variables in Description of Classes, Parameters in General Problems, General Expressions, Complex Numbers, Indeterminates in Polynomials.

A particular goal of the book is to resolve the many difficulties that arise through the use of the words "variable", "indeterminate", "parameter", "con-

stant". Professor Menger accomplishes this aim with charm and skill by means of numerous examples and also by references to the words' mathematical origin and subsequent history.

The book's tone is conversational and informal. Examples are discussed in detail and exercises are provided at appropriate places. It is a very readable introduction to mathematical semantics. The author has accomplished his objective in that this book should appeal greatly to the persons to whom it is addressed.

E. H. CRISLER
University of Notre Dame

Geometric Algebra. By E. Artin. Interscience, New York, 1957. x+214 pp. $6.00.

Most of this book is devoted to the study of algebraic structures arising from various geometries. The approach is algebraic rather than geometric. A good deal of the material was previously unavailable in English or in book form. The author has taken pains to write a wholly self-contained and readable exposition of this subject on the first year graduate level. He is to be congratulated on his success in this endeavor.

The author has adopted the wise course of including in Chapter I a large body of algebraic background material, some elementary and some not so elementary, which is used in the later portions of the book. The reader would do well to follow the author's suggestion to start with Chapter II and refer back to Chapter I when needed.

In Chapter II, affine geometry is introduced axiomatically and then coordinatized. Even here the approach is algebraic, *i.e.* a Desarguian plane is obtained by assuming that there exist "enough" translations and dilatations. The Pappus and Desargues theorems are then considered. The connection between ordered geometries and ordered coordinate fields is studied using material from Chapter I on ordered fields. The fourth harmonic point construction is given as a computation in the coordinate field. This computation gives the criterion for the field to have characteristic 2 and is also used, together with a theorem of Hua (proved in Chapter I), to prove a generalization of von Staudt's theorem concerning maps of a line preserving harmonic points.

Projective geometry is introduced, already coordinatized, as the geometry of subspaces of a vector space over a field. The author then proves the fundamental theorem connecting the collineations of the geometry with semilinear transformations of the underlying vector space. This, in turn, is used to obtain an exact diagram which concisely shows the connections between the various groups involved (collineation group, general linear group, field automorphism group, *etc.*). Chapter II closes by relating the affine and projective geometries.

Symplectic and orthogonal geometries are introduced in Chapter III by considering a vector space (over a commutative field) with an inner product. There is a nice concise section in Chapter I concerning pairings and duality which is necessary for most of the results of Chapter III. Chapters II and III are independent.

Orthogonal and symplectic geometries are studied together, until the former is identified as an orthogonal sum of lines and the latter (in the nonsingular case) as a sum of hyperbolic planes. Looking at these geometries separately, he considers a number of results on how the group of isometries acts on the geometry, *e.g.* an isometry on orthogonal geometry is almost determined by its action on a hyperplane, the symplectic group contains only rotations, *etc.* Chapter III ends with the computation of the orders of the various groups in case the field is finite.

Chapter IV, The General Linear Group, begins with a discussion of determinants over noncommutative fields. This is used to study the relations between the general linear group, the group generated by transvections, their centers, centralizers, *etc.* The orders of these groups are computed in the case of a finite field.

A similar study is undertaken in Chapter V on the structures of the symplectic and orthogonal groups. The theory here is not as yet complete, so the results are necessarily more complicated. It is necessary to study a number of special cases.

All in all, the book is delightful. The author's choice of material is sound; for example, he avoids the peculiarities of non-Desarguian planes, orthogonal geometry over a field of characteristic 2. His presentation is clear and well motivated. Indeed, one of the best things about the book is that it sounds like the author talking, complete with chatty remarks (see his comment on matrix methods in Chapter I). This is a book which can and will be enjoyed by a large section of the mathematical public.

JAMES P. JANS
Ohio State University

Elements of Mathematics (*2nd Edition*). By Helen Murray Roberts and Doris Skillman Stockton. Addison-Wesley, Reading, Mass. 1956, x+308 pp. $3.50.

The book, according to the preface, was written for the purpose of preparing students deficient in secondary school mathematics for college mathematics. It fulfills this purpose rather well on the whole.

There is a clear and careful development of basic ideas which are too often assumed to be a part of a student's background. Examples of this are plentiful; the discussion of our number system, the necessary extensions of the system from positive integers, to negative integers, to rational numbers, *etc.*, the discussion of the decimal number system are all well organized and meaningful to the student. There is a thorough and clear explanation of dependent and inconsistent equations. The confusion concerning rational exponents is handled very well.

This reviewer would question the use of the word "remainder" as a substitute for "difference" in subtraction. It does not make the process more clear and could cause confusion with other uses of the word remainder. Also the practice of avoiding the use of the standard names for Associative, Commutative, and

Distributive Laws does not seem logical, since they are properly descriptive terms.

The text defines a monomial in the following manner: "Any combination of numbers and literal numbers formed only by multiplication and division has been called a term. Frequently a single term is called a monomial." Then it defines a polynomial as "the sum of two or more monomials." This meaning is usually reserved for the word *multinomial*, with *polynomial* being defined in a much more restrictive sense.

On the whole, however, the care used in the development of fundamental concepts places the book among the best recent efforts.

DONALD R. LINTVEDT
Upsala College

Irrational Numbers. By Ivan Niven. Carus Mathematical Monograph No. 11. The Mathematical Association of America (distributed by Wiley, New York), 1956. 164 pp. $3.00.

The investigation of the nature of irrational numbers may well serve as a guide to number theory. After all, a good deal of the theory of divisors and prime numbers in Euclid's *Elements* may have been obtained with the aim of understanding the irrationality of surds. Nowadays, since we distinguish within the domain of irrational numbers between algebraic and transcendental ones, a discussion of irrational numbers must necessarily include a fair amount of the theory of algebraic number fields. A book on irrational numbers has therefore a natural place in a collection of expository articles such as the *Carus Monographs*.

The present book fulfills its purpose excellently. It covers the most important facts of the theory itself and contains references and suggestions for further reading concerning those topics, which surpass in extent and intricacy the scope of such a small volume. At the end of each of the ten chapters the author has appended notes of historical and bibliographical character and informative hints about related and more complicated matters. Commendable also is the list of notations and the glossary with skillfully worded explanations at the end of the book.

The contents of the book, chapter by chapter are: I. Rationals and Irrationals. The noncountability of the irrationals is shown by an argument of measure. Decimal expansions and the Cantor expansion give criteria for irrationality. II. Irrationality of $\cos r$ for rational $r\neq 0$ is proved by a variant of the author's well-known proof of the irrationality of π, using Hermite's device $F(x) = \sum(-1)^{\nu}f^{(2\nu)}(x)$, where $f(x)$ is a suitable polynomial. The proof that e is transcendental is also given here since all preparations are at hand. Chapter III deals with the algebraic nature of the roots of unity and with a result of D. H. Lehmer about $\cos(2\pi k/n)$. IV. The pigeon-hole principle is used for the discussion of the approximation of irrationals by rationals and also simultaneous approximation of several irrationals. Chapter V gives a concise little exposition

of continued fractions, especially periodic ones. Chapter VI discusses the goodness of approximation with Hurwitz's theorem about best approximation. It also contains two proofs of the uniform distribution modulo 1 of the sequence $\{n\xi\}$, where ξ is irrational. Chapter VII on algebraic and transcendental numbers contains, in particular, Liouville's theorem. The Thue-Siegel theorem is also mentioned in the notes. Chapter VIII deals with normal numbers. Borel's definition is compared with the more natural one which considers the frequency of blocks of k digits. That both definitions are equivalent is shown by theorems of Pillai and of Niven and Zuckerman. The last two chapters are less elementary. Chapter IX gives the general Lindemann theorem. For the algebraic tools needed here the book refers to the *Carus Monograph* by Pollard. In Chapter X, Gelfond's and Schneider's theorem that α^β is transcendental for algebraic α and β (with the obvious trivial exceptions) is proved. The proof follows Siegel's presentation of Gelfond's proof, but gives many more details than Siegel's concise version. The notes to this last chapter refer also to other classes of transcendental numbers which are considered, in particular, in Siegel's treatise on transcendental numbers.

This survey shows that all important features of the theory of irrational numbers are treated or at least touched upon. Niven's book is written in a clear and attractive style. It is a very valuable addition to the still too small number of expository books in our science.

HANS RADEMACHER
University of Pennsylvania

BRIEF MENTION

Theory of Lie Groups. By Claude Chevalley. Princeton, 1957. viii+217 pp. \$2.75.

It is welcome news that this out-of-print volume is again available in the paper back edition.

Integration. By Edward J. McShane. Princeton, 1957. viii+394 pp. \$2.95.

Another welcome reissue which needs no further review other than to announce its availability (paper back).

Applied Mathematics in Chemical Engineering. By H. S. Mickley, T. K. Sherwood, C. E. Reed. McGraw-Hill, 1957. xii+413 pp. \$9.00.

This reviewer is rather disappointed that so few *modern* additions appear in the revision of this excellent test. No mention, for example, is made of the modern applications of matric algebra to chemical engineering. On the other hand, the inclusion of the Laplace transform and of finite differences, and some work on numerical methods is most welcome.

Descriptive Geometry. By Steve M. Slaby. Barnes and Noble, 1956, 353 pp. \$2.25.

Automation: Its Purpose and Future. By M. Pyke. Philosophical Library, 1956, 191 pp. $10.00.

The Electrical Production of Music. By Alan Douglas. Philosophical Library, 1957, 223 pp. $12.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ASEE MATHEMATICS DIVISION

The Mathematics Division of the American Society for Engineering Education met on June 17–21, 1957 at Cornell University. Three well-attended sessions were held, including a joint conference with the Upper New York State Section of the Mathematical Association of America. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor C. R. Wylie, Jr., University of Utah; Vice-Chairman, Professor H. A. Giddings, New York University; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Directors, Professors W. G. Warnock, Rensselaer Polytechnic Institute, G. B. Thomas, Massachusetts Institute of Technology, C. O. Oakley, Haverford College; Representative to ASEE General Council, Professor Haim Reingold, Illinois Institute of Technology. The next annual meeting of the Mathematics Division will be held in June 1958 at Berkeley, California. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

NATIONAL SCIENCE FOUNDATION PUBLICATIONS

National Science Foundation publications are available to all scientists who have need for them. Lists of publications are available from the Foundation. Requests should be addressed to the Publications Office, National Science Foundation, Washington 25, D. C.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1957 Preliminary Actuarial Examination are as follows:

First Prize of $200.

Solovay, Robert	Harvard University

Additional Prizes of $100 each.

Flittie, John H.	Drake University
Gardner, John R.	University of Toronto
Kandall, Geoffrey A.	Princeton University
Lakser, Harry	University of Manitoba
Lichtenbaum, Stephen	Harvard University
Posner, Paul	Princeton University

Sadowsky, George	Harvard University
Zvengrowski, Peter D.	Rensselaer Polytechnic Institute

The Society of Actuaries has authorized a similar set of nine prizes for the 1958 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination.
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. Special Mathematics Examination.
(Finite differences, probability and statistics.)

The 1958 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 14, 1958. The closing date for applications is April 1, 1958.

PERSONAL ITEMS

Dr. M. H. Martin, Executive Secretary, Division of Mathematics, National Academy of Sciences-National Research Council, has resigned. Professor H. W. Kuhn, Bryn Mawr College, is replacing Dr. Martin.

Professor Everett Pitcher, Lehigh University, received the honorary degree of Doctor of Science from Western Reserve University in June 1957 on the twenty-fifth anniversary of his degree of Bachelor of Arts.

Antioch College: Professor Parker Hamilton has retired as Head of the Department of Mathematics; Associate Professor J. H. Blau has been appointed Chairman of the Department; Associate Professor W. K. Smith, Bucknell University, has been appointed Associate Professor; Assistant Professor Daniel Sokolowsky has been promoted to Associate Professor.

Brown University: Assistant Professors David Gale and John Wermer have been promoted to associate professorships; Dr. W. D. Barcus has been promoted to Assistant Professor; Mr. D. K. Harrison, Intern in Mathematics, has been appointed Instructor; Mr. A. M. Duguid, Cambridge University, has been appointed Instructor for the academic year 1957–58; Dr. E. H. Brown, Jr., University of Chicago, is holding a research associateship during the academic year 1957–58 under the support of ONR; Professor Herbert Federer has received a grant from the Sloan Foundation and is on leave of absence during 1957–58; Professor David Gale is on leave of absence with the RAND Corporation, Santa Monica, California.

Cornell University: Visiting Assistant Professor S. S. Abhyankar, Columbia University, has been appointed Assistant Professor; Associate Professor I. N. Herstein, University of Pennsylvania, has been appointed Associate Professor; Dr. Alfred Aeppli, Assistant, Swiss School of Technology, has been appointed Instructor; Dr. G. R. Livesay, Research Associate at the University, has been appointed Visiting Assistant Professor; Associate Professor Daniel Gorenstein, Clark University, has been appointed Visiting Lecturer; Assistant Professor H. D. Block has been appointed Associate Professor, Department of Mechanics; Dr. Walter Feit has been promoted to Assistant Professor; Associate Professors G. A. Hunt and Paul Olum have been promoted to Professors; Mr. Hung Ching Chow, Acting Director, Institute of Mathematics, Academia

Sinica, Taipei, Taiwan, China, is a Visiting Scholar on a Mundt, Smith, and Fulbright grant; Assistant Professor C. S. Herz is on leave during 1957–58 at the Institute for Advanced Study. An Institute for Symbolic Logic, sponsored by the American Mathematical Society under a grant from the National Science Foundation, was held at the University during the summer of 1957.

Illinois Institute of Technology: The Carnegie Corporation of New York has awarded the Institute a three years grant which will be used by Professor Karl Menger in developing a new approach to teaching of mathematics.

Montana State College: Associate Professor A. L. Hess has been promoted to Professor; Assistant Professor J. E. Whitesitt has been promoted to Associate Professor.

New York University, School of Commerce, Accounts, and Finance: Associate Professor H. E. Wahlert has been promoted to Professor; Mr. Stefan Bauer-Mengelberg has accepted a fellowship in music from the Frank Huntington Beebe Fund; Mr. Nathan Newman, Cooper Union, has been appointed Instructor.

University of Nebraska: Assistant Professor W. E. Mientka, University of Nevada, and Mr. Hubert Schneider, University of Munster, have been appointed Assistant Professors; Mr. S. E. Bohn and Mr. Konrad Suprunowicz, Graduate Assistants at the University, have been appointed Instructors; Dr. G. C. Cree, Dr. M. L. Keedy, and Dr. D. W. Miller have been promoted to Assistant Professors.

Professor C. B. Allendoerfer, who is on leave of absence from the University of Washington for the year 1957–58, is a Fulbright lecturer at Cambridge.

Dr. K. A. Brons, University of Illinois, has accepted a position as applied science representative for International Business Machines Corporation, River Forest, Illinois.

Mr. P. L. Chessin, Senior Engineer, Westinghouse Electric Corporation, Baltimore, Maryland, has received a National Science Foundation Science Faculty Fellowship and is at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

Associate Professor N. E. Dodson, Lenoir Rhyne College, has been appointed Assistant Professor at Wittenberg College.

Assistant Professor A. V. Fend, New Mexico College of Agriculture and Mechanic Arts, is now employed in Operations Research, Technical Operations, Monterey, California.

Mr. D. A. Franks, Chief, Ordnance Guided Missiles School, Internal Guidance Section, SAM Division, Redstone Arsenal, Huntsville, Alabama, has a position as a mathematician at Westinghouse Electric Corporation, Air Arm Division, Baltimore, Maryland.

Dr. A. J. Goldstein, Polytechnic Institute of Brooklyn, has accepted a position as a member of the technical staff, Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. Donald Greenspan, Research Engineer, Systems Development Laboratories, Hughes Aircraft Company, Culver City, California, has been appointed Assistant Professor at Purdue University.

Dr. Juris Hartmanis, Cornell University, has been appointed Assistant Professor at Ohio State University.

Dr. J. B. Johnston, Cornell University, has been appointed Assistant Professor at the University of Kansas City.

Dr. R. F. King, Argonne National Laboratories, has accepted a position as Scientist, Computer Division, Midwestern Universities Research Association, Madison, Wisconsin.

Mr. H. G. Loomis, Pennsylvania State University, has been appointed Instructor at Amherst College.

Dr. Joseph Mayer, Department of Economics, Miami University, has been appointed Professor of Mathematics at the University.

Dr. W. O. J. Moser, University of Saskatchewan, has been promoted to Assistant Professor.

Dr. J. P. Nash, University of Illinois, has accepted a position as Manager, Information Processing Division, Missile Systems Division, Lockheed Aircraft Corporation, Sunnyvale, California.

Dr. J. W. Neuberger, University of Texas, has been appointed Instructor at Illinois Institute of Technology.

Dr. J. C. C. Nitsche, Technical University of Berlin, has been serving as Visiting Associate Professor at the Institute of Technology, University of Minnesota, since February, 1957.

Dr. L. B. Rall, Mathematician, Shell Development Company, Emeryville, California, has been appointed Associate Professor at Lamar State College of Technology.

Professor Edward Saibel, Department of Mechanics, Carnegie Institute of Technology, has been appointed Visiting Professor of Mechanics at Rensselaer Polytechnic Institute.

Assistant Professor R. L. San Soucie, University of Oregon, has accepted a position as Head, Mathematics Group, Applied Research Department, Sylvania Electric Products, Buffalo, New York.

Dr. E. J. Schweppe, University of Nebraska, has been appointed Assistant Professor at Iowa State College.

Mr. Franklin Sheehan, San Francisco State College, has a position as Operations Analyst with Technical Operations, Monterey, California.

Dr. N. B. Stein, National Science Foundation Fellow, Cornell University, has been appointed Instructor at Yale University.

Dr. R. D. S. Tuan, Head, Department of Mathematics and Physics, Tougaloo Southern Christian College, has been appointed Professor at Indiana Technical College.

Dr. W. F. Whitmore is now on leave of absence from the Operations Evaluation Group to serve as Chief Scientist, Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D. C.

Mr. J. W. Young, Jr., National Security Agency, has accepted a position as Chief, Advanced Systems Research Section, Electronics Division, National Cash Register Company, Hawthorne, California.

Mr. J. R. Ziegler, University of California at Los Angeles, has a position as Manager, Hawthorne Branch Electronic Sales, National Cash Register Company, Hawthorne, California.

Rev. L. H. Dubé, Ottawa University, died on May 29, 1957. He was a member of the Association for thirty-seven years.

Dean J. A. Hardin, Centenary College, died in August, 1955. He was a member of the Association for thirty-three years.

Mr. J. V. Longenecker, Wichita, Kansas, died on March 15, 1956. He was a member of the Association for thirty-one years.

Assistant Professor D. S. Nathan, City College of the City of New York, died on July 1, 1957.

Dr. A. R. Schweitzer, Lake Forest, Illinois, died on June 12, 1957. He was a charter member of the Association.

Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts, died on July 3, 1957. He was a member of the Association for twenty-one years and served as an officer of the Southwestern Section for six years.

Professor Emeritus A. E. Whitford, Alfred University, died on April 14, 1957. He

was a charter member of the Association.

Dr. G. H. Wilson, Furlong, Pennsylvania, died on December 4, 1956. He was a member of the Association for twenty-three years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 103 persons have been elected to membership by the Board of Governors on applications duly certified.

WILLIAM R. ALFORD, Student, The Citadel.

RICHARD A. BACH, Student, Harpur College.

WILLIAM BENDER, Ph.D. (Yale) Professor & Head, Department of Physics, University of South Dakota.

MICHAEL BERNKOPF, B.A. (Dartmouth) Grad. Student, Columbia University.

MICHAEL H. BERNSTEIN, Student, Princeton University.

MRS. LAVERNA BOYLES, M.Ed. (Oklahoma) Teacher, Taft Junior High School, Oklahoma City, Okla.

EMIL W. BROWN, Traffic Engr., Mountain States Tel. & Tel. Co., Phoenix, Ariz.

THOMAS E. CALDWELL, B.A. (Illinois) Teaching Asst., University of Arizona.

FRANCIS P. CALLAHAN, JR., M.A. (Columbia) Math., Philco Corp., Philadelphia, Pa.

TIEN C. CHEN, Ph.D. (Duke) Asso. Physicist, I.B.M. Research Lab., Poughkeepsie, N. Y.

VIRGINIA B. CHRISTIAN, M.S. in Ed. (Eastern Illinois) Instr., Eastern Illinois State College.

DONALD A. COLE, Student, Georgia Institute of Technology.

GEORGE COPP, Ph.D. (Texas) Asso. Professor, North Texas State College.

RICHARD M. CORRY, Field Engr., Ebasco Services, Estacada, Ore.

GLENN R. COWGILL, B.S. (Kent) Math., National Advisory Committee for Aeronautics, Cleveland, Ohio.

GEOFFREY CROFTS, B.Com. (Manitoba) Asso. Professor of Actuarial Science, Occidental College.

JOHN H. DAUWALDER, Student, Occidental College.

JOHN E. DENES, B.S. (C.C.N.Y.) Statistician, National Industrial Conference Board, New York, N. Y.

DOMER V. DOUGHERTY, M.Ed. (Oklahoma) Asso. Professor, Phillips University.

FRANCIS A. DUNN, Student, LaSalle College.

EARL O. EMBREE, M.S. (Illinois) Instr., Morgan State College.

GEORGE EPSTEIN, M.S. (Illinois) Res. Asst., University of California at Los Angeles.

HARRY D. EYLAR, B.A. (Montana S.U.) Grad. Asst., Montana State University.

ANN FAUCHALD, Student, University of Washington.

RICHARD W. FELDMANN, JR., Student, University of Buffalo.

JOHN R. FLORENCE, JR., B.S. in M.E. (Colorado) Instr., University of Colorado.

MARY E. GABBERT, Student, Baylor University.

AUDLEY D. GASTON, JR., B.A. (Texas) Austin, Texas.

HAL M. GILMORE, M.A. (Western Kentucky) Instr., Illinois State Normal University.

JUDITH GLEASON, Student, Baylor University.

FRED G. GUSTAVSON, Student, Rensselaer Polytechnic Institute.

KATHRYN B. HANCHON, M.S. (Michigan) Engg. Asst., General Electric Co., Cincinnati, O.

PATRICIA A. HANDRICKEN, Student, Regis College.

HAROLD B. HANES, JR., Student, Texas Christian University.

DUNSTAN HAYDEN, O.S.B., Ph.B. (Catholic) Teacher, St. Anselm's Priory School, Washington, D. C.
JAMES B. HILDEBRAND, Student, Albion College.
SHELBY K. HILDEBRAND, B.A. (North Texas) Teaching Fellow, North Texas State College.
JOEL W. HOLLENBERG, Student, Cooper Union.
HOWARD T. HUMPHREY, Student, University of Buffalo.
RORA F. IACOBACCI, M.A. (Fordham) Instr., Manhattanville College of the Sacred Heart.
MRS. SONDRA O. JAFFE, Student, Brooklyn College.
JAMES H. JORDAN, B.S. (Southern Oregon) Grad. Asst., University of Oregon.
ALLEN J. KAPLAN, Student, Boston University.
STANLEY KAPLAN, Student, Cornell University.
ARTHUR F. KAUPE, JR., Student, Carnegie Institute of Technology.
REV. JOHN F. KELLER, S.J., M.S. (St. Louis) Asst. Professor & Chm., Department of Mathematics, Loyola University, La.
LOUIS J. KIJEWSKI, Student, LaSalle College.
GEORGE D. KING, M.A. (Alabama) Hd., Department of Mathematics, Brevard College.
JACOB O. KOEHL, Student, Agricultural and Mechanical College of Texas.
ELAINE H. KOPPELMAN, Student, Brooklyn College.
KENNETH S. KRETSCHMER, M.S. (Stanford) Grad. Student, Carnegie Institute of Technology.
DON R. LICK, B.S. (Michigan S.U.) Grad. Asst., Michigan State University.
JOE LIPMAN, Student, University of Toronto.
EDMONDO M. MASTROIANNI, M.S. (St. John's U.) Math., I.B.M. Corp., Endicott, N. Y.
ROBERT L. MCFARLAND, Student, Case Institute of Technology.
EARL H. MCKINNEY, M.S. (Pittsburgh) Instr., University of Pittsburgh.
BENJAMIN H. MCLEMORE, JR., M.A. (Illinois) Asso. Professor, Jackson State College.
RICHARD J. MILLER, M.A. (Syracuse) Instr., Clarkson College of Technology.
ENNIS J. MONTELLA, M.A. (Boston C.) Asst. Professor, Merrimack College.
INA MORAN, Student, Seton Hill College.
MERLYND K. NESTELL, Student, Emmanuel Missionary College.
DEWAYNE S. NYMANN, Student, Iowa State Teachers College.
ENUENWEMBA OBI, B.S. (Kansas City) Grad. Asst., University of Nebraska.
GARY H. OSBORN, Student, State College of Washington.
MRS. PATRICIA M. OVERDEER, M.S. (Delaware) Instr., Pennsylvania State University.
CLARENCE W. PATTY, B.S. (Georgia) Grad. Student, University of Georgia.
LYMAN C. PECK, Ph.D. (Ohio State) Asso. Professor, Ohio Wesleyan University.
WILLIAM F. POHL, Student, University of Chicago.
MATTHIAS F. REESE, B.S. (Houston) Houston, Texas.
WAYNE A. RHEA, Student, Baylor University.
HUBERT L. RICHARDS, Student, Eastern Kentucky State College.
MARY E. RIETMAN, Student, West Texas State College.
MRS. ETHEL A. ROBINSON, M.A. (Stanford) Instr., Fresno State College.
ERNEST L. ROETMAN, Student, University of Minnesota.
FRED D. ROSE, Student, University of Georgia.
REUBEN I. SANDLER, Student, Reed College.
ROBERT L. SAN SOUCIE, Ph.D. (Wisconsin) Asst. Professor, University of Oregon.
RICHARD E. SARBER, B.S.M.E. (Indiana Tech. C.) Instr., Indiana Technical College.
RONALD A. SCHAUFELE, B.Ed. (Alberta) Grad. Student, University of Washington.
RAYMOND H. SCHULZ, B.S. (Sam Houston S.C.) Grad. Student, Sam Houston State College.
MACEO T. SCOTT, M.A. (Columbia) Math., Flight Simulation Lab., White Sands Proving Ground, N. Mex.
M. THERESE SHEEHAN, A.B. (Dunbarton) Math., Army Map Service, Washington, D. C.
OSCAR D. SHELLEY, Res. Engr., North American Aviation, Downey, Calif.
ARNOLD SINGER, Student, Yeshiva University.
MARVIN B. SLEDD, Ph.D. (M.I.T.) Professor, Georgia Institute of Technology.
DONALD L. SMITH, Ph.D. (Rochester) Quality Engr., Eastman Kodak Co., Rochester, N. Y.
J. DAVID STANBERRY, Minneapolis, Minn.

PAULINE P. STEINBECK, Student, Central College, Mo.

JACK H. TAUB, Student, Rutgers University.

ROBERT G. TOBEY, Student, College of Wooster.

MARTIN L. USSERY, Student, Arkansas Polytechnic College.

SHERMAN B. VANAMAN, JR., M.S. (Kentucky) Chm., Department of Mathematics, Carson-Newman College.

ELDON J. VOUGHT, Student, Manchester College.

M. CURTIS WALKER, M.A. (Michigan) Math., General Motors Institute.

WILLIAM R. WHEELER, Student, Gannon College.

WILLIAM H. WHITEHEAD, III, Student, Ursinus College.

JACOB A. WILLIAMS, M.A. (Denver) Res. Math., Denver Research Institute.

ROBERT J. WINTERBOTTOM, III, Student, Ursinus College.

ALFRED H. WITTE, JR., Student, University of Nebraska.

ELAINE G. YODICE, B.A. (Hunter) Teaching Asst., University of Wisconsin.

JAMES M. YOHE, Student, DePauw University.

ROBERT R. A. YOUNG, Student, University of New Mexico.

HORAN B. ZARIAN, M.A. (California) Topographic Computer, U. S. Army.

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The sixteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hunter College, New York City, April 27, 1957. Dean Mina Rees of Hunter College, Chairman of the Section, opened the meeting and presided at the morning session which was devoted to papers on some mathematical developments related to computer applications. Dr. Irving Dodes, High School Vice-Chairman of the Section, was moderator for the afternoon session which was devoted to papers on curriculum trends in high school and college mathematics. There were 122 persons in attendance, including 94 members of the Association.

The following officers were elected for the years 1957–59: Chairman, Professor J. N. Eastham, Cooper Union; Vice-Chairman for Colleges, Professor E. R. Stabler, Hofstra College; Vice-Chairman for High Schools, Dr. Reinhold Walter, Manhattan School of Aviation Trades; Secretary, Dr. Azelle B. Waltcher, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting reports were given by the Treasurer, the Committee on Contests and Awards, and the Committee on Coordination of Mathematics Training. The motion was unanimously passed that the Treasurer of the Section, Mr. Aaron Shapiro, receive an expression of appreciation and gratitude for his service to the Section.

At the Executive Committee Meeting of the Metropolitan New York Section in November, 1956 a resolution was adopted concerning the willingness of the Committee on Contests and Awards to work with the National Contest Committee. As a result of this action, the Committee on Contests and Awards brought to the annual meeting the following resolution, which was unanimously adopted:

Whereas, the Mathematical Association of America has set up a National Committee on Contests, and

Whereas, the Committee on Contests and Awards of the Metropolitan New York Section has been conducting annual contests since 1949, and

Whereas, the contest booklets and experience of the Metropolitan New York Section have been used, and are being used, by the Contest units in eight or nine states and provinces,

Therefore, be it resolved that the Metropolitan New York Section offers the services of its Committee on Contests and Awards to the National Committee in the promotion of the objectives of the annual high school contest.

The By-Laws of the Section were amended by unanimous vote to provide for two-year terms of office for the Section officers.

The Committee on Coordination of Mathematics Training of the Metropolitan New York Section formulated three resolutions which received the approval of the Executive Committee at the November 1956 meeting. The resolutions, which follow, were adopted by the Section at the annual meeting.

1. The Metropolitan New York Section of the M.A.A. recommends to all high school students who plan to study calculus in college that they should have at least three years of high school mathematics, and the Section recommends to all high school advisors that they strongly advise this course of study.

2. The Metropolitan New York Section of the M.A.A. recommends to the colleges that three years of high school mathematics be required as a minimum for all college entrants intending to study the calculus, and that high school mathematics through advanced algebra be required as a minimum for all college entrants planning courses of study in mathematics, science, or engineering.

3. The Metropolitan New York Section of the M.A.A. recommends to the high schools that prospective college entrants be separated from other students in high school mathematics classes wherever feasible, and that the respective teaching methods and course contents be chosen in accordance with the objectives for each group of students.

It was moved and unanimously carried that the Metropolitan New York Section go on record as approving the sentiments expressed in a telegram sent by Dean Mina Rees to Chancellor John P. Myers, New York State Board of Regents. The telegram stated in effect the opinion that an increase in the course requirements in professional education courses without consideration of the adequacy of subject matter preparation would weaken the preparation of teachers, and would influence adversely serious efforts of educational leaders to improve the quality of instruction. The opinion was also expressed in this telegram, that faculty leaders outside of professional education circles should be called upon to participate in decisions which affected the course requirements in the preparation of teachers.

The following papers were presented at the meeting:

1. *The UNIVAC election forecasts*, by Dr. M. A. Woodbury, College of Engineering, New York University.

The mathematical and statistical problems involved in applying probability models to forecasting the 1952, 1954, and 1956 elections using an electronic computer are discussed. The problems of data selection and rejection, transformation into appropriate form, inferences, and decisions based on the probability model are set forth.

2. *Linear programming*, by Dr. Philip Wolfe, Princeton University.

An example of a production scheduling problem is taken as typical of the management problems formulable in the linear programming model. Procedures for converting such problems into computationally accessible problems are illustrated by means of the example. The theory of positive linear dependence is sketched and applied to the linear programming problem, and used to motivate the computational algorithm of the simplex method.

3. *New trends in the undergraduate college mathematics curriculum*, by Dr. Seymour Schuster, Polytechnic Institute of Brooklyn.

The traditional college curriculum in mathematics is briefly analyzed with regard to its present shortcomings. Particular emphasis is given to the failure of the traditional program to serve the principles of liberal education, which require fluency in language and mathematics. The apparent trends which seek revision of the curriculum are described through a consideration of the efforts of the M.A.A. Committee on the Undergraduate Mathematical Program and the many faculties which are experimenting with their respective curricula. The influence of several of the modern texts is discussed through a brief analysis of each. Finally, hope is extended for a trend which would

manifest itself in a concerted effort (with all departments joining forces) to assume responsibilities to the discipline of language, as well as mathematics—since communication depends on both.

4. *Preparing the bright student in twelfth year high school mathematics*, by Mr. H. D. Ruderman, The Bronx High School of Science.

Students preparing for the Advanced Standing Examination in Mathematics must take analytic geometry and calculus. For those not taking the Examination, the following five criteria might be considered as a basis for selection of the course. The mathematics for the course should: 1) emphasize ideas having wide and important applications, 2) contain a good sampling of significant mathematical ideas, 3) aim at a better understanding of the foundations of mathematics, 4) be learnable without undue hardship but rather with considerable interest and excitement on the part of the student, 5) be teachable by the best prepared teachers in every high school, again without undue hardship but rather with interest and enthusiasm. At The Bronx High School of Science there is a course that meets these conditions. The first half consists of advanced algebra with a large portion of differential calculus limited to algebraic functions. The second half follows closely chapters 1 to 6 and 12 of "Principles of Mathematics" by Allendoerfer and Oakley published by McGraw-Hill.

AZELLE B. WALTCHER, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held on April 27, 1957, at Southeast Missouri State College, Cape Girardeau, in conjunction with a meeting of the Missouri Council of Teachers of Mathematics. Professor W. R. Utz, Vice-Chairman of the Section, presided at the morning session of the Section. Professor R. J. Michel, Chairman of the Section, presided at the business meeting and the joint afternoon session. There were 54 members in attendance.

The following officers were elected: Chairman, Professor H. D. Brunk, University of Missouri; Vice-Chairman, Professor J. D. Elder, St. Louis University; Secretary-Treasurer, Miss Mary L. Cummings, University of Missouri.

The following papers were presented:

1. *Some Roman mathematics*, by Professor J. F. Daly, St. Louis University.

From an unpublished manuscript of an unknown author of the late twelfth or early thirteenth century an account is given of how a Roman could find the sum of an arithmetic progression of consecutive numbers and of consecutive even numbers. Also included are the six rules whereby Roman numerals can be multiplied. These rules for multiplication demand the use of an abacus, which is likewise furnished in the text.

2. *The Lebesgue integral for sophomores*, by Professor H. M. MacNeille, Washington University.

The possibility of defining the definite integral in the initial calculus course in a manner that will lead to the Lebesgue rather than the Riemann integral is discussed.

3. *Two-fold generalization of Cauchy's lemma*, by Professor D. E. Coffey, Missouri School of Mines and Metallurgy.

There exist integral solutions x_i of equations

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a, \qquad x_1 + x_2 + x_3 + x_4 = b,$$

if and only if $a \equiv b \pmod 2$ and $4a - b^2$ is a sum of 3 squares. If $b \geqq 4(1-k)$ and if $b^2 + 2bk + 4k^2 > 3a$, each $x_i > -k$.

4. *A unique construction*, by Mr. H. J. Johnson, Engineer, American Telephone and Telegraph Company, St. Louis.

On one side of a random angle, with a compass lay off points of 1 unit, 2 units and 3 units distance from the vertex of the angle and draw arcs within the angle through these points. Bisect the angle and bisect each half angle limiting the bisecting lines of the half angle to two units radius.

Place a half angle in a new position with its vertex on the intersection of the original bisector with the unit arc keeping the side of the half angle equidistant from the bisecting line. In this position the termini of the sides of the half angle determine two points in free space.

Pivot the compass on the vertex of the original angle, adjust its radius so that a concentric arc may be drawn through these terminal points. This is the unique arc. In case $P=3$, the unique arc is divided at these terminal points into 1/3, 2/3 sections. In case $P=5$, the fifth angle is isolated, *etc.*

5. *Continued fractions, an elementary treatment,* by Mr. C. A. Bridger, Missouri Division of Health, Jefferson City.

Continued fractions have held the attention of mathematicians for centuries. Their properties have aided in the solution of Diophantine equations, in the development of a theory of irrationals, and in the solution of the moment problem in probability, as examples.

Beginning with the long division algorithm, one can develop the c.f. for the rational fraction X in the form

$$a_0 + \frac{1}{a_1} \, \frac{1}{+ a_2} \, \frac{1}{+ a_3} \cdots \frac{1}{+ a_n},$$

where the a_i are the partial quotients. The convergents to X are P_i/Q_i, where $P_i=P_{i-1}a_{i-1}+P_{i-2}$ and $Q_i=Q_{i-1}a_{i-1}+Q_{i-2}$, $P_0=1$, $Q_0=0$, $P_1=a_0$, $Q_1=1$.

Similar results are derived for the general c.f. constructed from the sequence $\{a_i, b_i\}$ where a_i and b_i are not zero for all i equal to or greater than unity, $a_0=1$, and b_0 may be zero. The transformation for converting a c.f. into an equivalent c.f. is $a_i' = c_{i-1}c_ia_i$ and $b_i' = c_ib_i$, where $c_0=1$ and the other c_i are arbitrary. The convergent 13/8 for the c.f. for the Golden Section is used in art. The c.f. that has only positive integers as its convergents is

$$\frac{1}{1} \, \frac{1}{- 2} \, \frac{1}{- 2} \, \frac{1}{- 2} \, \frac{1}{- 2} \cdots.$$

6. *The teaching of elementary mathematics,* by Professor A. H. Copeland, Sr., University of Michigan. (By invitation.)

C. H. DALTON, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The thirty-first meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, on May 4, 1957. Dean L. T. Moston, Chairman, presided during the morning and business sessions, and Dr. Morris Ostrofsky presided during the afternoon session. There were 89 persons present, including 52 members of the Association.

During the business session, the following officers were elected for two year terms: Chairman, Professor I. Dee Peters, West Virginia University; Secretary-Treasurer, Dr. B. H. Mount, Westinghouse Electric Corporation, Pittsburgh; Executive Council, Professor J. H. Neelley, Carnegie Institute of Technology, and Professor H. B. Curry, Pennsylvania State University. Also, the Section voted to participate in the national contest program for secondary school students.

The following papers were presented:

1. *On solution of systems of linear equations,* by Dr. A. S. Householder, Army Research Center, Madison, Wisconsin, and Oak Ridge National Laboratories.

Most closed methods of solving linear algebraic equations or inverting matrices can be

classified as methods of factorization and methods of modification. In methods of factorization the matrix is expressed as a product of two matrices, each of which is readily inverted. Generally one of these is triangular, and the other either also triangular or else one with orthogonal rows or orthogonal columns. In methods of modification one proceeds from a matrix of known inverse and modifies an element or group of elements at a time. This method is thought to include Kron's methods of "tearing".

2. *Introduction to electronic digital computers*, by Dr. Ruth O'Donnell Goodman, Westinghouse Research Laboratories, Pittsburgh.

Electronic digital computers, although barely a dozen years old, have already had tremendous influence on the scientific and mathematical world. Each machine is an artfully designed complex of binary decision elements which can carry out arithmetic instructions, making logical decisions during the course of the computation. Since they perform at electronic speeds, even the slowest of these machines can execute hundreds of arithmetic operations per second. When a problem is to be solved by electronic computation, precise instructions for the obtaining of the solution must first be written down, then be translated into the exact "numerical language" which the machine's electronic circuitry can "understand". The preparation of problems for solution by electronic computation is a growing profession for mathematically trained personnel. The remaining papers will outline this work by following through the successive steps which must be taken between the statement of the problem and its final solution.

3. *Mathematical analysis of the problem—the numerical calculation of the Riemann Zeta function near unity*, by Dr. Lowell Schoenfeld, Westinghouse Research Laboratories, introduced by Dr. Ruth O. Goodman.

Consider the calculation, for $s>1$, of

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{M-1} n^{-s} + \sum_{n=M}^{\infty} n^{-s} \equiv Z_M(s) + R_M(s).$$

In order that the truncation error incurred by dropping $R_M(s)$ should not exceed ϵ, it is necessary that $M \geqq m(s, \epsilon)$. For $s=1.1$ and $\epsilon=10^{-3}$, we have $m(s, \epsilon)=10^{40}$ so that the computation cannot be carried out in this manner by any present or projected computing machine. By using the Euler-Maclaurin sum formula, we find that for each $N \geqq 1$ and some θ_N satisfying $0 \leqq \theta_N \leqq 1$

$$R_M(s) = \frac{1}{M^{S-1}} \left\{ \frac{1}{S-1} + \frac{1}{2M} + \sum_{j=1}^{N-1} b_j C_j + \theta_N b_N C_N \right\},$$

where the b_j are modified Bernoulli numbers, $C_1 = s/M^2$, and $C_j/C_{j-1} = (s+2j-3)(s+2j-2)/M^2$ for $j \geqq 2$. It is shown how to determine M and N so as to minimize the work of computation. In particular, if $\epsilon \geqq 10^{-8}$ and $s \leqq 22.1$, then $M \leqq 5$ and $N \leqq 13$, so that, on dropping the term $\theta_N b_N C_N / M^{s-1}$ (whose absolute value now does not exceed ϵ), we have only $(M-1)+1+1+(N-1) = M+N \leqq 18$ terms to compute. The calculation now becomes feasible and much computation time can even be saved for values of s ranging up to 3 and beyond.

4. *Observations on high school mathematics contests*, by Professor E. F. Myers, University of Pittsburgh.

In order to acquaint the members of the Section with the contests held in the (Pittsburgh) area, a report was given on the State Mathematics Tournament in 1955 and a Ninth Grade Algebra Contest, conducted by the Western Pennsylvania Association of the Teachers of Mathematics in five counties in April 1956. Difficulties encountered in administering such tests and tentative plans for future contests by local organizations were presented.

5. *Coordination, entrance conditions and curriculum of high schools*, by Professor J. H. Neelley, Carnegie Institute of Technology.

Mine is a voice "crying in the wilderness", the wilderness of the "wastelands" of American Education. My plea is that we rise up and cut the unimportant and repetitive from our high school mathematics courses and replace them by those topics which lead to a better understanding of college mathematics. Also, that high school mathematics teachers be told what (and in many cases how) to teach these important topics. In consequence, 84 topics desirable for prospective college students were proposed.

6. *Logistics of the computation*, by Dr. R. C. Bollinger, Westinghouse Research Laboratories.

It is the purpose of the talk to try to make clear to those having no previous familiarity with digital computing machines just what one must do to plan a computer program. The various ideas and devices necessary to the planning are illustrated by considering the programming of an actual computer routine to compute values of the Riemann Zeta function. A flow chart which expresses the organization of the computation is constructed in this, the logical analysis, stage of the planning.

7. *Arithmetic, bit by bit*, by Mrs. Aiko Hormann, Westinghouse Research Laboratories, introduced by Dr. R. C. Bollinger.

After a problem is programmed for the machine, it must be further broken down into simple arithmetic operations and then translated into machine code. This translation is comparable to the translation from one human language to another. The similarity and also the difference between the two types of translation are discussed with a few illustrations.

8. *Why doesn't it work?*, by Dr. H. C. Rice, Westinghouse Research Laboratories, introduced by Dr. M. Ostrofsky.

After a suitable emphasis on the inevitability of mistakes in programming a problem for a computer, a survey is made of the kinds of mistakes which can occur, their effect on the behavior of the computer when the program is run, and some of the techniques for finding and correcting them.

I. D. PETERS, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-fourth annual meeting of the Indiana Section of the Mathematical Association of America was held at Purdue University, Lafayette, Indiana on May 11, 1957. Professor C. F. Brumfiel of Ball State Teachers College, Chairman of the Section, presided at both morning and afternoon sessions. There were 77 in attendance including 54 members of the Association.

The following officers were elected: Chairman, Professor C. B. Gass, DePauw University; Vice-Chairman, Professor G. N. Wollan, Purdue University Center, Fort Wayne; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

Chairman Brumfiel announced that the Committee on Awards had awarded three Association medals during the year for high mathematical achievement in the Indiana Science Talent Search.

It was voted that a fall meeting be held this year on October 18 in joint session with the Mathematics Section of the Indiana Academy of Science.

Professor A. W. Tucker of Princeton University, National Lecturer for the Association, gave the invited hour address on the topic, "New Patterns in Mathematical Education."

The following short papers were presented:

1. *Mathematical instruction in Dutch high schools*, by Professor Philip Dwinger, Purdue University.

Some aspects of the teaching of mathematics in Dutch high schools were discussed, such as the subjects taught and the programs for the several final examinations. In addition, attention was called to an important report in 1954 of a committee of the Association of Mathematics Teachers, which, among other things, recommended the introduction of statistics into the program and an intuitive course in plane geometry to precede the regular course in that subject.

2. *The Council, the Association, and the Society*, by Professor C. F. Brumfiel, Ball State Teachers College.

Mathematicians recognize the need for a reorganization of high school mathematics. The content of current texts bears little relation to modern mathematics. These texts abound in gross errors, and an archaic terminology is employed. The organization which sees most clearly the need for the development of a new high school curriculum is the Council, and to effect a change the support of both the Association and the Society is needed. Changes in high school must match the changes that are occurring in colleges and universities. It is to be hoped that these three organizations will, in cooperation, persuade some of the best mathematicians to write texts on the high school level.

3. *Problems of criteria and evaluation*. by Professors M. W. Keller and C. L. Kaller, Purdue University, presented by Professor Keller.

The authors discussed some of the problems inherent in obtaining criteria for admission to and prognosis of success in graduate courses and teaching for mature individuals whose formal training was obtained approximately thirty years ago. A brief report on earlier studies was supplemented by observations based on the current experience of the authors.

4. *Classroom administration*, by Professor G. H. Graves, Valparaiso University.

Since the purposes of the class meeting are to further the student's mastery of material and to increase his ability to gain returns from study, the first essential is to see that his questions are answered, ordinarily by other students. The advantages gained by seat work, board work, and outside work handed in were contrasted. The author felt that note-taking should be discouraged since it distracts from the mental concentration required to take greatest advantage of class work.

5. *Proper cyclic elements, fine cyclic elements, and Lebesgue area*, by Professor C. J. Neugebauer, Purdue University, introduced by the Secretary.

Let Q be a unit square in E_2, and, for (T, Q) a continuous mapping from Q into E_3, let $(T, Q) = lm$, $m: Q \to M$, $l: M \to E_3$ be a monotone-light factorization. For C a proper cyclic element of M, let r_C be the monotone retraction from M onto C. For $L(T, Q)$, the Lebesgue area of (T, Q), the following cyclic additivity formula subsists:

(1) $L(T, Q) = \sum L(r_C m, Q)$, $C \subset M$ (T. Rado, *Length and Area*, Amer. Math. Col. Publ., 30, 1948).

The formula (1) has been generalized and extended by the introduction of a *fine cyclic element* of a mapping (T, J), where J is a closed finitely connected Jordan region (L. Cesari, *Fine cyclic elements of surfaces of the type* γ, Riv. Mat. Univ. Parma). If J is a 2-cell, the fine cyclic elements coincide with proper cyclic elements. In the other cases a fine cyclic element constitutes a suitable decomposition of a proper cyclic element. The above concept of a fine cyclic element can be extended to Peano spaces, and fine cyclic additivity theorems similar to those in paper by E. J. Mickle and T. Rado (*On cyclic additivity theorems*, Trans. Amer. Math. Soc., vol. 66, 1949, pp. 347–365) can be established.

6. *Critical thinking values in introductory modern mathematics*, by Sister Gertrude Marie, Marian College.

Elementary phases of number theory, group theory, the algebra of classes, and modern geometries are cited as source materials for basic experience with definition, undefined terms, relation-

ships expressed in postulates, and theorems resulting from deductive reasoning. The nature of inductive thinking is exemplified by statistical inference. Both induction and deduction are shown to fill important roles in scientific thought, while, in the symbolic formulation of logic, mathematics is identified with critical thinking in its purest interpretation.

7. *Do machines think?*, by Professor R. E. Baer, Purdue University, introduced by Professor Arthur Rosenthal.

Reference is made to papers under similar or related title by Turing, Wilkes, Oettinger, *et al.*, as well as the recent work of Hagelbarger, and Simon and Newell, and that of the Purdue Computation Laboratory. A thinking-like behavior on the part of the universal computer, barring metaphysical but not metamathematical considerations, requires emulation by the machine of both inductive and deductive behavior. The increasing degree of success of machine performance in the two directions is discussed.

8. *Order among complex numbers*, by Mr. Merl Kardatzke, student at Anderson College, introduced by Professor Gloria Olive.

This paper first orders complex numbers by a rule which does not seem to lend itself to a one-to-one correspondence between complex numbers and real numbers. In search for this relationship an analytic expression is found which can order a special set called "semi-countable complex numbers". Finally, binary numbers are used to construct a function which sets up the correspondence which is sought. In conclusion the concept of order is extended to n-dimensional space.

9. *An experiment in teaching calculus over closed-circuit television*, by Professor John Dyer-Bennet, Purdue University, introduced by Professor Arthur Rosenthal.

This paper is a brief report of an experiment conducted at Purdue University, comparing the effectiveness of teaching calculus to small groups over closed-circuit television with that of teaching large groups in lectures. Although the results have not yet been analyzed statistically, they appear to indicate that if effectiveness is measured by the sort of examination commonly used to determine grades, the two methods are about equally good.

10. *The differential*, by Professor H. L. Hunzeker, DePauw University.

The implications arising from the existence of differentials for real functions of one and of several real variables as well as for functions of a complex variable were summarized. An application for the differential of a function of a complex variable was shown in a rather direct proof of the Cauchy Integral Formula.

11. *Some additional remarks on a function defined by means of an infinite radical*, by Professor G. N. Wollan and Mr. D. M. Mesner, Purdue University Center, Fort Wayne, presented by Professor Wollan.

This paper presents some additional properties of the function $f(x)$ defined on $0<x\leqq 1$ by the relation $f(x)=\lim_{n\to\infty} f_n(x)$ where

$$f_1(x) = \sqrt{k+\alpha_1\sqrt{k}},\quad f_2(x) = \sqrt{k+\alpha_1\sqrt{k+\alpha_2\sqrt{k}}},$$
$$f_n(x) = \sqrt{k+\alpha_1\sqrt{k+\cdots+\alpha_n\sqrt{k}}} \text{ with } n \text{ nested root signs,}$$

$n=1, 2, \cdots$, and $\alpha_n=(-1)^{a_n}$ where a_n is the nth digit in the nonterminating binary representation of x. (See this MONTHLY, vol. 63, 1956, p. 614.) The author shows that when $k>2+\sqrt{2}$, although the function has a denumerably infinite set of discontinuities and is not monotone in any subinterval, it is of bounded variation; although it has a value at each point of the interval with $f(x_1)\neq f(x_2)$ when $x_1\neq x_2$, yet the set of values of the function is of measure zero. Furthermore the derivative exists almost everywhere and whenever it exists its value is zero, but there is a nondenumerable set of points (of measure zero) at which the derivative does not exist.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The spring meeting of the Minnesota Section of the Mathematical Association of America was held on May 11, 1957 at Carleton College, Northfield, Minnesota. Professor K. O. May of Carleton College presided at the morning session and Professor Walter Fleming, Chairman of the Section, presided at the afternoon session. There were 72 persons present including 55 members of the Association.

The following officers were elected for 1957–1958: Chairman, Professor O. E. Stanaitis, St. Olaf College; Members of the Executive Committee: Dr. G. E. Baxter, University of Minnesota; Professor Walter Fleming, Hamline University; Mr. F. A. Kros, Remington Rand Univac, St. Paul; Secretary-Treasurer, Professor F. L. Wolf, Carleton College.

It was resolved that the Section express its appreciation to Professor F. C. Smith of the College of St. Thomas for his long and able service as Secretary to the Minnesota Section. This appreciation was expressed by a standing round of applause.

By unanimous vote of the members present the Section resolved to sponsor in the Minnesota region the high school mathematics contest for 1958 as proposed by the National Standing Committee on High School Contests. The chair appointed the following Sectional Committee on High School Contests: G. K. Kalisch (Chairman), F. Hatfield, Donovan Johnson, Walter Fleming, F. L. Wolf.

The following papers were presented:

1. *Trigonometric values that are algebraic numbers*, by Professor K. W. Wegner, Carleton College.

Seventeen equations were presented which have integral coefficients and whose roots include the ninety values cos 1°, cos 2°, $\cdots$, cos 89°, cos 90°. Ten are of degree less than eight. These ten along with six others that were presented are the only equations with integral coefficients and of degree less than eight with roots that are cosines (or sines) of a rational number of degrees. These sixteen equations were suggested as good ones to use in the algebra and trigonometry classroom.

2. *A derivative relationship*, by Mr. R. P. Winter, College of St. Thomas.

This paper called to attention and reviewed the proof of the interesting derivative relationship $d^2x/dy^2 = -(d^2y/dx^2)(dx/dy)^3$.

3. *Numerical analysis, electronic computers, and information*, by Professor P. C. Hammer, University of Wisconsin. (By invitation.)

In the development of electronic computers some mathematicians have made remarkable contributions while others have held aloof or even actively condemned participants. While the computers cannot live up to all extravagant claims, they do provide mathematicians with means of relieving some of the tedium of their work, and future developments need the guidance of good mathematicians.

The theoretical possibility of duplicating any abstract structure in a computing machine is an indication of a better future for these machines. While application of existing mathematical theories in numerical analysis is by no means complete, more exciting is the fact that numerical analysis has opened new vistas of theoretical mathematics. Specific examples include asymmetric metric functions, spaces of metric functions, and the theory of subadditive functions. Curve fitting is a science in its infancy. The value of computers in general education was discussed.

4. *Remarks on a limit problem*, by Professor O. E. Stanaitis, St. Olaf College.

The limit problem $F(n) = \sum_{p=0}^{N} u_p(n)$, where $u_p(n)$ is a function of n and N is also a function of n that tends to infinity with n was discussed. If $\lim_{n\to\infty} u_p(n) = v_p$ exists, $|u_p(n)| \leqq w_p$, where w_p is independent of n and $p = 0, 1, \cdots$, and $\sum_0^\infty w_p$ converges; then $F(n) \to \sum_{p=0}^{\infty} v_p$ as $n \to \infty$. The theorem is closely related to the M-test for uniform convergence and, in fact, can be considered as a particular case of that test. The superiority of the theorem to the methods of our undergraduate textbooks was demonstrated by examples.

5. *The generalized coconut problem*, by Mr. R. B. Kirchner, Carleton College.

Notions from the calculus of finite differences were applied to the old problem about three sailors with a monkey who gathered a pile of coconuts. The generalization is to n men who divide the pile $n+1$ times, the last division being a division of the remaining coconuts. At each division an arbitrary predetermined number is thrown to the monkey. The sailor then takes one nth of what remains. A solution consists of finding the smallest possible initial amount so that each man receives an integral number of coconuts. A method for solution was given, and it was shown how compact formulas could be derived in special cases.

6. *Completeness and Parseval's equation*, by Professor John M. H. Olmsted, University of Minnesota.

An orthonormal sequence in the space L^2, for an interval $[a, b]$, is called complete in case it is maximal. Equivalent formulations are well known, one being Parseval's equation. If L^2 is replaced by the space RI of Riemann integrable functions on $[a, b]$, these formulations fall into two groups, Parseval's equation implying completeness. Proof that completeness no longer implies Parseval's equation is achieved by extending the space RI to L^2 and showing that the Riemann-integrable functions lying in the orthonormal complement Π of a non-Riemann function are dense in Π.

7. *A minimization problem over a finite set*, by Mr. William J. Hardell, Remington-Rand Univac, St. Paul, Minnesota.

A stepwise procedure for assigning nodes of a finite connected graph G to a finite set of lattice points L of E_n defines a mapping F of G into L. Choice of node and coordinate at each step is such that the maximum length of those edges with one end point placed and the other the node to be placed is a minimum. In the dynamic programming sense F is optimal. Attempts to obtain the mapping of G such that the maximum edge length over all mappings of G is a minimum, by means other than considering all mappings, have been unsuccessful.

F. L. WOLF, *Secretary*

THE MAY MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-seventh regular meeting of the Southern California Section of the Mathematical Association of America was held at San Diego State College, San Diego, California, on May 11, 1957. Professor H. F. Bohnenblust, Chairman of the Section, presided. The attendance was 80, including 63 members of the Association.

At the business meeting it was announced that the mail ballot had resulted in the election of the following officers for the next academic year: Chairman, Professor P. B. Johnson, Occidental College; Vice-Chairman, Professor P. J. Kelly, Santa Barbara College, University of California; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. These elected officers appointed the following Program Committee: Mr. Clark Lay (Chairman), Pasadena City College; Dr. C. J. A. Halberg, Jr., University of California, Riverside; Professor L. J. Paige, University of California, Los Angeles; and Professor A. L. Whiteman, University of Southern California.

The following program was presented:

1. *Some remarks on Menger's calculus*, by Dr. D. H. Potts, Naval Electronics Laboratory, San Diego.

The three salient features of Menger's presentation of calculus are: (1) his notational innovations, (2) his discussion of *variable* and *function*, and (3) his approach to the problem of the application of calculus to science. These topics are discussed and their basic pedagogic implications brought forth. In particular, the need for notations which clarify concepts yet automatize procedure is stressed. It is becoming increasingly apparent that the student must be taught *mathematics* and not merely mathematical techniques. Yet in introducing rigor we must avoid "rigor mortis". Menger's work is a gigantic step in this direction.

2. *Theory of games and some models of warfare*, by Dr. Melvin Dresher, Rand Corporation, Santa Monica, introduced by the Secretary.

As examples of zero-sum two-person infinite games—*i.e.*, each player has a continuum of strategies—four warfare models were presented:

(1) Defense of targets against attack. Within this model it is shown that only the most valuable targets should be defended. The attacker, however, should use a mixed strategy—concentrate his attack on some one of the defended targets chosen at random subject to a given probability distribution.

(2) Allocation of resources in time. With this model it is shown that during the early stages of a campaign the weaker side should bluff and the stronger side has an optimal pure strategy.

(3) Duels. These analyze the timing of decisions.

(4) Reconnaissance. This evaluates the effect of information on the payoff.

3. *Sets of integers*, by Professor Bodo Volkmann, University of California at Los Angeles, introduced by Professor M. R. Hestenes.

If A and B are infinite sets of nonnegative integers, the sum set $A+B$ is defined as the set of all sums $a+b$, $a\in A$, $b\in B$. A brief survey of some recent results on such sum sets is given, in particular with regard to bases (*i.e.*, sets B for which some "multiple" $hB=B+B+\cdots+B$ contains all nonnegative integers), asymptotic densities of sum sets, and essential components (*i.e.*, sets B satisfying $D(A+B)>D(A)$ for any set A with $0<D(A)<1$, D denoting Schnirelmann density). Some unsolved problems are mentioned, including the following, due to A. Stöhr: Let C_h be the class of bases of order h. For $B\in C_h$, let σ_1, σ_2, and σ_3 denote the $\underline{\lim}$, $\overline{\lim}$, and l.u.b., of $B(n)\cdot n^{-1/h}$, respectively. What are the greatest lower bounds for σ_1, σ_2, and σ_3 as B runs through the class C_h?

4. *Mathematics is changing*, by Professor P. H. Daus, University of California, Los Angeles.

The speaker reviewed the changes that have taken place in the attitude towards mathematics and the contents of various multi-track courses given in the High Schools of California, since the Southern California Section was formed. He also called attention to the changes in the first two years of College Mathematics courses that have already taken place, and the changes that are seriously being considered and are sure to come—for good or ill.

5. *Analytic solutions of interface problems*, by Dr. W. C. Sangren, General Atomic Corporation, San Diego.

By an interface problem is meant a classical boundary value problem with additional internal requirements, or interface conditions, resulting from two or more different internal materials. Many of these problems can be solved analytically by the conventional techniques of the Laplace transform or separation of variables. It is even possible to solve certain problems where the internal boundaries lie along two or more coordinate lines.

6. *Comma-free codes*, by Dr. Basil Gordon, California Institute of Technology.

Let A be an alphabet of n letters, and let S be the set of all k-letter words $(a_1\, a_2 \cdots a_k)$, $a_i\in A$. A subset $D\subseteq S$ is called a comma-free dictionary if, whenever $(a_1 \cdots a_k)$, $(b_1 \cdots b_k)$ are in D, the "overlaps" $(a_2 \cdots b_1), \cdots (a_k \cdots b_{k-1})$ are not in D. Let $W_k(n)$ be the greatest number of words such a dictionary can have. The value of $W_k(n)$ is determined for $k=1, 2, 3, 5, 7, 9, 11, 13, 15, 17$, and arbitrary n. It is conjectured that for odd k, $W_k(n)=k^{-1}\sum_{d|k}\mu(d)n^{k/d}$, where $\mu(d)$ is the Möbius function. This expression is shown to be an upper bound, but is never attained for k even and $n>3^{k/2}$.

7. *A course in actuarial science*, by Professor Geoffrey Crofts, Occidental College.

An actuary is a scientist and a professional man. As a scientist, he is interested in discovering the pattern of death rates or other rates of decrement. He sets up mathematical models of these rates and draws implications from them. There is now, as always, a need for better models. As a professional man he uses the findings of science and the implications of mathematics to give answers and judgments on matters concerning the operation of pension and insurance funds and related fields.

P. H. DAUS, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The thirteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Skidmore College, Saratoga Springs, New York, on May 4, 1957. The Chairman of the Section, Professor A. J. Coleman of the University of Toronto, presided at the morning session, and the Vice-Chairman, Professor E. E. Haskins of Clarkson College of Technology presided at the afternoon session. There were 90 persons in attendance, including 56 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor E. E. Haskins, Clarkson College of Technology; Vice-Chairman, Professor Caroline A. Lester, College for Teachers at Albany, State University of New York; Secretary, Professor N. G. Gunderson, University of Rochester.

The Secretary reported that on September 25, 1956, Col. Bessell resigned as Vice-Chairman, and that the Executive Committee then filled the vacancy by choosing Professor E. E. Haskins of Clarkson College of Technology to be Vice-Chairman.

Professor Edith R. Schneckenburger presented the report of the Committee on Mathematics Contests. A resolution was passed to the effect that the Section continue to co-sponsor with the Buffalo Public Schools a contest for the western part of the state, and that the extension of the contest to other areas be investigated.

The appointment of two committees was authorized, one on the strengthening of mathematics in the section, and the other, composed of department chairmen present, to study the specific requirements that have been added by the New York State Education Department for accreditation in the teaching of mathematics in New York State.

The program was as follows:

1. *The training and development of the professional mathematician*, a panel discussion with Professor E. E. Haskins, Clarkson College of Technology, as moderator.

The members of the panel were Mr. W. E. Andrus, Jr., International Business Machines Corporation, Endicott, N. Y., representing industry; Mr. H. E. Webb, Jr., Rome Air Development Center, Rome, N. Y., representing government service; Mr. Frank Hawthorne, New York State Education Department, representing public school teaching; and Professor Paul Olum, Cornell University, representing graduate study.

Mr. Andrus stated that a thorough course in advanced calculus is an absolute "must", and explained how topology is rapidly becoming of greater importance in engineering applications. Mr. Webb spoke of the factors involved in the formulation of the mathematical model representing the problem being dealt with, and especially the importance of probability theory. Mr. Hawthorne reviewed the N. Y. State requirements for high school certification, together with the proposed changes. He was, however, more concerned about the attitude of the teacher toward his subject and teaching, than with the specific preparatory courses. The young teacher should realize that mathematics is a living entity with esthetic qualities of a high order. Professor Olum thought that the undergraduate should have a solid two year course in advanced calculus (analysis), a one year course in modern algebra, and at least an introduction to modern geometry, although he thought that a broad foundation course in modern geometry was probably not very easy to find.

2. *The College Entrance Examination Board's advanced placement program,* by Professor C. R. Keller, College Entrance Examination Board and Department of History, Williams College. (By invitation.)

Under the College Board's Advanced Placement Program able and ambitious students take advanced, college-level work in secondary school followed by advanced placement examinations. These students are then considered by colleges and universities for credit and advanced placement, and many take sophomore courses in college.

Students are challenged in both school and college, school curricula are being thoughtfully revised and standards are being raised, school teachers are being excited by new experiences, college curricula are developing a much needed flexibility, schools and colleges are cooperating in a heartening fashion. For most of these students the Program means educational enrichment rather than acceleration, although acceleration is possible for a few.

In mathematics, the most popular advanced placement subject, advanced placement work means analytic geometry and calculus in the senior year of high school.

3. *The mathematics of "operations research",* by Dr. A. W. Jones, Bell Telephone Laboratories, New York, New York. (By invitation.)

A brief résumé of the history and nature of operations research was given. Differential and difference equations arising in inventory problems, matrix theory and topology involved in competition models, linear programming, quantity scheduling, probability theory in queuing, renewal, and maintenance problems, symbolic logic in routing and sequence scheduling were given. A new elementary derivation of a linear programming algorithm and a use of complementary solutions in transportation problems were shown to demonstrate the availability of interesting and challenging problems from Operations Research for use in the standard undergraduate and graduate mathematics courses.

4. *A proof of the isoperimetric property of the circle,* by Professor A. G. Davis, Clarkson College of Technology.

The speaker gave a brief review of the history of the isoperimetric problem. The history of integral geometry and the relevant definitions and results were presented. These results were then used to prove the classical isoperimetric inequality that $L^2-4\pi A \geqq 0$. Then it was proved that the equality holds only in the case of a circle. This proof follows the proof given by L. A. Santalo and is of interest because the methods of integral geometry involved use nothing more than elementary calculus.

5. *The matrix equation $X^2=I$ over a finite field,* by Professor J. H. Hodges, University of Buffalo.

Let $GF(q)$, $q=p^f$ where p is prime, denote the finite field of q elements. Consider the problem of finding the number $N(m, x^2-1)$ of $m\times m$ matrices X with elements in $GF(q)$ which satisfy $X^2-I=0$. It can be shown that X is a solution of the equation if and only if X is similar to one of a certain set of canonical matrices. For $p>2$, the canonical matrices are $J_t=\text{diag}\ (I_t,\ -I_{m-t})$ for $0\leqq t\leqq m$. For $p=2$, they are $H_t=\text{diag}\ (I_{m-2t}, K_1, \cdots, K_t)$ for $0\leqq 2t\leqq m$ and $K_i=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ for all i. The number $N(m, x^2-1)$ is found by finding the number of distinct X's similar to each canonical matrix and summing over the set of canonical matrices.

6. *Vector methods in elementary analytic geometry,* by Professor Viktors Linis, University of Ottawa.

The author contends that the traditional approach to the analytic geometry with its heavy emphasis on Cartesian coordinate methods and on treatment of conics is of little value from the mathematical and pedagogical point of view. The vector method which introduces such basic concepts as line, parallelism, order, linear dependence, orientation, signed areas and volumes, has the advantages of simplicity, flexibility and applicability. Axiomatic and informal treatments can

be easily blended to suit the needs of students and the temperament of the instructor. The introduced concepts permit natural extension into the theory of linear equations, matrices, transformations, trigonometry, theory of curves and mechanics. The method offers good training in useful formalisms, and at the same time there is no lack of supporting visual interpretations. A course based on the outlined ideas has been prepared by Professor O. Biberstein and has been taught successfully at the University of Ottawa.

7. *Strategy and tactics in mathematics*, by Professor R. B. Davis, Syracuse University.

There are two possible approaches to a mathematical problem; one, by classifying problems and learning specific procedures for their solutions (well-illustrated by many elementary differential equations courses); second, by locating clues within the problem which indicate an appropriate line of attack. This second method, discussed notably by Pólya, involves seeing a problem in such a way that it itself implies a method of attack. What is good or bad about the problem? What is new, or else familiar? One utilizes "approximate "or rough thinking to outline a strategy, then implements this strategy by developing requisite tactical details and devices.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-fifth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Wisconsin State College, Whitewater, Wisconsin, on May 11, 1957 with Mr. C. J. Vanderlin, Chairman of the Section, presiding. There were 72 persons in attendance, including 41 members.

The following officers were elected for 1957–58; Chairman, Professor R. D. Wagner, University of Wisconsin; Vice-Chairman, Professor John Finch, Beloit College; Secretary-Treasurer, Sister Mary Felice, Mount Mary College.

After a short address of welcome, the following papers were presented:

1. *New patterns in geometry*, by Professor R. H. Bruck, University of Wisconsin.

There is evidence that geometry has almost disappeared from the undergraduate curriculum. Nevertheless, a recent revival in geometric research offers hope of a return to geometry during the next decade. Towards this end, much experimenting along expository lines will be necessary. The present paper represents such an experiment; namely, a partial account of the theory of finite euclidean and projective planes with emphasis on non-standard pictorial representations.

2. *Topology of a line*, by Professor C. E. Burgess, University of Utah, Visiting Lecturer at the University of Wisconsin.

This address consisted of a brief expository discussion of the topological characterization of a line as a connected, separable, linearly ordered space which has neither a first point nor a last point.

3. *Polynomials and Baire's theorem*, by Professor R. P. Boas, Jr., Northwestern University.

The speaker proved the following theorem, discovered by Corominas and Sunyer Balaguer: *If a function $f(x)$ defined on a real interval has derivatives of all orders, and if for each x there is a derivative of some order which vanishes at x, then $f(x)$ is a polynomial.* The proof depends on Baire's category theorem, for which the speaker outlined the necessary preliminaries.

4. *Contest Committee report*, by Professor R. D. Wagner, University of Wisconsin.

Over 1,000 high school students from 132 of Wisconsin's 535 secondary schools took the contest examination which was held on April 6 in twenty cities throughout the state. The top ranking 10 per cent of the students received recognition in the form of certificates, pins, and small cash awards. A set of *World of Mathematics* was awarded to the schools having the 20 top ranking students.

The contest examination was open to any high school students who had completed one year of elementary algebra and was at least taking his second semester of plane geometry. The problems were in part of the short answer type. Over 60 per cent of the examination consisted of essay type questions calling for considerable understanding and ingenuity.

5. *A plea for balance*, by Mr. Arthur Adkins, Wisconsin State Department of Public Instruction, Madison, Wisconsin, introduced by the Secretary.

While recognizing the shortages of trained personnel in mathematics, the sciences, and other technical fields, and admitting the obligation of schools and colleges in helping to meet them, this paper makes a plea for perspective in responding to the pressures on the schools. Feeling that there is a danger, and that there are signs that schools may be yielding to this pressure for more mathematics and science in the curriculum at the expense of other areas and other needs that are also important, the paper asks for balance among specialized fields, and between specialized training and general education, in the public school curriculum.

6. *The new entrance requirements in mathematics of the College of Engineering*, by Professor H. A. Peterson, University of Wisconsin, introduced by the Secretary.

The United States is facing a critical shortage of scientists and engineers. This shortage is not one of *quantity* only, but *quality* as well. It is important that the gifted student be stimulated early and made aware of the opportunities for him in science and engineering. Educators are becoming increasingly aware of the problem and are taking steps to meet the challenge presented. Mindful of the ever increasingly complex problems being encountered by scientists and engineers requiring greater proficiency in the use of mathematics, the College of Engineering of the University of Wisconsin has recently raised the college entrance requirements in mathematics.

7. *Correspondence study courses in relation to the new entrance requirement in mathematics of the College of Engineering*, by Professor H. P. Evans, University of Wisconsin.

The new entrance requirement in mathematics of the University College of Engineering, becoming effective in 1959, includes trigonometry, college algebra, and either solid or analytic geometry. Although most large high schools will adjust to this change in stride, it is anticipated that many small high schools will be unable to offer the necessary courses or to secure adequate staff. Correspondence courses may alleviate these difficulties under a plan which is currently in use and may be expanded. The student carries a correspondence course as part of his high school program and the school system pays the cost of instruction.

8. *Implications for Wisconsin colleges of the 4-yr. mathematics requirement of the University of Wisconsin College of Engineering*, by Dr. L. F. Wahlstrom, Wisconsin State College, Eau Claire.

Seven state and private colleges in Wisconsin brought out the fact that the four-year requirement in high school mathematics which has been adopted by the University of Wisconsin College of Engineering will (1) cause the colleges to re-evaluate their courses and arrange different sequences beginning with calculus and analytic geometry; (2) create a demand for additional upper-level courses in mathematics in the colleges; (3) require better preparation of prospective high school mathematics teachers to teach the mathematics; and (4) probably result in larger enrollments in the upper-level courses.

9. *Implications of the Engineering School's requirement for high schools and report of State Curriculum Committee*, by Mr. W. B. White, North High School, Sheboygan, introduced by the Secretary.

The four-year mathematics requirement for entrance into the University of Wisconsin's School of Engineering, the nationwide emphasis on science and mathematics, and increasing enrollments have brought problems to the state's high schools. Mathematics classes will be of lower aver-

age ability, less experienced teachers may be utilized to teach the additional classes, and there is difficulty in selecting candidates for four years of mathematics at the eighth grade level. A bulletin being prepared by the Statewide Mathematics Curriculum Committee may help solve these problems and will provide suggestions on course content.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The tenth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at State College of Washington, Pullman, Washington on June 14, 1957 in conjunction with the 536th meeting of the American Mathematical Society. Professor D. C. Murdoch, Chairman of the Section, presided over the meetings. There were 95 persons in attendance, including 65 members of the Association.

Following a joint dinner meeting with the American Mathematical Society a business meeting was held. The following officers were elected: Chairman, Professor S. G. Hacker, State College of Washington; Vice-Chairman, Professor K. A. Bush, University of Idaho; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon.

After some discussion of the High School Mathematics Contest which the Association plans to sponsor in 1958, the secretary was instructed to report to the national secretary that the contest will be offered in 1958 in British Columbia, Oregon and Washington as in past years, and that there is a possibility that the contest may be offered in Idaho and in Montana. Professor Bush of Idaho and Professor Hurst of Montana State College agreed to investigate the possibility of offering the contest in their states. Professor Murdoch in British Columbia, Professor Chapman at the University of Washington and Professor Ghent in Oregon plan to operate the contests in their respective states as in recent years. Professor Keeping reported that the contest will not be offered in Alberta since the University of Alberta is already sponsoring a provincial contest with the assistance of the Canadian Mathematical Congress.

The afternoon session consisted of an invited address delivered by Professor R. H. Bruck of the University of Wisconsin and a symposium on computing machines. Professor Ostrom introduced Professor Bruck. Professor Lonseth acted as moderator for the symposium.

1. *New patterns in geometry*, by Professor R. H. Bruck, University of Wisconsin. (By invitation.)

See The May Meeting of the Wisconsin Section, abstract 1, p. 627.

2. *Symposium on computing machines*, Moderator, Professor Arvid Lonseth, Oregon State College.

Speakers:

(1) Professor T. E. Hull, University of British Columbia, *The University of British Columbia Computing Center.*

After making plans for more than a year, the University of British Columbia established a computing centre in March, 1957. An Alwac III-E computer is maintained by the Electrical Engineering department. A small staff in the computing centre gives programming courses, develops basic programs and helps users of the equipment. Research projects are being initiated. Besides the need for a more widespread knowledge of computer techniques, there is a need for the modification of existing undergraduate courses and for the development of research in numerical analysis.

(2) Dr. R. E. Gaskell, Boeing Aircraft Company, *Large Scale Industrial Computers and the Universities.*

There is no longer any argument about the fact that the high-speed digital computer has "arrived". The acceptance of the computer as a scientific and business tool is a sign of maturity. Perhaps, then, along with selling the value of the computer, we should be educating potential

consumers of computation in its skillful and economic use. Included in this consumer education might be (1) a notion of the cost of computation, (2) the need for careful preliminary analysis and proper choice of the mathematical model, and (3) the need for complete usage of the computer, for analysis and interpretation as well as for calculation.

(3) Professor O. W. Rechard, State College of Washington, *The computers and the curriculum.*

In this paper the impact of large-scale digital computers on the mathematics curriculum is examined. New courses are called for in programming and coding for digital computers and in the logical theory of automata. Standard courses in numerical analysis and applied mathematics should be adjusted to reflect the changing emphasis in these subjects. Finally, with most of our undergraduate mathematics majors and many of our graduate students taking employment involving the use of digital computers there should be more emphasis on numerical methods in the standard algebra and analysis courses. A computer can make this emphasis feasible and can also serve as an instructional tool to help illuminate certain basic mathematical concepts.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 30–31, 1958.

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

The following is a list of the Sections of the Association with dates of future meetings, so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, DePauw University, Greencastle, October 18, 1957.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA, State Teachers College, Mankato, October 5, 1957.

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA, Oklahoma City University, October 25, 1957.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, Haverford College, Haverford, Pennsylvania, November 30, 1957.

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

TO LESTER R. FORD
ON HIS SEVENTIETH BIRTHDAY

The Sixth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

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PART II

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Number 6
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

OCTOBER 1957

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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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DEDICATION

October 25, 1956, was the seventieth birthday of Lester R. Ford, President of the Mathematical Association of America (1947–1948), editor of the AMERICAN MATHEMATICAL MONTHLY (1942–1948). In the spring of 1956, some of the numerous friends and former students of Dr. Ford decided to dedicate papers to him on this occasion as tokens of their appreciation and friendship. The collection of manuscripts was presented to Dr. Ford on his birthday by Drs. W. L. Duren, Jr., Karl Menger, and G. T. Whyburn. Some of the papers are related to the fields of Ford's major interests: complex functions, interpolation, differential equations, and numerical analysis. Other papers were inspired by remarks that Ford made in talks and lectures. Still others have to do with Ford's former activity as editor of the MONTHLY, where he started the series of papers with the titles "What is . . . ?"

In this number of the Slaught papers, the papers dedicated to Dr. Ford are arranged alphabetically according to the authors' names. All authors are united in the feeling expressed in the final remark contained in the paper of G. T. Whyburn.

KARL MENGER
GORDON PALL

DISTANCE GEOMETRY OF METRIC ARCS*

WILLIAM R. ABEL and LEONARD M. BLUMENTHAL,
University of Missouri

1. Introduction. This paper, the first of a series, investigates metric arcs that possess at each point a finite Menger curvature. The properties of this class of arcs that are established here center about those finite subsets of an arc that form *lattices* and *homogeneous* chains. It is shown, for example, that for all but a finite number of positive integers n, each n-lattice L_n of an arc A (with finite Menger curvature at each point) is (i) unique, and (ii) a homogeneous $\lambda(n)$-chain, where $\lambda(n)$ is the distance of two consecutive points of L_n. The concept of *metrically monotone arcs* is introduced in Section 5, and it is seen that each metric arc with everywhere finite Menger curvature is the sum of a finite number of such arcs.

Other theorems established are (1) the length of *any* metric arc (rectifiable or not) is the *limit* of the lengths of inscribed n-lattices (a result that is not valid for semimetric arcs) and (2) every metric ptolemaic arc that is a geodesic (that is, *locally*, a metric segment) is a metric segment. These two results fill lacunae in the proof of the fundamental theorem that characterizes metric segments as arcs with everywhere vanishing Menger curvature which Schoenberg obtained by extending Menger's procedure for euclidean arcs to a more general environment. It is indicated, finally, how our study leads to a new approach to that characterization theorem.†

2. Definitions and preliminary results. A metric arc is a homeomorph of a line segment, with a distance defined that satisfies the postulates for a metric space. Denoting such an arc, with initial point a and terminal point b by $A(a, b)$, a finite subset $P=(p_1, p_2, \cdots, p_m)$ is *normally ordered* provided these points are encountered in the order of their subscripts as $A(a, b)$, is traversed from a to b. The length $L[A]$ is the least upper bound of the numbers $L[P]=p_1p_2+p_2p_3 + \cdots +p_{m-1}p_m$ for all normally ordered subsets $P=(p_1, p_2, \cdots, p_m)$ of $A(a, b)$.

A normally ordered subset P of $A(a, b)$ is a homogeneous ϵ-chain, $\epsilon>0$, provided (1) $ap_1<\epsilon$, $p_mb<\epsilon$, (2) $p_ip_j=\epsilon$ for $|i-j|=1$, $(i, j=1, 2, \cdots, m)$ and (3) $p_ip_j\geqq\epsilon$, for $|i-j|>1$, $(i, j=1, 2, \cdots, m)$.

It is easily proved that for each positive ϵ, each metric arc contains a homogeneous ϵ-chain.‡

A normally ordered $(n+1)$-tuple $p_0, p_1, \cdots, p_n$ of $A(a, b)$ forms an n-lattice provided $p_0=a$, $p_n=b$, and $p_0p_1=p_1p_2= \cdots =p_{n-1}p_n$.

An elegant proof by Schoenberg shows that for every positive integer n, every continuous curve with distinct endpoints, in any metric space (indeed in

* Presented to the American Mathematical Society, October 27, 1956.

† For references to the literature, as well as to those concepts of distance geometry not defined in this paper, see L. M. Blumenthal, Theory and Applications of Distance Geometry, The Clarendon Press, Oxford, 1953, referred to in the following footnotes as Distance Geometry.

‡ Distance Geometry, pp. 60–61.

any semimetric space with continuous distance function) contains an n-lattice.† We shall refer to this important result as the *n-lattice theorem*. The distance p_ip_{i+1} of consecutive points p_i, p_{i+1} $(i=0, 1, \cdots, n-1)$ of an n-lattice is denoted by $\lambda(n)$.

The distinction between an n-lattice and a homogeneous chain is clear. Apart from the fact that the initial and terminal points of a homogeneous chain need not coincide with the initial and terminal points of the arc (while this is required for an n-lattice) it is important to observe that condition (3) for a homogeneous chain has no counterpart in an n-lattice. It should also be remarked that neither homogeneous ϵ-chains nor n-lattices are necessarily uniquely determined by the given ϵ and n, respectively.

If $p\in A(a, b)$, the metrization of the curvature $K(p)$ of the arc at p, due to Menger, is expressed by

$$K(p) = \lim_{q,r,s\to p} \frac{\{-D(q, r, s)\}^{1/2}}{qr\cdot rs\cdot qs}, \qquad q \neq r \neq s \neq q,$$

where $q, r, s, \in A(a, b)$ and

$$D(q, r, s) = (qr + rs + qs)(qr + rs - qs)(qr - rs + qs)(qr - rs - qs).‡$$

The expression whose limit is taken is called the *curvature of the point triple* q, r, s and is denoted by $K(q, r, s)$. It is seen that $K(q, r, s)$ is the reciprocal of the radius (perhaps infinite) of a circle (in the euclidean plane) that contains three points q', r', s' with $qr=q'r'$, $rs=r's'$, $qs=q's'$. Clearly, $K(q, r, s)=0$ if and only if q', r', s' are on a line.

3. Lattices and arc length. The principal result of this section exhibits arc length as the limit of the lengths of n-lattices inscribed in the arc. This extension to metric arcs of a classical property of euclidean arcs is very useful in our study. It is observed that the result is not valid in general semimetric spaces with continuous distance function.§

LEMMA 3.1. *If A is any metric arc and $\lambda(n)$ is the side of any n-lattice of A, then* $\lim_{n\to\infty} \lambda(n) = 0$.

Proof. Let f denote a homeomorphism from the unit interval $I=[0, 1]$ to the arc A. Since f is uniformly continuous in I, there corresponds to each positive ϵ a positive δ such that if $p, q\in I$ and $pq<\delta$, then $f(p)f(q)<\epsilon$. Now if N is a positive integer such that $N\delta>1$, then for every $n>N$, each n points of I has at

† Distance Geometry, pp. 73–74.

‡ Distance Geometry, p. 75.

§ Consider, for example, the semimetric space (an arc) with continuous distance function obtained by squaring the euclidean metric of the segment $[0, 1]$. For every positive integer n, the points $0, 1/n, 2/n, \cdots, (n-1)/n, 1$ form an n-lattice of the arc, but $\lim_{n\to\infty} n\cdot\lambda(n) = \lim_{n\to\infty} 1/n = 0$. Note that $\lim_{n\to\infty} n^2\cdot\lambda(n) = 1 = \text{l.u.b.}\, L[P]$, for all finite normally ordered subsets P of the arc; that is, $\lim_{n\to\infty} n^2\cdot\lambda(n) = L(\text{arc})$. This suggests associating an integer k with a semimetric space provided $L[A] = \lim_{n\to\infty} n^k\cdot\lambda(n)$ for each arc A of the space.

least one consecutive pair with distance less than δ. Consequently, the side of any n-lattice of A, with $n > N$, is less than ϵ and so $\lim_{n\to\infty} \lambda(n) = 0$.

LEMMA 3.2. *Let C denote a constant and Q any normally ordered subset of a metric arc $A(a, b)$ with $L[Q] > C$. There exists a positive integer N such that for every $n > N$, $n\cdot\lambda(n) > C$, where $\lambda(n)$ is the side of any n-lattice of $A(a, b)$.*

Proof. Let $Q = \{q_0, q_1, \cdots, q_m\}$ and consider the point q_i, where i is an arbitrary but fixed one of the indices $0, 1, \cdots, m$. We show that corresponding to each $\sigma > 0$ there is a positive integer N_i^* such that for every $n > N_i^*$ each n-lattice of $A(a, b)$ has a point on $A(q_i, b)$ (the sub-arc of $A(a, b)$ with endpoints q_i, b) with distance from q_i less than σ. If $i = 0$ and $q_0 = a$ then a is itself such a point, while b serves in case $i = m$ and $q_m = b$. Supposing that q_i is neither a nor b, consider the two subsets $A(a, q_i)$ and $A(q_i, b) - U(q_i; \sigma)\cdot A(q_i, b)$ of $A(a, b)$, where $U(q_i; \sigma)$ is the (open) spherical neighborhood of q_i with radius σ. If $A(q_i, b) - U(q_i; \sigma)\cdot A(q_i, b) = 0$, then b again serves as the desired point; in the contrary case, the two closed and compact sets have a positive distance δ.

Since $\lim_{n\to\infty} \lambda(n) = 0$, there exists a positive integer N_i^* such that $\lambda(n) < \delta$ whenever $n > N_i^*$. Clearly, for each n-lattice L_n, each of the sets $L_n\cdot A(a, q_i)$, $L_n\cdot[A(q_i, b) - U(q_i; \sigma)A(q_i, b)]$ is non-null, and consequently for every $n > N_i^*$, every n-lattice of $A(a, b)$ contains a point with the desired property. In a similar manner, to each $\sigma > 0$ there is a positive integer N_i^{**} such that every n-lattice with $n > N_i^{**}$ has a point on $A(a, q_i)$ with distance from q_i less than σ.

Now if $n > N(\sigma) = \max[N_0^*, N_0^{**}, N_1^*, N_1^{**}, \cdots, N_m^*, N_m^{**}]$, then every n-lattice L_n has points on each side of q_i with distances from q_i less than σ. Let the first such point on $A(q_i, b)$ be labelled $q_{j_i}^*$, and the last such point on $A(a, q_{i+1})$ be denoted by $q_{j_{i+1}}^{**}$ ($q_{j_i}^*$ is the first point of L_n to the right of q_i, with distance from q_i less than σ, and $q_{j_{i+1}}^{**}$ is the last point of L_n to the left of q_{i+1}, with distance from q_{i+1} less than σ). Then $q_iq_{i+1} \leqq q_iq_{j_i}^* +$(length of L_n from $q_{j_i}^*$ to $q_{j_{i+1}}^{**}$) $+ q_{j_{i+1}}^{**}q_{i+1}$; that is,

$$q_iq_{i+1} < 2\sigma + (\text{length of } L_n \text{ from } \overset{*}{q_{j_i}} \text{ to } \overset{**}{q_{j_{i+1}}}).$$

It follows that for every $n > N(\sigma)$, and each n-lattice L_n

$$n\cdot\lambda(n) + 2m\sigma > L[Q] > C.$$

Take $\sigma = (L[Q] - C)/2m$ and put $N = N(\sigma)$. Then for $n > N$, $n\cdot\lambda(n) + (L[Q] - C) > L[Q]$; that is, $n\cdot\lambda(n) > C$, and the lemma is proved.

THEOREM 3.1. *If A is any metric arc, then $L[A] = \lim_{n\to\infty} n\cdot\lambda(n)$.*

Proof. If $L[A]$ is finite then to each $\epsilon > 0$ there is a normally ordered subset Q of A such that $L[Q] > L[A] - \epsilon$. Letting $L[A] - \epsilon$ be the constant C of the above lemma, there exists a positive integer N such that for every n-lattice with $n > N$, $L[A] \geqq n\cdot\lambda(n) > L[A] - \epsilon$; that is, $L[A] = \lim_{n\to\infty} n\cdot\lambda(n)$.

If $L[A]$ is infinite, for every positive constant C there exists a normally

ordered subset Q of A with $L[Q]>C$. Then by Lemma 3.2, there exists a positive integer N such that for $n>N$, the length of every n-lattice of A exceeds C. Hence $\lim_{n\to\infty} n\cdot\lambda(n)=\infty=L[A]$.

Remark. It may be of interest to note that if A is a metric arc and Q is a finite normally ordered subset of A, there might not exist any n-lattice of A such that $n\cdot\lambda(n)\geqq L[Q]$. For let a, b, c be three non-collinear points of the euclidean plane with ac/bc irrational. Then no n-lattice of the arc $A(a, b)$ $=$seg $[a, c]+$seg $[c, b]$ contains the point c and consequently for every n-lattice of $A(a, b)$, $n\cdot\lambda(n)<L[Q]=ac+cb$, where $Q=\{q_0, q_1, q_2\}$, $q_0=a$, $q_1=c$, $q_2=b$.

4. Lattices and homogeneous chains. The principal results of this section, as well as Section 5, are established with the aid of the following lemma.

LEMMA 4.1. *If a metric arc A has finite Menger curvature at each point, then a positive number ϵ exists such that for every point p of A, r, s, $t\in U(p;\epsilon)\cdot A$, $rs=st$ imply $rs<rt$.*

Proof. If the lemma is false then for each positive integer n there exists a point p_n of A and three points r_n, s_n, t_n of $U(p;\epsilon_n)\cdot A$ with $r_ns_n=s_nt_n\geqq r_nt_n$, where $\{\epsilon_n\}$, $(n=1, 2, \cdots)$, is an infinite sequence with limit zero. Since A is closed and compact, there is a point p of A such that $\lim_{i\to\infty} p_{n_i}=p$, where $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$. Clearly $\lim_{i\to\infty} r_{n_i}=\lim_{i\to\infty} s_{n_i}=\lim_{i\to\infty} t_{n_i}=p$.

Now

$$K^2(r_{n_i}, s_{n_i}, t_{n_i}) = [4(r_{n_i}s_{n_i})^2 - (r_{n_i}t_{n_i})^2]/(r_{n_i}s_{n_i})^4 \geqq 3/(r_{n_i}s_{n_i})^2,$$

and consequently

$$K^2(p) = \lim_{i\to\infty} K^2(r_{n_i}, s_{n_i}, t_{n_i}) = \infty,$$

in contradiction to the assumed finiteness of the curvature of A at each point.

THEOREM 4.1. *If a metric arc $A(a, b)$ has finite Menger curvature at each point, then a positive integer N exists such that for $n>N$ every n-lattice of $A(a, b)$ is a homogeneous $\lambda(n)$-chain (that is, "almost all" n-lattices of the arc are homogeneous chains).*

Proof. According to Lemma 3.1, a positive integer N exists such that for $n>N$, $\lambda(n)<\min(\epsilon, ab)$, where ϵ is the epsilon of the preceding lemma, and $\lambda(n)$ is the side of any n-lattice $L_n=(p_0^n, p_1^n, \cdots, p_n^n)$ of $A(a, b)$. To prove that L_n is a homogeneous $\lambda(n)$-chain, it suffices to show that $p_i^np_{i+j}^n\geqq\lambda(n)$, $(i, j=0, 1, \cdots, n;\ 0<i+j\leqq n)$.

Suppose two distinct points p_i^n, p_{i+j}^n of L_n exist such that $p_i^np_{i+j}^n<\lambda(n)$. Then $j>1$ and the sub-arc $A(p_{i+1}^n, b)$ contains p_{i+j}^n. Let $f(p_i^n)$ denote a foot of p_i^n on this sub-arc.

Case 1. *The point $f(p_i^n)$ is distinct from b.* Since $p_i^np_{i+1}^n=\lambda(n)>p_i^np_{i+j}^n$, then $f(p_i^n)\neq p_{i+1}^n$ and $p_i^nf(p_i^n)\leqq p_i^np_{i+1}^n=\lambda(n)<\min(\epsilon, ab)$. For each positive integer k

select points x_k, y_k such that $x_k \in A(p_{i+1}^n, f(p_i^n))$, $y_k \in A(f(p_i^n), b)$, $0 < x_k f(p_i^n) = y_k f(p_i^n) < 1/k$. If $x_k p_i^n \leqq y_k p_i^n$ put $x_k' = x_k$ and choose $y_k' \in A(f(p_i^n), y_k)$ so that $y_k' p_i^n = x_k' p_i^n$; similarly, if $x_k p_i^n > y_k p_i^n$, define $y_k' = y_k$ and select a point x_k' of $A(x_k, f(p_i^n))$ such that $x_k' p_i^n = y_k' p_i^n$. It is easily seen that such selections are possible, and so two infinite sequences $\{x_k'\}$, $\{y_k'\}$ of points are obtained with $x_k' p_i^n = y_k' p_i^n$, $(k=1, 2, \cdots)$ and $\lim_{k\to\infty} x_k' = \lim_{k\to\infty} y_k' = f(p_i^n)$.

Since $p_i^n f(p_i^n) < \min(ab, \epsilon)$ and $\lim_{k\to\infty} x_k' y_k' = 0$, then for k sufficiently large, $p_i^n x_k' = p_i^n y_k' < \epsilon$, but $x_k' y_k' < p_i^n x_k'$, in contradiction to the preceding lemma.

Case 2. *The point $f(p_i^n)$ coincides with b.* Since $p_i^n b \leqq p_i^n p_{i+j}^n < \lambda(n) < ab$, then $p_i^n \neq a$ and so $i \neq 0$. Denote by $f(b)$ a foot of b on $A(a, p_i^n)$. Then $bf(b) \leqq bp_i^n \leqq p_i^n p_{i+j}^n < \lambda(n) < ab$, and consequently $f(b) \neq a$. The same argument used in Case 1 may be applied to show the existence of two points x, y of $A(a, b)$, on opposite sides of $f(b)$, such that $bx = by < \epsilon$, and $xy < bx$, in contradiction to Lemma 4.1.

If P is any finite subset of a metric space, let R denote any non-reflexive, symmetric, binary relation defined in P, with respect to which each two elements of P are comparable. The subset P is R-connected if and only if P contains for each pair s, t of distinct points, a subset $p_1, p_2, \cdots, p_n$ with $s = p_1$, $t = p_n$ and p_i, p_{i+1} in the relation R for every $i = 1, 2, \cdots, n-1$. The R-content $\mu_R(P)$ is defined to be the sum of the distances of all point-pairs of P that are in the relation R, while the *content* $\mu(P)$ of P is the smallest of the numbers $\mu_R(P)$ for all of the (obviously finite) relations R with respect to which P is R-connected.

COROLLARY. *If A is a metric arc with finite Menger curvature at each point, then for almost all positive integers n, the length of L_n equals $\mu[L_n]$, where $\sum_{i=1}^n p_{i-1}^n p_i^n$ defines the length of the n-lattice L_n.*

It is known that $L[P] = \mu(P)$ for any homogeneous chain P of a metric arc, and for almost all natural numbers n, each n-lattice of arc A is, as proved above, a homogeneous chain.†

5. Uniqueness of lattices. Let A be an arc of a metric space M, and f a homeomorphism from $I = [0, 1]$ to A. If $x, y \in A$, we write $x \prec y$ if and only if $f^{-1}(x) < f^{-1}(y)$. Obviously, A is ordered with respect to the relation $\prec$.

If $p \in A$ and $\delta > 0$, $U^R(p; \delta)$ denotes the set $[q \mid q \in U(p; \delta) \cdot A \ \& \ p \prec q]$; similarly $U^L(p; \delta) = [q \mid q \in U(p; \delta) \cdot A \ \& \ q \prec p]$. The arc A is called *metrically monotone* if and only if $x, y, z \in A$, $x \prec y \prec z$ imply $xz \geqq \max(xy, yz)$.

LEMMA 5.1. *If A is a metric arc with finite Menger curvature at each point, then positive numbers δ_R, δ_L exist such that for every point p of A,* (i) *$x, y \in U^R(p; \delta_R)$, $x \prec y$ imply $px < py$, and* (ii) *$x, y \in U^L(p; \delta_L)$, $x \prec y$ imply $px > py$.*

Proof. To prove (i), assume the contrary. Then a sequence $\{\delta_n\}$ of positive numbers exists with $\lim_{n\to\infty} \delta_n = 0$, such that for each positive integer n, A con-

† The proof that $L[P] = \mu[P]$ for any homogeneous chain P is due to Menger. See Distance Geometry, pp. 64–65.

tains a point p_n and $x_n, y_n \in U^R(p_n; \delta_n)$ such that $x_n \prec y_n$, but $p_n x_n \geqq p_n y_n$. Since A is closed and compact, a point p of A and a sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$ exist with $p = \lim_{k\to\infty} p_{n_k}$.

Now corresponding to each positive integer k, the set $U^R(p_{n_k}; \delta_k)$ contains a point y'_{n_k} such that $p_{n_k}y'_{n_k} = p_{n_k}y_{n_k}$, where $p_{n_k} \prec y'_{n_k} \prec x_{n_k}$ in case $p_{n_k}x_{n_k} > p_{n_k}y_{n_k}$, and $y'_{n_k} = x_{n_k}$ if $p_{n_k}x_{n_k} = p_{n_k}y_{n_k}$. The function $\theta(z) = p_{n_k}z$ is continuous on $A(y'_{n_k}, y_{n_k})$, the sub-arc of A with endpoints y'_{n_k}, y_{n_k}, which consequently contains a point m_k that maximizes $\theta(z)$. Consider the infinite sequence $\{m'_k\}$, where $m'_k = m_k$ in case m_k is an interior point of $A(y'_{n_k}, y_{n_k})$. In the contrary case, let f_k denote a foot of p_{n_k} on $A(y'_{n_k}, y_{n_k})$. If f_k is an interior point of that sub-arc, put $m'_k = f_k$, while if f_k is an endpoint select any point in the interior of the sub-arc as m'_k.

Clearly $\lim_{k\to\infty} y'_{n_k} = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} m'_k = p$, and so, for k_0 sufficiently large, pm'_{k_0} is less than the ϵ of Lemma 4.1. It is now easily seen that $A \cdot U(p; pm_{k_0})$ contains points p_{n_k}, r, t (for k sufficiently large) such that $r \in A(y'_{n_k}, m'_k)$, $t \in A(m'_k, y_{n_k})$, $r \neq t$, $p_{n_k}r = p_{n_k}t > rt$, in contradiction to Lemma 4.1.

The argument for (ii) is entirely analogous.

COROLLARY. *If A is a metric arc with finite Menger curvature at each point, then A is the sum of a finite number of non-overlapping, metrically monotone arcs.*

THEOREM 5.1. *If a metric arc A has finite Menger curvature at each point, then there exists a positive integer N such that for $n > N$ the arc A contains exactly one n-lattice.*†

Proof. The existence of at least one n-lattice in A for every natural number n was established by Schoenberg, as remarked above. To prove uniqueness, select N so that $n > N$ implies that $\lambda(n)$ is less than either of the numbers δ_R, δ_L of the preceding lemma, and suppose there are two n-lattices $L_n^1 = (p_0, p_1, \cdots, p_n)$, $L_n^2 = (q_0, q_1, \cdots, q_n)$ of A with $\lambda_1(n) \leqq \lambda_2(n)$, and $n > N$.

If $\lambda_1(n) < \lambda_2(n)$ then since $p_0 = q_0 = a$ and $\lambda_1(n) < \delta_R$, $\lambda_2(n) < \delta_R$, the points p_1, q_1 belong to $U^R(a; \delta_R)$ and hence $ap_1 < aq_1$ implies $p_1 \prec q_1$. Now $p_1q_2 > \lambda_2(n)$, for in the contrary case, $p_1, q_1 \in U^L(q_2; \delta_L)$ and $p_1 \prec q_1$ gives $p_1q_2 > q_2q_1 = \lambda_2(n)$. It follows that $p_2 \prec q_2$, for, by what has just been established, $p_2 \neq q_2$ while if $q_2 \prec p_2$ then the interior of $A(p_1, q_2)$ contains a point p'_2 such that $p_1p'_2 = p_1p_2 = \lambda_1(n)$. Since $p'_2 \prec q_2$ and $q_2 \prec p_2$, then $p'_2 \prec p_2$. But $p_2, p'_2 \in U^R(p_1; \delta_R)$ and so $p_1p'_2 < p_1p_2$.

Making the inductive hypothesis that $p_i \prec q_i$ (anchored for $i = 1$), the above procedure shows how $p_n \prec q_n$ follows from $p_{n-1} \prec q_{n-1}$. But since $p_n = q_n = b$, this is impossible.

Finally, use of Lemma 5.1 shows that $p_i = q_i$ ($i = 0, 1, \cdots, n$) in case $\lambda_1(n) = \lambda_2(n)$, and the theorem is proved.

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† In a forthcoming paper, the writers obtain the conclusions of Theorems 4.1, 5.1 with considerably weaker hypotheses.

Remark. The restriction of finite Menger curvature at each point enters in the application of Lemma 4.1 to the proof of Lemma 5.1. That the condition is not necessary for uniqueness of n-lattices, for all sufficiently large values of n, is shown by considering the metric arc obtained from the unit segment $[0, 1]$ by defining the distance of each two points as the positive square root of their euclidean distance. Clearly an $(n+1)$-tuple of this arc is an n-lattice if and only if it is an n-lattice of the segment, and hence n-lattices are unique for every n. On the other hand, the metric arc (which, as is well-known, is congruent with an arc of Hilbert space) has *infinite* Menger curvature at each point.

6. Arcs with zero Menger curvature. *Historical remarks.* In this section we are concerned with metric arcs A such that $p \in A$ implies $K(p) = 0$. Calling such arcs "un-curved," the question immediately arises: *Are un-curved arcs straight*; that is, if A is a metric arc, and $p \in A$ implies $K(p) = 0$, is A a metric segment (*i.e.*, congruent with a line segment whose length equals the distance of the end-points of A)? An easy example shows that this question must be answered in the negative. For the circle S_1 (curve), with the distance of two distinct points defined as the length of the shorter arc joining them, is a metric space. Any arc A of S_1, *greater than a semi-circle*, is clearly not a metric segment, though the arc has zero Menger curvature at each point. This example is instructive from another aspect, for not only is arc A un-curved, it is even *locally straight* (that is, A is a *geodesic* arc) but fails to be a metric segment. Menger has given an example of an un-curved metric arc that is not locally straight, and Schoenberg has shown how a wide class of un-curved, not straight, metric arcs may be obtained.

In what metric spaces are un-curved arcs metric segments? The first results in this direction were obtained by Menger, who showed that in any euclidean space (indeed, in any space with each four points congruently imbeddable in euclidean three-space) metric segments are characterized among all arcs by the property of being un-curved.† The highly ingenious proof establishing this important result is based upon two lemmas. One of these lemmas is the n-lattice theorem, referred to in Section 2; the other one (of an *ad hoc* nature) we are not concerned with here.

It was soon pointed out that the argument given for the n-lattice theorem was not valid, and a proof for euclidean arcs was sketched by Alt and Beer. When Schoenberg established the theorem in its present general environment, he showed also that the form in which the second of Menger's two lemmas entered into the proof of the characterization theorem was valid in spaces more general than those possessing the euclidean four-point property; namely the so-called metric ptolemaic spaces.‡ These are metric spaces with the property that for every quadruple p, q, r, s, of points, the three products $pq \cdot rs$, $pr \cdot qs$, $ps \cdot qr$ of

† Distance Geometry, pp. 80–84.

‡ Distance Geometry, pp. 79–80.

"opposite" distances satisfy the triangle inequality. It was concluded, therefore, that in every metric ptolemaic space, un-curved arcs are straight.

Concerning this conclusion, two comments seem pertinent. (1). Statements occur in Menger's proof of the theorem itself (apart from the two lemmas) that are readily acceptable in the case he treated (euclidean space), but which require substantiation when the space is made more general. An instance of this is the expression for arc length that we have obtained in Theorem 3.1. (2). It is tacitly assumed in Menger's proof that an infinite sequence of n-lattices L_n of A exists such that, for each lattice of the sequence, not every three consecutive points have zero curvature. Though no mention of it is made, this assumption need not be verified; but it is easy to show that a *euclidean* arc for which it is not valid is a segment. It turns out (as we shall show) that metric ptolemaic arcs for which this assumption is not valid are also metric segments. It happens that metric betweenness possesses a transitive property in metric ptolemaic spaces that it does not have in merely metric spaces. By virtue of this transitivity, the desired property of arcs will be established in the next section.

The problem of characterizing metric segments as un-curved arcs of a restricted class of metric spaces was studied also by Haantjes.†

In the second section of Finsler's dissertation a definition of curvature is formulated that, as Haantjes observed, is directly applicable to rectifiable arcs of any metric space, though it is not, perhaps, so purely a metrization of the notion as is Menger's definition. If $p \in A(a, b)$, a rectifiable metric arc, the Finsler-Haantjes curvature $k(p)$ of $A(a, b)$ at p is given by

$$k(p) = \left\{24 \lim_{q,r \to p} (L[A(q, r)] - qr)/L^3[A(q, r)]\right\}^{1/2},$$

q, $r \in A(a, b)$ where $L[A(q, r)]$ denotes the length of that sub-arc of $A(a, b)$ with endpoints q, r.

This notion of curvature is defined only for rectifiable arcs, while no such restriction is explicit in the definition of Menger curvature (the notion of which is meaningful at each accumulaton point of any metric space, provided the limit involved exists). But the greater generality of the latter definition is more apparent than real, for Pauc has shown that if p is a point of any metric continuum and $K(p)$ exists, then in a neighborhood of p the continuum is a rectifiable arc.‡ It follows that if $K(p)$ exists at each point p of a metric arc $A(a, b)$, then the arc is rectifiable and the Finsler-Haantjes curvature notion is applicable.

Haantjes proved that for metric arcs the existence of both $K(p)$ and $k(p)$ implies their equality; that if $K(p)$ exists so does $k(p)$, but not conversely; while for euclidean arcs the two notions are equivalent. On the basis of Pauc's theorem and the second of the above results, he proved that a metric, ptolemaic arc with

† Distance Geometry, pp. 76–78.

‡ C. Pauc, Courbure dans les espaces métriques, Rend. Accad. Naz. die Lincei, vol. 24, 1936, pp. 109–115. The theorem is announced without proof. In a forthcoming paper by the writers, the theorem is seen to follow as a corollary of a much more general theorem.

$K(p)\equiv 0$ is a metric segment. His proof of this fundamental theorem is quite rigorous, but in view of the number of preliminary theorems upon which it is based (some of which are not easily established) it is far from being direct or complete in itself.

7. A new approach to the fundamental theorem. In view of the remarks made in the preceding section, it seems desirable to obtain a proof of the theorem characterizing segments in metric ptolemaic spaces as un-curved arcs that is direct, complete in itself, and free from lacunae. We indicate now how this may be done; the details will appear in another publication.

An arc is called a *geodesic* provided each of its points is contained in a subarc that is a metric segment. The following lemma, needed in our approach, is of intrinsic interest and fills the gap (mentioned in Section 6) in the Menger-Schoenberg procedure.

LEMMA 7.1. *In a metric ptolemaic space, each geodesic arc is a metric segment.*

Proof. If $A(a, b)$ is a geodesic arc, it follows from its definition and a standard compactness argument that a positive number δ exists such that for each point p of the arc, $U(p;\ \delta)\cdot A(a, b)$ is linear (that is, congruently imbeddable in a straight line). Let N be a positive integer such that $n>N$ implies that for every n-lattice L_n of $A(a, b)$, $\lambda(n)<\delta$. Then each consecutive triple of points in every n-lattice L_n, $(n>N)$, is linear and, since $p_i^n p_j^n=\lambda(n)$ for $|i-j|=1$, it follows that p_{i+1}^n is metrically between p_i^n and p_{i+2}^n for $i=0, 1, \cdots, n-2$ (that is, $p_i^n\neq p_{i+1}^n\neq p_{i+2}^n$ and $p_i^n p_{i+1}^n+p_{i+1}^n p_{i+2}^n=p_i^n p_{i+2}^n$). Symbolize this betweenness relation by $p_i^n p_{i+1}^n p_{i+2}^n$.

Consider the quadruple p_0^n, p_1^n, p_2^n, p_3^n. Since the space is ptolemaic,

$$p_0^n p_1^n\cdot p_2^n p_3^n + p_0^n p_3^n\cdot p_1^n p_2^n \geqq p_0^n p_2^n\cdot p_1^n p_3^n.$$

Now $p_0^n p_1^n=p_1^n p_2^n=p_2^n p_3^n=\lambda(n)$, and $p_0^n p_2^n=p_1^n p_3^n=2\lambda(n)$ (since $p_0^n p_1^n p_2^n$ and $p_1^n p_2^n p_3^n$ hold), and so the above inequality yields $p_0^n p_3^n\geqq 3\lambda(n)$. But since the space is metric, $p_0^n p_3^n\leqq p_1^n p_1^n+p_1^n p_2^n+p_2^n p_3^n=3\lambda(n)$; that is, $p_0^n p_3^n=3\lambda(n)$.

Making the inductive assumption that $p_0^n p_k^n=k\cdot\lambda(n)$ for $k=2, 3, \cdots, n-1$, and applying the above procedure to the quadruple p_0^n, p_{n-2}^n, p_{n-1}^n, p_n^n, we obtain $p_0^n p_n^n=n\cdot\lambda(n)$. Hence for $n>N$,

$$p_0^n p_n^n/[n\cdot\lambda(n)] = ab/[n\cdot\lambda(n)] = 1.$$

Taking the limit as $n\rightarrow\infty$, use of Theorem 3.1 gives $L[A(a, b)]=ab$; that is, the length of $A(a, b)$ is the distance of its end-points, and hence $A(a, b)$ is a metric segment.

The argument used in the proof of Lemma 7.1 shows that if $A(a, b)$ is a metric ptolemaic arc and there is an infinite sequence of n-lattices of $A(a, b)$ such that each consecutive triple of points of each n-lattice of the sequence has zero curvature, then $A(a, b)$ is a metric segment.

Remark. To prove that a metric ptolemaic arc with everywhere vanishing Menger curvature is a metric segment, it suffices to show that it is a geodesic; moreover, according to the Corollary of Lemma 5.1, we may restrict our attention to metrically monotone arcs.

LEMMA 7.2. *If A is any metric arc with zero curvature at each point, then to each $\epsilon>0$ there corresponds a positive integer N such that for $n>N$ the curvature of each consecutive triple of points of every n-lattice of A is less than ϵ.*

Proof. Since $K(p)=0$, $p\in A$, there corresponds to each p of A a positive number $\sigma(p)$ such that every three points of $A\cdot U(p;\sigma(p))$ has curvature less than ϵ, and the usual compactness argument shows that $\sigma(p)>\sigma>0$ for every p of A. It suffices, then, to take N such that $n>N$ implies $\lambda(n)<\sigma$, for then $p_i^n, p_{i+1}^n, p_{i+2}^n \in A\cdot U(p_{i+1}^n;\sigma)$, and consequently $K(p_i^n, p_{i+1}^n, p_{i+2}^n)<\epsilon$ for $n>N$.

Now let $A(a, b)$ be a monotone arc of a metric ptolemaic space, and for a given positive (arbitrarily small) ϵ, let N be a positive integer such that (1) for $n>N$ *each* consecutive triple $p_i^n, p_{i+1}^n, p_{i+2}^n$ of each n-lattice $L_n=(a=p_0^n, p_1^n, \cdots, p_n^n=b)$ has curvature less than ϵ, and (2) $\lambda(n)<\epsilon/2^m$, (m arbitrarily large). It follows that

$$1 \geqq p_i^n p_{i+2}^n/[2\lambda(n)] \geqq 1 - \epsilon^2\lambda^2(n).$$

Application of the ptolemaic inequality to an arbitrary quadruple p_i^n, p_{i+1}^n, p_{i+2}^n, p_{i+3}^n of consecutive points of L_n gives

$$(*) \qquad 1 \geqq p_i^n p_{i+3}^n/[3\lambda(n)] \geqq 1 - (4/3)\lambda^2(n)[\epsilon_{i+1} + \epsilon_{i+2} - \lambda^2(n)\epsilon_{i+1}\epsilon_{i+2}],$$

where $\epsilon_{i+j}=\epsilon_{i+j}(n)\geqq 0$, $\lim_{n\to\infty}\epsilon_{i+j}=0$, $(j=1, 2)$.

The proof is completed by extending (*) inductively to obtain

$$\lim_{n\to 8} p_0^n p_n^n/n\cdot\lambda(n) = 1.$$

THE APPROXIMATION OF LOGARITHMIC NUMBERS

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1. Introduction. The coefficients of the powers of x in the expansion of $x/\log(1+x)$ are called *logarithmic numbers*. Let us denote these by L_n, so that

$$\frac{x}{\log(1+x)} = 1 + L_1x + L_2x^2 + L_3x^3 + \cdots + L_nx^n + \cdots, \tag{1}$$

where $L_1 = \frac{1}{2}$, $L_2 = -1/12$, $L_3 = 1/24$, $L_4 = -19/720$, *etc.*

These numbers to $n=20$ have been computed by A. N. Lowan and H. E. Salzer,* who terminated their table with the value

$$L_{20} = -\frac{12365\ 72232\ 34699\ 80029}{48\ 17145\ 97618\ 97472\ 00000} = -0.00256\ 7022545 \cdots.$$

The size of the numerator and denominator of this fraction and the obvious, though not insuperable, difficulties of extending the computation to higher values of n suggest the problem of searching for an asymptotic formula for logarithmic numbers. Such a formula was published in 1924 by J. F. Steffensen,† who gave as an asymptotic approximation the following:

$$L_n \sim \frac{(-1)^{n+1}}{n \log^2 n} = S_n. \tag{2}$$

Unfortunately the ratio L_n/S_n converges very slowly toward 1 as one sees from the values $L_{20}/S_{20} = 0.461 \cdots$, $L_{100}/S_{100} = 0.545 \cdots$. In fact, for the ratio L_n/S_n to vary from 1 by as little as 0.01, n would have to exceed 29×10^{12}.

The principal object of this paper is to improve the formula of Steffensen and to suggest other methods by means of which more satisfactory approximations can be made to L_n for large values of n. Incidentally, these numbers will be generalized by means of what we shall call the *logarithmic number function*, from which a companion set of numbers corresponding to $n = m + \frac{1}{2}$ will be defined, where m is an integer. These numbers will be found to have asymptotic properties similar to those of L_n.

2. Properties of logarithmic numbers. Logarithmic numbers find their principal use in integration formulas. Thus, if we denote by x_m the quantity $x_0 + m\omega$ and by f_m the value $f(x_m)$, where $f(x)$ is a function integrable over the interval (x_0, x_p), then the Gregory-Newton formula for finite integration is written as follows:

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* Tables of coefficients in numerical interpolation formulae, Journal of Math. and Physics, vol. 22, 1943, pp. 49–50.

† On Laplace's and Gauss' summation-formulas, Skandinavisk Aktuariettidskrift, 1924, pp. 2–4. See also Interpolation, Baltimore, 1927, pp. 106–107.

$$\frac{1}{\omega}\int_{x_0}^{x_p} f(s)ds = \sum_{m=0}^{p} f_m - L_1(f_0 + f_p) + L_2(\Delta f_{p-1} - \Delta f_0) \tag{3}$$
$$- L_3(\Delta^2 f_{p-2} + \Delta^2 f_0) + L_4(\Delta^3 f_{p-3} - \Delta^3 f_0) + \cdots.$$

Logarithmic numbers are also special values of Bernoulli polynomials of nth order and nth degree, that is,

$$L_n = \frac{B_n^{(n)}(1)}{n!}, \tag{4}$$

where $B_m^{(n)}(x)$ is the coefficient of $t^m/m!$ in the expansion of the function $t^n e^{xt}/(e^t - 1)^n$.

Observing the expansion

$$\frac{x}{(1+x)\log(1+x)} = \sum_{m=0}^{\infty} \frac{x^m B_m^{(m)}(0)}{m!},$$

we have from comparison with (1) the equation

$$L_n = [nB_{n-1}^{(n-1)}(0) + B_n^{(n)}(0)]/n!. \tag{5}$$

From the identity

$$B_n^{(n)}(x) = \int_x^{x+1} (s-1)(s-2)\cdots(s-n)ds,$$

we then obtain, by means of (5), the following essential formula:

$$L_n = \int_0^1 \frac{s(s-1)\cdots(s-n+1)}{n!}\,ds. \tag{6}$$

In order to attain a more satisfactory form for L_n, we write

$$s(s-1)\cdots(s-n+1) = \Gamma(s+1)/\Gamma(s-n+1), \qquad n! = \Gamma(n+1).$$

By means of these identities we can express (6) as follows:

$$L_n = \int_0^1 \frac{\Gamma(s+1)}{\Gamma(n+1)\Gamma(s-n+1)}\,ds. \tag{7}$$

If, furthermore, we observe the identity

$$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x,$$

and use it to write

$$(8)\qquad \begin{aligned}\Gamma(s-n+1) = \Gamma[1-(n-s)] &= \frac{\pi}{\sin \pi(n-s)\Gamma(n-s)} \\ &= \frac{\pi}{(-1)^{n+1}\sin \pi s\, \Gamma(n-s)},\end{aligned}$$

then the integral (7) can be reduced to the following attractive form

$$(9)\qquad L_n = \frac{(-1)^{n+1}}{\pi}\int_0^1 \frac{\Gamma(s+1)\sin \pi s\, \Gamma(n-s)}{\Gamma(n+1)}\, ds.$$

If one makes the observation that the integrand of (9) is actually the function $\sin \pi s B(s+1, n-s)$, where $B(p, q)$ is the Beta function, then one can express L_n as the double integral

$$L_n = \frac{(-1)^{n+1}}{\pi}\int_0^1\int_0^1 \sin \pi s\, t^s(1-t)^{n-s-1}dtds.$$

The representation of L_n given by (9) is readily adapted to the evaluation of L_n by means of finite integration. Within the limits of integration $\Gamma(s+1)$ varies from a minimum value

$$(10)\qquad \gamma_0 = \Gamma_n(1+s_0) = 0.88560\ 31943, \text{ at } s_0 = 0.46163\ 21450,$$

to a maximum value of 1 at $s=0$ and $s=1$. The ratio $\Gamma(n-s)/\Gamma(n+1)$ varies from a maximum of $1/n$ at $s=0$ to a minimum of $1/n(n-1)$ at $s=1$.

The rapidity of the convergence of the integral is shown from the fact that the values of L_{20} to 9 significant figures and of L_{100} to 7 significant figures were computed by means of formula (3) in which only eleven values of the integrand were used, namely, those over the range $s=0(.1)1$. For future reference the latter value to the indicated approximation is recorded as follows:

$$(11)\qquad L_{100} = -\ 0.000297\ 4763\cdots.$$

3. Asymptotic formulas for logarithmic numbers. We now have in (9) a formula readily adapted to achieve the desired asymptotic representation of L_n. We first replace $\Gamma(n-s)$ and $\Gamma(n+1)$ by their Stirling approximations, namely,

$$\Gamma(x) \sim x^{x-1}(2\pi x)^{1/2}e^{-x}\left[1+\frac{1}{12x}+\cdots\right].$$

Making the proper substitutions and simplifying the calculation, we obtain

$$\frac{\Gamma(n-s)}{\Gamma(n+1)} \sim \frac{e^{-as}}{n}\left(1-\frac{s}{n}\right)^{-s+1/2} \sim \frac{e^{-as}}{n},$$

where $a=\log n$.

When this value is substituted in (9), we obtain the following asymptotic expression for L_n:

$$(12) \qquad L_n \sim \frac{(-1)^{n+1}}{n\pi} \int_0^1 \Gamma(s+1) \sin \pi s \, e^{-as} ds.$$

Since the integrand is positive we can now apply the law of the mean and thus obtain

$$(13) \qquad L_n \sim \frac{(-1)^{n+1}}{n\pi} \Gamma(\xi+1) \int_0^1 \sin \pi s \, e^{-as} ds, \qquad 0 \leqq \xi \leqq 1,$$

and hence,

$$(14) \qquad L_n \sim \frac{(-1)^{n+1}\Gamma(\xi+1)}{n(\log^2 n + \pi^2)} = T_n \Gamma(\xi+1).$$

Using the extreme values of $\Gamma(\xi+1)$, namely 1 and γ_0 given by (10), we obtain the following inequalities for $n=20$ and $n=100$:

$$(15) \qquad 0.002350 < L_{20} < 0.002653, \qquad 0.0002850 < L_{100} < 0.0003218.$$

Since $L_{20}/T_{20}=0.967$ and $L_{100}/T_{100}=0.924$, one conjectures that the limiting value of L_n/T_n will be close to γ_0. This has not been proved, however.

Although the asymptotic formula given by (14) is a reasonably satisfactory one, and does indeed establish the ultimate correctness of Steffensen's approximation given in Section 1, it would be useful to achieve a sharper one if possible. An obvious suggestion is to replace $\Gamma(s+1)$ in (12) by its power series representation. Unfortunately, the proximity of the singular point at $s=-1$ makes the series converge very slowly in the interval (0, 1) and actually diverge at $s=1$. These difficulties can be largely surmounted, however, if a polynomial approximation to $\Gamma(s+1)$ is made by means of Lagrange's interpolation formula, or some other similar device.

Actually, it was found desirable to compute the cubic polynomial which has the value 1 at $s=0$ and $s=1$, which has the value γ_0 at $s=s_0$, and the derivative of which is zero at $s=s_0$. This polynomial is readily found to be

$$y = 1 - a_1 s + a_2 s^2 - a_3 s^3,$$

where $a_1=0.52590\ 56748$, $a_2=0.66802\ 76014$, $a_3=0.14212\ 19266$.

When this function is introduced into (12), we then have

$$(16) \qquad L_n \sim \frac{(-1)^{n+1}}{n\pi} [I_0(a) - a_1 I_1(a) + a_2 I_2(a) - a_3 I_3(a)],$$

where we introduce the function

$$I_p(a) = \int_0^1 s^p \sin \pi s \, e^{-as} ds.$$

It will be useful also to introduce the companion function

$$J_p(a) = \int_0^1 s^p \cos \pi s \, e^{-as} ds.$$

These functions are readily shown to satisfy the following relationships:

$$(17)\qquad \begin{aligned} I_p(a) &= \frac{\pi e^{-a}}{a^2+\pi^2} + \frac{p}{a^2+\pi^2}\left[aI_{p-1}(a) + \pi J_{p-1}(a)\right],\\ J_p(a) &= \frac{\pi e^{-a}}{a^2+\pi^2} + \frac{p}{a^2+\pi^2}\left[aJ_{p-1}(a) - \pi I_{p-1}(a)\right].\end{aligned}$$

They also satisfy separately the following iterative equations:

$$(18)\qquad \begin{aligned} I_{p+2}(a) &= \frac{\pi e^{-a}}{a^2+\pi^2} + \frac{2a(p+2)}{a^2+\pi^2} I_{p+1}(a) - \frac{(p+1)(p+2)}{a^2+\pi^2} I_p(a).\\ J_{p+2}(a) &= \frac{a-p-2}{a^2+\pi^2} e^{-a} + \frac{2a(p+2)}{a^2+\pi^2} J_{p+1}(a) - \frac{(p+1)(p+2)}{a^2+\pi^2} J_p(a).\end{aligned}$$

Beginning with the initial values,

$$(19)\qquad \begin{aligned} I_0 &= \frac{\pi(e^{-a}+1)}{a^2+\pi^2}, & J_0 &= \frac{a(e^{-a}+1)}{a^2+\pi^2},\\ I_1 &= \frac{\pi e^{-a}}{a^2+\pi^2} + \frac{1}{a^2+\pi^2}(aI_0 + \pi J_0), & J_1 &= \frac{ae^{-a}}{a^2+\pi^2} + \frac{1}{a^2+\pi^2}(aJ_0 - \pi I_0),\end{aligned}$$

the functions of higher subscript are readily computed from (18). Equations (17) are useful in checking the computations.

By means of these formulas the efficiency of the approximation function (16) was checked numerically for $n=20$ and $n=100$. The following values for I_p ($p=0, 1, 2, 3$) were obtained:

$n = 20$	$n = 100$
$I_0 = 0.17505\ 14455$	$I_0 = 0.10210\ 08618$
$I_1 = 0.01390\ 15517$	$I_1 = 0.03127\ 05043$
$I_2 = 0.03045\ 03062$	$I_2 = 0.01297\ 53716$
$I_3 = 0.01700\ 51089$	$I_3 = 0.00651\ 01003$

When these values are substituted in (16), the approximations $L_{20} \sim .00254 \cdots$ (with an error of 3 in the last place) and $L_{100} \sim .0002973$ (with an error of 1 in the last place) are obtained.

4. Inequalities satisfied by logarithmic numbers. In order to establish an inequality between successive logarithmic numbers, we shall write $n+1$ for n in formula (9) and replace $(-1)^{n+1}$ by $\cos (n+1)\pi$. We then have

$$
\begin{aligned}
L_{n+1} &= -\frac{\cos(n+1)\pi}{\pi\Gamma(n+2)}\int_0^1 \Gamma(s+1)\sin\pi s\,\Gamma(n+1-s)ds,\\
&= -\frac{\cos(n+1)\pi}{\pi(n+1)\Gamma(n+1)}\int_0^1 \Gamma(s+1)\sin\pi s\,(n-s)\,\Gamma(n-s)ds,\\
&= -\frac{\cos(n+1)\pi}{\pi\Gamma(n+1)}\int_0^1 \Gamma(s+1)\sin\pi s\,\Gamma(n-s)ds \qquad (20)\\
&\quad + \frac{\cos(n+1)\pi}{\pi(n+1)\Gamma(n+1)}\int_0^1 \Gamma(s+2)\sin\pi s\,\Gamma(n-s)ds,\\
&= -L_n + \frac{\cos(n+1)\pi}{\pi(n+1)\Gamma(n+1)}\int_0^1 \frac{\Gamma(s+2)}{\Gamma(s+1)}\Gamma(s+1)\sin\pi s\Gamma(n-s)ds.
\end{aligned}
$$

But by the theorem of mean value, the last integral is equal to

$$
\frac{\Gamma(\xi+2)}{\Gamma(\xi-1)}\frac{L_n}{n+1} = \frac{\xi+1}{n+1}L_n, \qquad 0 \leqq \xi \leqq 1,
$$

so that, from (20) we get

$$
L_{n+1} = -L_n\left[1-\frac{\xi+1}{n+1}\right], \qquad 0 \leqq \xi \leqq 1. \tag{21}
$$

Letting ξ have successively the values 0 and 1, we obtain the following inequality:

$$
\frac{n}{n+1}\left|L_n\right| \geqq \left|L_{n+1}\right| \geqq \frac{n-1}{n+1}\left|L_n\right|. \tag{22}
$$

By an argument which does not differ essentially from that just given, we obtain the following expression for L_{n+p}:

$$
L_{n+p} = (-1)^p L_n\left[1-\frac{\xi_1+1}{n+p}\right]\left[1-\frac{\xi_2+1}{n+p-1}\right]\cdots\left[1-\frac{\xi_p+1}{n+1}\right],
$$
$$
0 \leqq \xi_i \leqq 1.
$$

If all the ξ_i are set equal to 1, we get the inequality

$$
\left|L_{n+p}\right| \geqq \frac{n(n-1)}{(n+p)(n+p-1)} \geqq \left|L_n\right|;
$$

and if all the ξ_i are set equal to 0, we have

$$
\left|L_{n+p}\right| \leqq \frac{n}{n+p}\left|L_n\right|;
$$

that is to say, we have established the following inequality, of which (22) is a special case:

$$\frac{n}{n+p}\,|L_n| \geqq |L_{n+p}| \geqq \frac{n(n-1)}{(n+p)(n+p-1)}\,|L_n|. \tag{23}$$

5. The logarithmic number function. If we return to equation (7) we see that the integrand is defined over a continuous range of values of n. This suggests the possibility of defining a logarithmic number function of the continuous variable n, which would, in particular, reduce to L_n when n is an integer. Denoting this function by $L(n)$, we thus write

$$L(n) = \int_0^1 \frac{\Gamma(s+1)}{\Gamma(n+1)\Gamma(s-n+1)}\,ds, \tag{24}$$

or, making use of (8),

$$\begin{aligned} L(n) &= \frac{1}{\pi\Gamma(n+1)}\int_0^1 \sin\pi(n-s)\,\Gamma(s+1)\Gamma(n-s)ds, \\ &= \pi n P(n) - \cos\pi n\, Q(n), \end{aligned} \tag{25}$$

where we abbreviate

$$P(n) = K(n)\int_0^1 \cos\pi s\,\Gamma(s+1)\Gamma(n-s)ds,$$

$$Q(n) = K(n)\int_0^1 \sin\pi s\,\Gamma(s+1)\Gamma(n-s)ds, \qquad K(n) = 1/[\pi\,\Gamma(n+1)].$$

We first observe the special values: $L(-n) = 0$, $L(0) = 1$, and $L(n) = (-1)^{n+1}Q(n) = L_n$, when n is a positive integer.

If n is replaced by $m+\frac{1}{2}$, where m is an integer, we obtain *half-logarithmic numbers*, which we denote by $L_{m+1/2}$, that is,

$$L(m+\tfrac{1}{2}) = (-1)^m P(m+\tfrac{1}{2}) = L_{m+1/2}. \tag{26}$$

These numbers can be computed from (24) by finite integration. Beginning with $m = -1$, we find the following seven-figure approximation of the first four numbers as follows:

$L_{-1/2} = 0.5097422$, $L_{1/2} = 0.9854485$, $L_{3/2} = 0.03483709$, $L_{5/2} = 0.007700113$.

From a consideration of (24) we readily prove that $L(n)$ has one zero in each of the intervals $p<n<p+1$ $(p=1, 2, 3, \cdots)$.

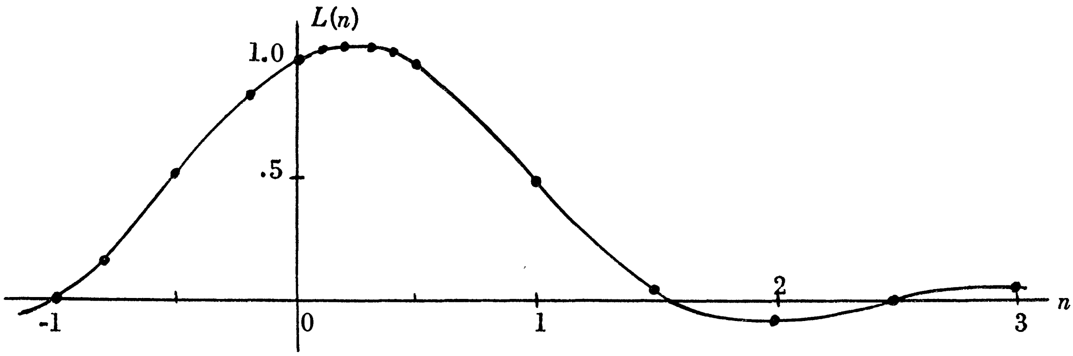
We also see that $L(n)$ has a maximum value between $n=0$ and $n=1$. This is readily shown from the derivative of $L(n)$, that is,

$$\frac{dL}{dn} = L'(n) = \int_0^1 \frac{\Gamma(s+1)}{\Gamma(n+1)\Gamma(s-n+1)}[\Psi(s-n+1) - \Psi(n+1)]ds, \tag{27}$$

which is obtained by differentiating (24) and introducing the psi-function defined by the ratio: $\Psi(x)=\Gamma'(x)/\Gamma(x)$.

Since $\Psi(s+1)-\Psi(0)$ is positive and $\Psi(s)-\Psi(2)$ is negative in the interval $0\leqq s\leqq 1$, the derivative changes from a positive to a negative value between $n=0$ and $n=1$. Hence a maximum value exists in the interval. This maximum is found to be in the neighborhood of $s=0.2$ and is approximately 1.068.

The graphical representation of $L(n)$ between $n=-1$ and $n=3$ is shown in the accompanying figure. The points on the curve were obtained by the finite integration of (24).



An asymptotic expansion for $L(n)$ for positive values of n is obtained if we replace $\Gamma(n-s)/\Gamma(n+1)$ in (25) by its asymptotic value, namely, e^{-as}/n, where $a=\log n$. We thus obtain

$$L(n) \sim \frac{\sin n\pi}{n\pi}\int_0^1 \cos \pi s\,\Gamma(s+1)\,e^{-as}ds - \frac{\cos n\pi}{n\pi}\int_0^1 \sin \pi s\,\Gamma(s+1)\,e^{-as}ds. \tag{28}$$

If in this expression we replace $\Gamma(s+1)$ by $1+A_1s+A_2s^2+A_3s^3+\cdots$, we obtain the further approximation,

$$\begin{aligned} L(n) &\sim \frac{\sin n\pi}{n\pi}(J_0+A_1J_1+A_2J_2+A_3J_3+\cdots) \\ &\quad - \frac{\cos n\pi}{n\pi}(I_0+A_1I_1+A_2I_2+A_3I_3+\cdots), \end{aligned} \tag{29}$$

where I_p and J_p are the functions defined in Section 3.

If in (29) n is replaced by $m+\frac{1}{2}$, where m is an integer, the coefficient of the second series vanishes and we have an asymptotic formula for the half-logarithmic numbers. Neglecting all but the first term, we obtain

$$L_{m+1/2} \sim \frac{(-1)^m}{(m+1/2)\pi}J_0 \sim (-1)^m\frac{a}{(m+1/2)\pi(a^2+\pi^2)}, \tag{30}$$

where $a=\log(m+\frac{1}{2})$.

MATHEMATICAL INDUCTION IN SETS

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Recent years have seen an increased use of a principle of proof which is analogous to ordinary mathematical induction over the natural numbers. This principle, known as the Hausdorff Maximal Principle or Zorn's Lemma, was first recognized to be equivalent to transfinite induction and the axiom of choice and so it was used in abstract algebra and general set theory [1], [2]. However, the same style of proof by a form of mathematical induction over the linearly ordered set of real numbers in an interval $[a, b]$ yields simple and elegant proofs in the theory of the calculus, as pointed out by L. R. Ford [3]. Thus the principle of mathematical induction unifies a diverse set of methods of proof at many levels of mathematics.

Let us first examine a palpably false theorem about natural numbers: *The set of all natural numbers is a finite set.* Attempting the proof by ordinary mathematical induction, we easily show that the set $\{1\}$ is finite and that the cardinal set $\{1, 2, \cdots, n\}$ is finite if the cardinal set $\{1, 2, \cdots, n-1\}$ is finite. Hence by the principle of mathematical induction, for every n the cardinal set $C_n = \{1, \cdots, n\}$ is finite. These cardinal sets C_n which have been proved finite form a chain $\mathfrak{C}$ of sets totally ordered by containment. That is, if C_n and C_m are two sets in the chain, either $C_n \subset C_m$ or $C_m \subset C_n$. Moreover, the chain $\mathfrak{C}$ is a *maximal* totally ordered chain of finite cardinal sets. That is, if another chain of finite cardinal sets exists one of whose members is contained in a member of $\mathfrak{C}$, then every member of this new chain is a member of $\mathfrak{C}$. Now to establish the conclusion of the theorem we must show that the union of all the sets in the chain $\mathfrak{C}$ also has the property of being a finite set. This certainly does not follow and, in fact, it is not true; so the "proof" fails.

In the proofs by general mathematical induction one forms a maximal chain of sets each of which has some property P and the issue which characteristically arises is whether the union of the chain has the property P. In proofs of theorems of the calculus the form of induction which is applicable is induction over the linearly ordered (but not well ordered) set of real numbers in an interval, and our characteristic issue is often settled by means of boundedness of the interval and the Dedekind property of the real numbers. Let us examine a proof of the Heine-Borel theorem which is a prototype for proofs of this sort. [4], [5].

The Heine-Borel theorem for the real numbers states that if a family $\mathfrak{F}$ of open intervals covers a closed and bounded interval $[a, b]$ of real numbers, then a finite subfamily of the covering family also covers the interval $[a, b]$. (Here a family of open intervals is said to *cover* the set $[a, b]$ if and only if every point x in the interval is contained in some open interval of the family.)

We make the proof by mathematical induction over the linearly ordered set of real numbers in the interval. First, by hypothesis, the initial point a is contained in some open interval of the family. Hence some non-empty open

interval $I_1=\{a\leqq x<c_1\}$ is covered by a finite subcollection (namely one) of the sets of the covering family. Now we form a maximal chain $\mathfrak{C}$ of sets $I=\{a\leqq x<c\}$ with $I_1\subset I\subset[a, b]$, each having the property that it is covered by a finite subcollection of open intervals in $\mathfrak{F}$. Does the union of this chain which is an interval $I_0=\{a\leqq x<c_0\}$ have a finite covering? We answer this question by the aid of the Dedekind property. The number c_0 is the least upper bound of numbers c such that $\{a\leqq x<c\}$ has a finite covering. Since the interval $[a, b]$ is closed the number c_0 is in $[a, b]$. Hence c_0 is in some open interval J of the family $\mathfrak{F}$. J intersects some one of the intervals $I=\{a\leqq x<c\}$ which has a finite covering and thus $I\cup J$ has a finite covering. Moreover, $I\cup J$ contains I_0 so that the union of the chain *does* have the property of being finitely covered. Now if $c_0=b$ the proof is complete and if $c_0<b$ interval $I\cup J$ properly contains I_0 so that the chain $\mathfrak{C}$ cannot be a maximal one. Hence the interval $[a, b]$ has a finite covering.

The above wording is not quite the simplest statement of the argument but it emphasizes the issue about the maximal chain of sets with the required property. It is worthy of note that the method of proof does not make use of the separability of the numbers, that is, the fact that the countable set of rational numbers is dense in the interval $[a, b]$. In fact, the method of proof yields actual generalizations of the Heine-Borel theorem for real numbers. In particular, we find that neither the statement nor the proof of the Heine-Borel theorem needs modification to apply it to the interval $[-\infty, +\infty]$ in the extended real numbers. For this interval is "bounded" in the sense required by the proof by mathematical induction.

It is also instructive to attempt to replace the covering family $\mathfrak{F}$ of open intervals, $\{x_1<x<x_2\}$ by a covering family $\mathfrak{F}'$ of half-open intervals $\{x_1\leqq x<x_2\}$. A statement analogous to the Heine-Borel theorem can be made and the proof proceeds as before until we reach the statement: "J intersects some one of the intervals $I=\{a\leqq x<c\}$, which has a finite covering." This is false for the half-open covering intervals. So we cannot establish that the union of the maximal chain of sets with finite covering also has a finite covering. In fact, the analogue of the Heine-Borel theorem for half-open intervals is false.

Let us apply the induction style of argument to obtain a direct proof of the theorem:

If f is a real valued function on the closed interval $[a, b]$ and if the derivative $f'=0$ at every point of $[a, b]$ then f is constant.

Proof: For every $\epsilon>0$ there is an interval $I_1=\{a\leqq x<c_1\}$ such that in I_1, $f(x)-f(a)<\epsilon(x-a)$. This follows from the definition of the derivative and the fact that $f'(a)=0$. As before we form a maximal chain $\mathfrak{C}$ of intervals $I=\{a\leqq x<c\}$ with $I_1\subset I\subset[a, b]$ in each of which $f(x)-f(a)<\epsilon(x-a)$. The union of the chain $\mathfrak{C}$ is an interval $I_0=\{a\leqq x<c_0\}$. Taking the least upper bound c_0 of I_0 we see that $f(c_0)-f(a)\leqq\epsilon(c_0-a)$. Since c_0 is in $[a, b]$ and hence $f'(c_0)=0$ we also have, for all x in some open interval J containing c_0, $f(x)-f(c_0)<\epsilon(x-c_0)$. Adding the last two inequalities we find that, for all x in J, $f(x)-f(a)<\epsilon(x-a)$. Moreover J

intersects one of the intervals I in $\mathfrak{C}$ and hence the same inequality holds in $I \cup J$ which properly contains I_0, so the union of the chain *does* have the property. If $c_0 < b$ the chain $\mathfrak{C}$ cannot be maximal. Hence $c_0 = b$ and for all x in $[a, b]$ we have $f(x) - f(a) \leqq \epsilon(b-a)$. Analogously for all x in $[a, b]$ we have $f(x) - f(a) \geqq \epsilon(b-a)$. Since ϵ is arbitrary this implies that $f(x) \equiv f(a)$.

The method can be generalized according to an observation of L. R. Ford [3], if we define an *extendable property* of open sets.

DEFINITION. *A property P of open sets is an extendable property if for every pair A, B of open sets each with property P, and having non-void intersection, their union $A \cup B$ also has property P.*

Now by applying the method of proof by induction over the linearly ordered set of real numbers in an interval $[a, b]$, we readily prove the following theorem due to L. R. Ford.

FORD'S THEOREM. *If P is an extendable property of open sets and if every point x in the bounded and closed interval $[a, b]$ is contained in an open set having property P, then the interval $[a, b]$ has property P.*

More generally, defining a set A to be *compact* if every covering of A by a collection of open sets contains a finite subcollection which covers A, we can prove by generalized mathematical induction the theorem: *If a compact set A is covered by a family of open sets each having the extendable property P, then A has the property P.* However, this generalization does not contain all instances of extension of an extendable property of sets by mathematical induction.

In applying mathematical induction over the linearly ordered interval of real numbers $[a, b]$ both the assertion of the existence of a maximal chain $\mathfrak{C}$ of sets having the extendable property P and the technique for showing that the union of the maximal chain has the property P depend upon the Dedekind property, the existence of the least upper bound. In ordinary (finite) induction over the natural numbers the existence of a maximal chain $\mathfrak{C}$ having property P depends upon the fact that the natural numbers are *well ordered*, which implies that if the theorem is false for some n, then there exists a first number n_1 for which the theorem is false. However, the fact that the union of any maximal chain of subsets of a finite set is a member of the chain comes from the finiteness itself. In the case of transfinite induction the existence of a maximal chain of sets having property P follows from the well-ordering but the technique of extension past the limit ordinals is similar to the calculus proofs using the Dedekind property. Finally, in general set theory [1], [2], the existence of the maximal chain of sets having the property P is a separate logico-mathematical principle, and the question whether the union of the maximal chain has property P, and the question of the existence of a proper extension of it having property P, are still ones for which there are no general techniques for answering.

Let us return in conclusion to one more example from the calculus. This is the theorem that a continuous function f on a bounded and closed real interval $[a, b]$ is uniformly continuous in $[a, b]$. To prove this theorem we must first

show that ordinary continuity implies local uniform continuity, which is a property of sets. That is, we first show that for every point x_0 in the interval and every $\epsilon>0$ there exists a $\delta>0$ such that if x_1 and x_2 are points of $[a, b]$ in a δ-neighborhood of x_0, then $|f(x_1)-f(x_2)|<\epsilon$. This is proved by observing that ordinary continuity at x_0 implies that for every $\epsilon/2>0$ there exists a $\delta/2>0$ such that if x_1 is in $[a, b]$ and $|x_1-x_0|<\delta/2$ then $|f(x_1)-f(x_0)|<\epsilon/2$. Similarly for x_2. Then the local uniform continuity at x_0 follows from combining these inequalities. But the uniform continuity of f is an extendable property of open intervals, hence by Ford's theorem it is true in the interval $[a, b]$.

References

1. E. R. Stabler, This MONTHLY, vol. 51, 1944, p. 129.
2. O. Frink, This MONTHLY, vol. 61, 1954, p. 223.
3. L. R. Ford, Interval additive propositions (abstract), Bull. Am. Math. Soc. vol. 62, 1956, p. 158; This MONTHLY, vol. 64, 1957, p. 107.
4. Veblen and Lennes. Infinitesimal Analysis. N. Y. 1907, p. 34.
5. T. H. Hildebrandt. Bull. Am. Math. Soc. vol. 32, 1926, p. 423.

CALCULATION OF MOMENTS FOR A CANTOR-VITALI FUNCTION

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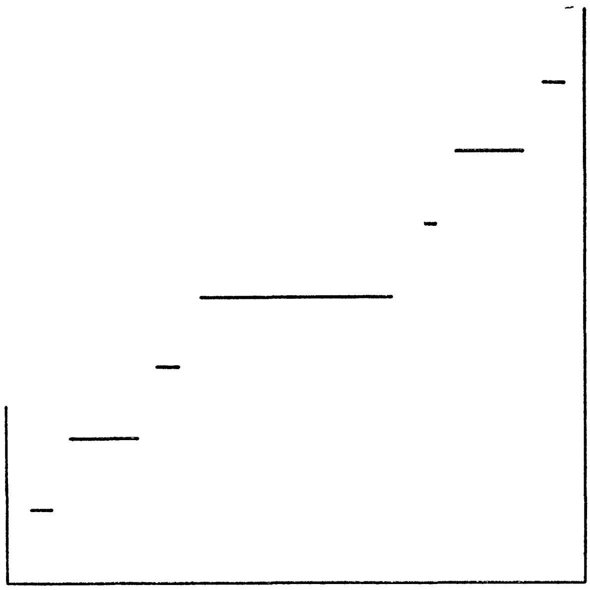
1. The "middle-third" function. Our task is to find moments of the form $M_n=\int_0^1 x^n p(x)dx$ and the related moments $m_n=\int_0^1 x^n dp(x)$ for functions $p(x)$ which have certain properties of symmetry. The most familiar of these is the one which is based on the Cantor "middle-third" set, which is obtained from the interval $0\leqq x\leqq 1$ by removing first the open middle third $1/3<x<2/3$, then, from the remaining closed intervals, their open middle thirds $1/9<x<2/9$ and $7/9<x<8/9$, and so on, from remaining closed intervals, open middle thirds indefinitely. The union of this denumerable collection of open intervals constitutes an open set E, of which the remaining complement F is the typical Cantor *discontinuum*. Between any two points of F there lie intervals of E; F is a perfect nowhere dense set. A simple calculation shows that the total of the lengths of the intervals of E is the value 1 itself.

The corresponding function $p(x)$ is defined first for points of E: as $1/2$ on $1/3<x<2/3$; as $1/4$ on $1/9<x<2/9$, as $3/4$ on $7/9<x<8/9$; as $1/8$ on $1/27<x<2/27$, as $3/8$ on $7/27<x<8/27$, $\cdots$. Thus on E, $p(x)$ is a nondecreasing

function of x, constant on each interval of E (see Figure). Next we write $p(0) = 0$, $p(1) = 1$, and to the end points of each interval of E assign to $p(x)$ the value it has for the interval itself. In order to complete the definition on remaining points of F we note that for the nondecreasing function so far defined the values $y = p(x)$ are dense in $0 \leqq y \leqq 1$; in fact, at a stage k of setting up the intervals of E, y has been given values, $m/2^k$, $m = 1, 2, \cdots, 2^k - 1$. Hence at a point of F



there is one and only one value y which will make the function $p(x)$ continuous at the point. We assign that value. It follows that $p(x)$ is monotonic-increasing on $0 \leqq x \leqq 1$.*

* If the reader has not run across the fact, it will interest him to prove that, in the scale of 3 the x-coordinates of the right hand end points of intervals of E are arbitrary decimal fractions composed of a finite number of digits 2 and 0. The corresponding y values, in the scale of 2, however, are given by the same sequences except that digits 2 are replaced by 1. For example, when $x = .2022$, $y = .1011$.

The points of F, in the scale of 3, are given by all decimal fractions, finite or infinite, which can be written as sequences of digits 2 and 0; from this it follows by a conventional argument that F is a nondenumerable infinite set of points. The points $.0222 \cdots = .1 = 1/3$, $.020222 \cdots = .021 = 7/27$ are in F as left end points of intervals of E, and $020202 \cdots = 1/4$ is in F but is not an end point. In fact the end points of intervals are only denumerable.

Functions of the kind considered in this article are merely very special examples of the type of nondecreasing continuous function, called by Vitali an *elementary discard*, for which is assigned a dense set of constant values of y on a sequence of intervals which exhausts in measure the fundamental interval. In terms of elementary discards Vitali analyzes the singular part of a continuous function of bounded variation (Rendiconti del Circolo Matematico di Palermo, vol. 46, 1922, p. 388).

The Stieltjes integrals $\int_0^1 f(x)dp(x)$, $\int_0^1 p(x)df(x)$, if $f(x)$ is continuous, exist as limits of Riemann sums, and

$$\int_0^1 f(x)dp(x) = f(x)p(x)\Big]_0^1 - \int_0^1 p(x)df(x). \tag{1}$$

From (1) we have

$$\int_0^1 x^{n+1}dp(x) = x^{n+1}p(x)\Big]_0^1 - \int_0^1 p(x)(n+1)x^n dx.$$

Hence

$$m_{n+1} = 1 - (n+1)M_n. \tag{2}$$

For this "middle-third" function $p(x)$, as a special case, we shall derive the formula

$$2M_n = \frac{1}{n+1} + \frac{1}{3^{n+1}-1}\left\{(2+M)^n - M_n\right\},$$

expressing in symbolic form a binomial development in which M^k stands for M_k, and M^0 for M_0 instead of 1.*

2. The class of functions $p(x)$. As a slight generalization, we consider the functions $p(x)$ in which the number 3 of §1 is replaced by an a, rational or irrational, with $a>1$.

From $0\leqq x\leqq 1$ we take the middle open interval of length $d_1=1/a$, and from the two remaining closed intervals, each of length $f_1=(1/2)(1-1/a)$, the middle open intervals of length $d_2=\left\{(a-1)/2a\right\}(1/a)$, and so on successively, from each of the closed intervals of length f_k the middle open portions of length $d_{k+1}=f_k/a$ thus, at stage k,

$$d_k = \left(\frac{a-1}{2a}\right)^{k-1}\frac{1}{a}, \qquad f_k = \left(\frac{a-1}{2a}\right)^k.$$

Since there are 2^{k-1} open intervals taken in passing from stage $k-1$ to stage k, the total of lengths d_k, $k=1, 2, 3, \cdots$, is

$$\sum_1^\infty 2^{k-1}d_k = 1.$$

The union of the open intervals of lengths d_1, d_2, $\cdots$ constitute the set E, and the intersection of the closed intervals (adding the points $x=0$, $x=1$) the set F, which is the complement of E.

The function $p(x)$ is then defined in complete analogy with the function of §1: first on the points of E, as $1/2$ on the central interval of stage 1, as $1/4$ and

* This MONTHLY (abstract) vol. 58, 1951, p. 593.

$3/4$ on those of stage 2, and so on, and then extended to the points of F, to obtain finally a monotonic increasing function, continuous for $0 \leqq x \leqq 1$.

We find that $p(x)$ has the following properties:

$$p(x) = \frac{1}{2} + p\left(x - \frac{a+1}{2a}\right), \qquad \frac{a+1}{2a} \leqq x \leqq 1, \tag{3}$$

$$p(x) = 2p\left(\frac{a-1}{2a}x\right), \qquad 0 \leqq x \leqq 1, \tag{4}$$

$$p(x) + p(1-x) = 1, \qquad 0 \leqq x \leqq 1. \tag{5}$$

Of these, (5) is immediate on account of the symmetry of the construction, and (3) is a statement that starting from the right-hand end point $x = 1/2 + 1/2a$, of the interval of stage 1 of E, the values assigned to $p(x)$ to the right are $1/2$ plus the values assigned to the right of $x = 0$.

The relation (4) also is made clear by the method of construction.* The restriction of the function $p(x)$ to the interval $0 \leqq x \leqq 1/2 - 1/2a$ is a scale model of $p(x)$ on $0 \leqq x \leqq 1$, with the x-axis in the scale $(a-1)/2a$: 1 and the y-axis in scale 1:2. Relation (4) is allied to the determination of the metrical dimension of F.†

3. Calculation of M_n, m_n. We use (3) and (4) to calculate M_n. We have

$$M_n = \int_0^1 x^n p(x)dx = \int_0^{(a-1)/2a} x^n p(x)dx + \int_{(a+1)/2a}^1 x^n p(x)dx + \int_{(a-1)/2a}^{(a+1)/2a} \frac{1}{2} x^n dx.$$

where by (3)

$$\int_{(a+1)/2a}^1 x^n p(x)dx = \int_{(a+1)/2a}^1 \frac{1}{2} x^n dx + \int_{(a+1)/2a}^1 x^n p\left(x - \frac{a+1}{2a}\right) dx.$$

By means of changes of variable, $t = x - (a+1)/2a$ in the last integral above, and $t = x$ in the first integral in the expression for M_n, we have

* The following sentence is a substitution, suggested by the referee, for a longer proof.

† Physical dimension is an analysis of the problem of change of units. Let new units of length, time and mass have the measures l, k, and n respectively in terms of given units. If the measure of a quantity in terms of the new units is $l^{-r}k^{-s}n^{-t}$ times its measure in terms of the old ones, the quantity has dimensions r, s, t respectively in length, mass and time. A generalization of this idea in terms of measure or mass of sets leads to the concept of metric, or Hausdorff, dimension. One sees that since "half" the set F lies on a portion $(a-1)/2a$ of the original interval, the metric dimension D of F is log 2 to the base $2a/(a-1)$. For $a=3$, $D = \log_3 2$; for $a=2$, $D=\frac{1}{2}$.

The easiest way to make the procedure legitimate is to define a mass measure of F. Let there be a uniform distribution of unit mass on $0 \leqq x \leqq 1$. At the first stage move this mass from the central interval of length $1/a$, half to be distributed uniformly on each of the adjacent intervals, constituting F_1, and continue this process indefinitely, in the stage m moving from a central interval of F_{m-1} half the mass uniformly to each of the contiguous intervals of F_m. Thus in the limiting situation the mass on the set $x_1 \leqq x \leqq x_2$ is $p(x_2) - p(x_1)$. Then (4) may be used.

In more general cases one may use methods associated with Vitali's theorem. For a discussion of metric dimension see Hurewicz and Wallman, Dimension Theory, Princeton, 1948, Chapter VII.

$$M_n = \int_0^{(a-1)/2a} \left\{ t^n + \left(t + \frac{a+1}{2a} \right)^n \right\} p(t)dt + \frac{1}{2} \int_{(a-1)/2a}^{1} x^n dx$$

$$= \frac{a-1}{2a} \int_0^1 \left(\frac{a-1}{2a} \right)^n \left\{ x^n + \left(x + \frac{a+1}{a-1} \right)^n \right\} p\left(\frac{a-1}{2a} x \right) dx$$

$$+ \frac{1}{2(n+1)} \left\{ 1 - \left(\frac{a-1}{2a} \right)^{n+1} \right\}.$$

By (4),

$$p\left(\frac{a-1}{2a} x \right) = \frac{1}{2} p(x).$$

With this substitution, and a rearrangement which moves the terms in x^n of the expansion in the right member into the left hand member, we obtain the equation

$$\left\{ \left(\frac{2a}{a-1} \right)^{n+1} - 1 \right\} M_n$$

$$= \frac{1}{2(n+1)} \left\{ \left(\frac{2a}{a-1} \right)^{n+1} - 1 \right\} + \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{a+1}{a-1} \right)^{n-k} \int_0^1 x^k p(x) dx.$$

Thus finally,

$$2M_n = \frac{1}{n+1} + \frac{1}{\left(\frac{2a}{a-1} \right)^{n+1} - 1} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{a+1}{a-1} \right)^{n-k} M_k. \tag{6}$$

This formula may be written in the form

$$2M_n = \frac{1}{n+1} + \frac{1}{\left(\frac{2a}{a-1} \right)^{n+1} + 1} \left\{ \left(M + \frac{a+1}{a+1} \right)^n - M_n \right\}, \tag{7}$$

where in the binomial development M^k stands for M_k and M^0 for M_0 instead of 1, as in the special case $a=3$ at the end of §1. The values m_n are then given by (2).

In particular, we find

$$M_0 = \frac{1}{2}, \qquad M_1 = \frac{1}{2} \frac{2a-1}{3a-1}, \qquad M_2 = \frac{9a-5}{12(3a-1)},$$

and from the fact that $p(0)=0$, $p(1)=1$,

$$m_0 = 1.$$

4. The symmetry relation (5). In the above calculation it has not been necessary to use (5). By means of it we may write $M_n=\int_0^1 x^n p(x)dx$, after change of variable $t=1-x$, in the form

$$M_n = \int_0^1 (1-t)^n p(1-t)dt = \int_0^1 (1-t)^n\{1-p(t)\}dt$$
$$= \int_0^1 (1-t)^n dt - \int_0^1 (1-t)^n p(t)dt.$$

In this way we have

$$\begin{aligned} M_n &= \frac{1}{n+1} + \sum_{k=0}^{n} (-1)^{k+1}\binom{n}{k} M_k, \\ M_n(1+(-1)^n) &= \frac{1}{n+1} + \sum_{k=0}^{n-1} (-1)^{k+1}\binom{n}{k} M_k, \end{aligned} \tag{8}$$

valid whenever (5) is valid. From (8), which does not involve a, the moments M_n of even subscripts are determined successively in terms of those of odd subscript. It may be used as a check on (6). Also, since the M_n itself is eliminated when n is odd, there are two expressions involving the same moments.

The three relations (3), (4), (5), with the requirement of continuity do not determine the function $p(x)$ uniquely, because there remains the possibility of assigning on the middle interval, of length $1/a$, other values than $y=1/2$. Evidently (3) and (5) do not imply (4). On the other hand, although (3) is convenient to use, it is a consequence of (4) and (5).

In fact, by application in succession of (4), (5), (4), and (5), with $(a+1)/2a \leqq x \leqq 1$, we have

$$\frac{1}{2}+p\left(x-\frac{a+1}{2a}\right) = \frac{1}{2}+\frac{1}{2}\,p\left(\frac{2a}{a-1}\,x-\frac{a+1}{a-1}\right),$$
$$0 \leqq x-\frac{a+1}{2a} \leqq \frac{a-1}{2a}, \quad 0 \leqq \frac{2a}{a-1}\,x-\frac{a+1}{a-1} \leqq 1,$$
$$= 1-\frac{1}{2}\,p\left(\frac{2a}{a-1}(1-x)\right), \qquad 0 \leqq \frac{2a}{a-1}(1-x) \leqq 1,$$
$$= 1-p(1-x), \qquad 0 \leqq 1-x \leqq \frac{a-1}{2a},$$
$$= p(x).$$

The ranges of the variables are noted in order to make clear the geometry of the transformations.

SOLUTION OF A RANKING PROBLEM FROM BINARY COMPARISONS

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1. Introduction. Occasionally one is faced with the problem of ranking a collection of objects on the basis of a number of binary comparisons, where these have not been, nor cannot be, chosen in a manner desirable to the experimenter. The standard procedures (*e.g.*, p. 217 of [1]) seem to require usually that the number of comparisons between any given pair be equal to the number between any other pair; for one reason or another it may be quite difficult in practice to achieve this equality for a suitably large sample size. For example, if one were attempting to rank various makes of automobiles by obtaining paired comparisons only from persons who have actually owned both makes one sees easily that it might be extremely difficult to obtain a large number of comparisons for such pairs as Cadillac *versus* Crosley. On the other hand, Chevrolet *versus* Ford comparisons would be available in large numbers.

The method proposed in this paper does not require any specific number of comparisons between pairs, and for this reason may have some application in contexts similar to the above. Another potential application is in ranking players of a two-person game on the basis of their records against each other. In collegiate football, for example, where matches are not scheduled in a manner suited to the experimenter, this technique could be used.

We assume as given a matrix $A=(a_{ij})$, where a_{ij} represents the number of times object i has been preferred to object $j(a_{ii}=0)$. The approach we shall use is that of maximum likelihood. We associate with the ith object a weight w_i. These weights will be interpreted as odds, in the sense that the probability of i being preferred to j in a future comparison will be taken to be $w_i/(w_i+w_j)$.

With these probabilities we may compute the *a priori* probability of obtaining precisely the matrix of results which we in fact did obtain (*i.e.*, the matrix A), under the assumption that each comparison that takes place between i and j is independent of all other comparisons, and is drawn randomly from a binomial distribution with $p=w_i/(w_i+w_j)$ and with the number of ij comparisons being required to be equal to $a_{ij}+a_{ji}$.

We shall then attempt to assign w_i that are positive and that maximize this probability. The function to be maximized is:

$$\Pr(A \mid w_1, \cdots, w_N) = \prod_{i<j} \binom{a_{ij}+a_{ji}}{a_{ij}} \left(\frac{w_i}{w_i+w_j}\right)^{a_{ij}} \left(\frac{w_j}{w_i+w_j}\right)^{a_{ji}}.$$

We shall actually solve the following equivalent problem: In the set $\{w_i>0; \sum w_i=1\}$, find values w_i that maximize

$$F_A(w) = \prod_{i<j} \left(\frac{w_i}{w_i+w_j}\right)^{a_{ij}} \left(\frac{w_j}{w_i+w_j}\right)^{a_{ji}}.$$

We shall make the following assumption about the matrix A:

ASSUMPTION. *In every possible partition of the objects into two nonempty subsets, some object in the second set has been preferred at least once to some object in the first set.*

This assumption is a requirement that one might reasonably expect. For if it is violated then there exist two subsets S_1 and S_2 with either no interset comparisons, or with all interset comparisons favoring S_1. In the former case one cannot expect to rank any object in S_1 against any object in S_2, since there is no basis for comparison. In the latter case the maximizing weights assigned to S_2 must turn out to be zero, since otherwise we could increase our function $F_A(w)$ by multiplying the S_1 weights by some common factor >1 and multiplying the S_2 weights by some common factor <1. These factors may be chosen to preserve a sum of 1. The factors of $F_A(w)$ involving ij both in S_1 or both in S_2 remain unchanged by this operation. However, the factors involving i in S_1 and j in S_2 increase, thereby increasing $F_A(w)$.

But if the S_2 weights must all be zero, we cannot hope to compare individual members of S_2 by this process.

The partition assumption is sufficient to establish the existence of a unique set of w_i in the region $\{w_i<0;\ \sum w_i=1\}$ which maximize $F_A(w)$. An equivalent form of the partition assumption which we shall find useful is the following: For each ordered pair ij there exists a sequence of indices $i_0, i_1, \cdots, i_n$ with $i_0=i$, $i_n=j$ and with $a_{i_k i_{k+1}}>0$ for $k=0, 1, \cdots, n-1$.

2. **Existence and uniqueness of the solution.** For the remainder of the paper, we shall assume that our result matrix $A=(a_{ij})$ satisfies the assumption of the preceding section.

We readily see that $F_A(w)$ is positive in the region $\{w_i>0;\ \sum w_i=1\}$. Thus to establish the existence of a maximum in the region it will suffice to show that defining $F_A(w)=0$ for w on the boundary gives a continuous extension of $F_A(w)$ to the closed region $\{w_i\geqq 0;\ \sum w_i=1\}$.

If $w^0=(w_1^0, \cdots, w_N^0)$ is on the boundary, then there is an index i such that $w_i^0=0$ and a j such that $w_j^0>0$. By our assumption on A we can find a sequence of indices $i_0, i_1, \cdots, i_n$ with $i_0=i$, $i_n=j$, and $a_{i_k i_{k+1}}>0$, $k=0, \cdots, n-1$. Clearly there are adjacent indices in the sequence, say m and k, with $w_m^0=0$, $w_k^0>0$, and $a_{mk}>0$. We may now write

$$F_A(w) = \left(\frac{w_m}{w_m + w_k}\right)^{a_{mk}} \cdot \phi(w),$$

where $0<\phi(w)\leqq 1$ in the interior. Hence $\lim_{w\to w^0} F_A(w)=0$ and continuity of the extension follows; hence also the existence of a maximum in the interior follows.

Taking logarithms and partial derivatives leads to the following system of equations which w must satisfy at a maximum point:

$$W_i = \sum_{j=1}^{N} \frac{A_{ij} w_i}{w_i + w_j}, \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $W_i = \sum_j a_{ij}$ and $A_{ij} = a_{ij} + a_{ji}$. Let $\bar{w}$ be any interior point satisfying equations (1). Define a new matrix $C = (c_{ij})$ by $c_{ij} = A_{ij}\bar{w}_i/(\bar{w}_i + \bar{w}_j)$. Using the matrix C we may compute Υ_i and C_{ij} (corresponding to W_i and A_{ij}) and we see that

$$\begin{aligned} \Upsilon_i &= \sum_j c_{ij} = \sum_j \frac{A_{ij}\bar{w}_i}{\bar{w}_i + \bar{w}_j} = W_i, \\ C_{ij} &= c_{ij} + c_{ji} = A_{ij}\bar{w}_i/(\bar{w}_i + \bar{w}_j) + A_{ji}\bar{w}_j/(\bar{w}_j + \bar{w}_i) = A_{ij}. \end{aligned} \tag{2}$$

However, note also that we may write, using relations (2),

$$F_A(w) \equiv \frac{\prod_i w_i^{W_i}}{\prod_{i<j} (w_i + w_j)^{A_{ij}}} \equiv \frac{\prod_i w_i^{\Upsilon_i}}{\prod_{i<j} (w_i + w_j)^{C_{ij}}} \equiv F_C(w), \tag{3}$$

so that $F_A(w) \equiv F_C(w)$ over the entire region. We shall now need the following result.

LEMMA. *If l_{ij} and l_{ji} are both positive, then*

$$\left(\frac{w_i}{w_i + w_j}\right)^{l_{ij}} \left(\frac{w_j}{w_i + w_j}\right)^{l_{ji}}$$

has a maximum (unique up to a proportionality factor) at the values $w_i = K \cdot l_{ij}$, $w_j = K \cdot l_{ji}$.

Our assumption on the A matrix now implies that $F_C(w)$ has a unique maximum at $\bar{w}$, as follows. Suppose we assign the weight $\bar{w}_1$ to the first object. For the jth object we have a sequence $i_0, i_1, \cdots, i_n$, with $i_0 = 1$, $i_n = j$, and $a_{i_k i_{k+1}} > 0$, $k = 0, 1, \cdots, n-1$. Hence, since $\bar{w}$ is an interior point, $c_{i_k i_{k+1}}$ and $c_{i_{k+1} i_k}$ are both positive. We proceed inductively along this sequence, assigning weights *via* the above lemma to maximize the factor of the form

$$\left(\frac{w_{i_k}}{w_{i_k} + w_{i_{k+1}}}\right)^{c_{i_k i_{k+1}}} \left(\frac{w_{i_{k+1}}}{w_{i_k} + w_{i_{k+1}}}\right)^{c_{i_{k+1} i_k}}$$

at each stage. We see that having assigned $w_{i_k} = \bar{w}_{i_k}$ we must assign $w_{i_{k+1}} = \bar{w}_{i_{k+1}}$ in order to maximize. Hence w_j must be taken to be $\bar{w}_j$, and thus uniqueness of the maximum of $F_C(w)$ at $\bar{w}$ follows.

3. The iteration and proof of convergence.* Since there appears to be no simple direct way of solving equations (1), we resort to the following iterative scheme:

* Standard conditions for the convergence of a scheme of this sort (*e.g.*, pp. 206–210 of [2]) are not satisfied in our case.

$$
\left.\begin{aligned}
w_i^{n+1} &= \frac{W_i}{\displaystyle\sum_j \frac{A_{ij}}{w_i^n + w_j^n}} \\
w_j^{n+1} &= w_j^n, \quad j \neq i.
\end{aligned}\right\} \quad i = 1, 2, \cdots, N \text{ cyclically.} \tag{4}
$$

Clearly, if we start with $w_i^0 > 0$ we retain this property in subsequent iterations. Furthermore, comparison of (4) with (1) indicates that if we can demonstrate convergence to an interior point (*i.e.*, a point in $\{w_i > 0;\ \sum w_i = 1\}$), the limit must be the desired solution. (Properly speaking, the limiting values will not satisfy $\sum w_i = 1$. Equations (1) are homogeneous of degree zero in the w_i, and we shall actually obtain convergence of the ratios of the w_i. We could normalize at each stage of the iteration; this was deemed unnecessary, since the w_i are important only up to a proportionality factor.)

Assume that we are about to apply (4) to compute a new w_i. The index i will remain fixed in what follows. Define

$$
\delta_i(w) = \frac{\partial \ln F_A(w)}{\partial w_i} = \frac{W_i}{w_i} - \sum_k \frac{A_{ik}}{w_i + w_k}, \tag{5}
$$

$$
D_i(w) = w_i \cdot \delta_i(w) = W_i - \sum_k A_{ik} \frac{w_i}{w_i + w_k}. \tag{6}
$$

From (6) we observe that $D_i(w)$ is monotone in w_i for $0 < w_i < \infty$. Hence if we can show that $\delta_i(w^n)$ and $\delta_i(w^{n+1})$ have the same sign, we may assert that this sign prevails along the entire line segment between w^n and w^{n+1}. For this purpose notice from (5) and (4) that

$$
\begin{aligned}
\delta_i(w^{n+1}) &= \frac{W_i}{w_i^{n+1}} - \sum_k \frac{A_{ik}}{w_i^{n+1} + w_k^{n+1}} \\
&= \sum_k \frac{A_{ik}}{w_i^n + w_k^n} - \sum_k \frac{A_{ik}}{\dfrac{W_i}{\displaystyle\sum_j \frac{A_{ij}}{w_i^n + w_j^n}} + w_k^n} \\
&= \sum_k A_{ik} \left\{ \frac{1}{w_i^n + w_k^n} - \frac{\displaystyle\sum_j \frac{A_{ij}}{w_i^n + w_j^n}}{W_i + w_k^n \displaystyle\sum_j \frac{A_{ij}}{w_i^n + w_j^n}} \right\} \\
&= \delta_i(w^n) \left\{ \sum_k \frac{A_{ik} w_i^n}{(w_i^n + w_k^n)\left(W_i + w_k^n \displaystyle\sum_j \frac{A_{ij}}{w_i^n + w_j^n}\right)} \right\}
\end{aligned}
$$

and thus that $\delta_i(w)$ does not change sign along the segment joining w^n to w^{n+1}.

Now we note from (4), (5), and (6) that

$$\delta_i(w^n) = \frac{[w_i^{n+1} - w_i^n]W_i}{w_i^{n+1} \cdot w_i^n} \tag{7}$$

and conclude from this, since our assumption guarantees that $W_i > 0$, that $\delta_i(w^n)$ is positive or negative according as $w_i^{n+1} - w_i^n$ is positive or negative. But now, using the fact that $\delta_i(w)$ is $\partial \ln F_A(w)/\partial w_i$ and does not change sign along the segment from w^n to w^{n+1}, we see that $F_A(w)$ is monotone non-decreasing as w_i moves from w^n to w^{n+1}, and is strictly monotone if $\delta_i(w^n) \neq 0$.

Since $F_A(w)$ is continuous and is equal to zero on the boundary, we see that the successive points arrived at under the iterations are bounded away from the boundary.

The iteration proceeds cyclically. Let w_0 be the initial normalized approximation. Let w_K be the normalized w-vector after K complete cycles from 1 to N. As we have just seen, $F_A(w)$ increases under an i change unless $\delta_i(w) = 0$, in which case none of the quantities involved is altered. Hence, if $F_A(w)$ is unchanged during a complete cycle, $\delta_i(w)$ must be zero for every i. However, $F_A(w_K)$ is monotone in K and bounded; hence $F_A(w_K)$ approaches a limit. Let $\bar{w}$ be any limit point of the w_K sequence.

Since the change in $F_A(w)$ during a complete cycle is a continuous function of w, the change in $F_A(\bar{w})$ must be zero, and thus $\delta_i(\bar{w}) = 0$, for all i. But this is precisely the statement that equations (1) are satisfied at $\bar{w}$. However, $\bar{w}$ was any limit point of w_K, and equations (1) have a unique solution in the interior.

Thus $w_K \to \bar{w}$ and the proof of convergence to the unique interior point satisfying (1) is complete.

4. Some properties and examples. We note the following: If each of the $(N^2 - N)/2$ possible comparisons has been made the same number of times (*i.e.*, $A_{ij} = Q$) then the ordering obtained from the weights is the same as that obtained from comparing the total percentages of favorable comparisons (the win-loss percentages).

For if $W_i > W_j$ then, at a solution,

$$0 < \sum_k \frac{Qw_i}{w_i + w_k} - \sum_k \frac{Qw_j}{w_j + w_k} = Q(w_i - w_j) \sum_k \frac{w_k}{(w_i + w_k)(w_j + w_k)},$$

and $w_i > w_j$ follows. This indicates, for example, that the system of ranking used in major-league baseball is in agreement with that proposed in this paper. In this case it should be noted, however, that the win-loss percentages are not in general a solution of the weight equations, but merely give the same ordering.

In case the A_{ij} are not constant, the example

$$A = \begin{bmatrix} 0 & 15 & 15 & 0 \\ 11 & 0 & 10 & 20 \\ 11 & 10 & 0 & 20 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

shows that the win-loss percentages do not necessarily represent the desired rankings:

Object	Win-Loss %	Weight
1	.576	.736
2	.611	.541
3	.611	.541
4	.047	.025

Here objects 2 and 3 have the highest win-loss percentages as a result of their many wins over the weak object 4; but object 1, which has consistently beaten objects 2 and 3, has the greatest weight.

References

1. Guildford, J. P., Psychometric Methods, McGraw-Hill Book Company, Inc., New York, 1936.

2. Scarborough, J. B., Numerical Mathematical Analysis, Johns Hopkins Press, Baltimore, 1950.

ON THE COMPUTATION OF CERTAIN ROOTS BY THE USE OF THE BINOMIAL SERIES*

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When the author was a student in Professor Ford's calculus class at the Rice Institute, the subject under discussion on one occasion was that of finding roots —in this instance, cube roots—of numbers by using the binomial expansion. The class was invited to propose numbers of which to compute the cube root, and the number 37 was suggested. A student's slide rule gave the approximate value 3 1/3 for the cube root of 37, and this was taken as point of departure for the binomial expansion:

$$
\begin{aligned}
\sqrt[3]{37} &= \frac{10}{3}\sqrt[3]{37\cdot\left(\frac{3}{10}\right)^3} = \frac{10}{3}\sqrt[3]{\frac{37\cdot 27}{1000}} \\
&= \frac{10}{3}\sqrt[3]{\frac{999}{1000}} = \frac{10}{3}\sqrt[3]{1-\frac{1}{10^3}} \\
&= \frac{10}{3}\left\{1 - \frac{1}{3\cdot 10^3} - \frac{1\cdot 2}{3^2\cdot 10^6\cdot 2!} - \frac{1\cdot 2\cdot 5}{3^3\cdot 10^9\cdot 3!} - \cdots\right\}.
\end{aligned}
$$

The series is a delight to compute and converges very rapidly; the first two terms give

$$\sqrt[3]{37} = 3.33222\cdots,$$

correct to five decimal places.

Presumably, the student's happy choice of the number 37 was a fortuitous one. It appears of some interest, if only pedagogical, to look into the reason why things worked out like they did, and to look for further examples of this sort of phenomenon. "This sort of phenomenon" will be defined as the occurrence of an integer K for which

$$\sqrt[3]{K} = \frac{p}{q}\sqrt[3]{1 \pm \frac{1}{10^n}}, \tag{1}$$

p, q, and n being integers.

To simplify the analysis, consider first the analogous problem for the square root:

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$$\sqrt{K} = \frac{p}{q}\sqrt{1 \pm \frac{1}{10^n}}\,. \tag{2}$$

We may assume without loss of generality that p and q have no other common divisor than 1 ($(p, q) = 1$) and that K is square free. From (2) it follows that

$$q^2K = p^2 \pm \frac{p^2}{10^n}\,. \tag{3}$$

Case 1. $n = 2m$ is even.

Clearly $p^2/10^n = p^2/10^{2m}$ is an integer, and, since it is the square of a rational number, it is the square r^2 of an integer r:

$$q^2K = p^2 \pm r^2. \tag{4}$$

From (4) one sees that any factor common to q and r would be a factor of p, and hence $(q, r) = 1$. From (3) and $p^2 = r^2 \cdot 10^{2m}$, it follows that

$$q^2K = r^2(10^{2m} \pm 1).$$

Since r^2 divides q^2K and $(K, q) = 1$, r^2 divides K. Since K has no repeated factors except 1, we conclude that $r = 1$ and

$$q^2K = 10^{2m} \pm 1. \tag{5}$$

Case 2. $n = 2m+1$ is odd.

In this case $p^2/10^{2m+1} = S$, an integer. Then $p^2/10^{2m} = 10S = r^2$, as in Case 1. Now the square of an integer r can not terminate in 0 unless r terminates in 0; thus $r = 10t$, and $p^2/10^{2m+1} = 10t^2$. We note that $(q, t) = 1$ and substitute for p^2 in (3). This gives

$$q^2K = 10t^2(10^{2m+1} \pm 1).$$

As in Case 1, $t = 1$ and we obtain

$$q^2K = 10(10^{2m+1} \pm 1). \tag{6}$$

From (5) and (6) it is evident that the phenomenon in which we are interested can occur when and only when $10^n \pm 1$ has a repeated factor q, and therefore to find examples of it, we must find n such that $10^n \pm 1$ contains a repeated factor.

For the cube root case, the conclusion is analogous. The situation described in (1) can occur only if $10^n \pm 1$ has a twice repeated factor. The case mentioned at the beginning of this article is that corresponding to $n = 3$; $10^3 - 1 = 3^3 \cdot 37$.

There appears to be no general method of factoring numbers of the form $10^n \pm 1$. One would not expect there to be, considering the similarity of these numbers to the Mersenne numbers. Therefore, examples of the phenomena we are seeking may be expected to be hard to come by. A considerable amount of attention, however, has been given to their factorization for special values of n,

and a number of tables exist giving their factors for small n. (See Lehmer, [2].) The author had available the table of Cunningham and Woodall, [1] which gives, among other things, all factors of $10^n \pm 1$ for $n \leqq 20$ except the case $10^{19}+1$.

Examination of this table reveals very few examples, and makes the student's choice of 37 appear even more astonishingly fortunate. In fact, for $n \leqq 20$, no other examples at all of triple factors of $10^n \pm 1$ appear, except for the obvious cases $n=3k$, which again have the factor 27. The case $n=6$ gives

$$\sqrt[3]{37037} = \frac{100}{3}\sqrt[3]{1-\frac{1}{10^6}},$$

and this is the smallest number except 37 for which (1) has been found to hold.

The square root problem also has few examples. Of course, for every n, 10^n-1 is divisible by 9, so there are a number of more or less equivalent examples:

$$\sqrt{11} = \sqrt{\frac{99}{9}} = \frac{10}{3}\sqrt{1-\frac{1}{10^2}},$$

$$\sqrt{1110} = \frac{100}{3}\sqrt{1-\frac{1}{10^3}},$$

etc. The only example of repeated roots other than 3 occurred in the case $n=11$,

$$10^{11} + 1 = 11^2 \cdot 826446281,$$

which gives the example

$$\sqrt{8264462810} = \frac{10^6}{11}\sqrt{1+\frac{1}{10^{11}}}\,.$$

To obtain further examples requires factorization of further large numbers. However, it may be possible to show by general methods of number theory that certain K may *not* be expressed in forms (1) or (2). For example, in the square root case, simple considerations show that any K satisfying (2) must terminate in digits 1 or 9 for the case n even, or in the digits 10 or 90 in the case n odd. It would be interesting to know that those K expressible in the forms (1) or (2) are really as rare as they seem to be.

Bibliography

1. A. J. C. Cunningham and H. J. Woodall, Factorization of $(y^n \pm 1)$, London, 1925.
2. D. H. Lehmer, Guide to Tables in the Theory of Numbers, Washington, D. C., 1941.

AN EXCERPT FROM THE WORKS OF EULER

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Among the phenomena that dramatically accent the history of mathematics is the abundance of genius that trod the stage of the eighteenth century. The era immediately preceding that had marked the birth of the great key ideas that unlocked the door to the mathematics of modern times. Mathematical analysis had suddenly been endowed with power and dependability beyond all that had previously been suspected. Seemingly limitless fields for development had been revealed all around, and opportunities for triumphs of ingenuity and invention were abundantly uncovered. The stimulus was matched by the response. Genius, ordinarily so rare, cropped up generously. Great masters of mathematical analysis were contemporaries. In many respects the greatest of these was Leonhard Euler.

Euler needed no throne to sit upon, nor any army to command, to leave his mark upon history. In an age which accorded mathematical ability the primacy as a mark of human greatness, he was the pre-eminent mathematician. Researches and books flowed from his pen in incredible volume. He was a superlative expositor, and an openhearted soul who corresponded—in Latin, German, French and Russian—with practically all of his great contemporaries. Adulation was his in overflowing measure, but left no tarnish upon him. A modest and generous personality, he was content to be a burgher, the jolly father of thirteen children. His afflictions, heavy though they were, did not embitter him. To his death, at the age of seventy-seven, the inspired enthusiasm of the explorer stayed with him. He enjoyed the discoveries of others as he did his own.

Euler was born in 1707. He was therefore nine when Leibniz died, and twenty when Newton followed. His early manhood came less than half a century after the invention of the calculus. For one of Euler's type of genius no time could have been better. Rich in his instinct for the mathematical character of a physical phenomenon, he was seemingly inexhaustible in his power for formulation and invention. With magnificent skill as a mathematical technician he combined an enduring memory for details in all their minuteness and a versatility of interests beyond that of any other. He has been called "the myriad-eyed Euler," and the appellation is fit, notwithstanding the fact that he worked during the last seventeen years of his life in total blindness.

It is distinctive of any epoch in which mathematical fields for new discovery are at hand, that the explorer is prone to hasten on, even when the advance must be made over ground that is only dubiously secure. The attention is then more readily given to the finding of new formulas and solutions, than to the slower and more painstaking reflections which are requisite to the construction of rigorous proofs. We say such a time is one of mathematical *formalism*. Euler's was such a time, and Euler was a characteristic formalist. His work, therefore, radiant

of ingenuity and imagination as it is, is also often naive and logically uncompelling. Astonishingly often he breaks out new paths and shows us the way, but fails to prove that the way is there with the conclusiveness that we demand in our time.

I propose, in the following pages, to present one of Euler's derivations which is typical in all these respects, and which shows his genius for solving a difficult problem by elementary means. My presentation will not be a facsimile. I have "edited" it rather drastically, but, I believe, without impairing either its essential spirit or content. The original is to be found in a longish memoir written in Latin [1], where it is imbedded in considerable discursive material. It has been noted that by this derivation Euler came to the brink of one of the greatest of mathematical discoveries, one that was made years later by Fourier. I shall point this out in its proper place.

There are two mathematical relations that become familiar even to college freshmen, for which we are indebted to Euler, and which have been called the "heroes" of his works. These are, namely, the definition of the exponential function which is basic to the elementary calculus

$$e^x = \lim_{n\to\infty}\left\{1+\frac{x}{n}\right\}^n, \tag{1}$$

and the cardinal formula of analytic trigonometry

$$e^{ix} = \cos x + i \sin x. \tag{2}$$

In what follows, these heroes may be seen in action.

The problem Euler set himself is the following one:

Given an arbitrary periodic function $f(x)$ with the period 1, to find a function $y(x)$ which is changed by the amount $f(x)$ when the variable is increased from x to $x+1$.

In symbols, this is a call for a solution of the *difference equation*

$$y(x+1) = y(x) + f(x). \tag{3}$$

Euler bases his attack upon the formula

$$F(a+h) = F(a) + F'(a)h + \frac{1}{2!}F''(a)h^2 + \frac{1}{3!}F'''(a)h^3 + \cdots$$

namely, upon the Taylor's series, which had been published in 1715. With F, a and h, taken respectively to be y, x and 1, this gives

$$y(x+1) = y(x) + y'(x) + \frac{1}{2!}y''(x) + \frac{1}{3!}y'''(x) + \cdots,$$

and this, by comparison with the equation (3), shows that the function $y(x)$ sought is one for which

$$y'(x) + \frac{1}{2!}y''(x) + \frac{1}{3!}y'''(x) + \cdots = f(x). \tag{4}$$

In other words, what is wanted is a solution of the equation (4), an ordinary linear differential equation of infinite order.

It is common practice in dealing with ordinary differential equations to symbolize the kth derivative of y by $D^k y$. The equation (4) is thereby given the form

$$\left\{D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right\}y = f(x),$$

and this is unambiguously indicated by writing it

$$\{e^D - 1\}y = f(x), \tag{5}$$

since for any value of z we have the power series expansion

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots.$$

Euler, therefore, seeks a solution to the symbolically expressed equation (5).

His appeal at this point is to his hero relation (1). In accordance with that the equation (5) is, in other terms,

$$\lim_{n\to\infty}\left\{\left(1 + \frac{D}{n}\right)^n - 1\right\}y = f(x).$$

He is led by this to explore the ground by considering the "approximating" equation

$$\left\{\left(1 + \frac{D}{n}\right)^n - 1\right\}u = f(x), \tag{6}$$

in which n is a specific but unspecified integer. He finds it convenient to take this integer to be odd, and since the purpose is only one of exploration, nothing is to be lost by taking it so. Euler's thought in this connection was, of course, the following. Whatever the integer n, the equation (6) is linear and has constant coefficients. It is therefore of an elementary type whose solution can be explicitly written out. By observing how that solution "matures" when n is permitted to become infinite, one should be able to discern the character of a solution of the equation (5).

The common method of dealing with an equation (6) is, first, to write it as

$$u = \frac{1}{\left(1 + \frac{D}{n}\right)^n - 1}f(x),$$

then, to treat the multiplier of $f(x)$ in this as an algebraic fraction in D, and as such to resolve it into partial fractions:

$$\frac{1}{\left(1+\dfrac{D}{n}\right)^n - 1} = \sum_{j=1}^{n} \frac{c_j}{D - a_j}. \tag{7}$$

Thereby the equation (6) is made to appear as

$$u = \sum_{j=1}^{n} \frac{c_j}{D - a_j} f(x),$$

and a solution of it is accordingly

$$u(x) = \sum_{j=1}^{n} u_j(x), \tag{8}$$

where the terms $u_j(x)$ are respectively solutions of the equations

$$u_j(x) = \frac{c_j}{D - a_j} f(x),$$

which is to say of the first order differential equations

$$\{D - a_j\} u_j = c_j f(x). \tag{9}$$

To carry this program out the constants a_j and c_j must be determined. We proceed to do that.

The constants a_j are the values of D for which the denominator of the left-hand member of the equation (7) is zero, namely are the roots of the equation

$$\left(1+\frac{D}{n}\right)^n = 1. \tag{10}$$

Now since $\cos 2\pi x$ and $\sin 2\pi x$ have the values 1 and 0 respectively when x has an integral value k, Euler's relation (2) shows that

$$e^{2k\pi i} = 1,$$

and hence that the equation (10) can be written as

$$\left(1+\frac{D}{n}\right)^n = e^{2k\pi i}.$$

On dividing the exponents by n and solving for D, we find as the roots the values

$$n\{e^{2k\pi i/n} - 1\}.$$

Precisely n of these are distinct. We may label them thus:

$$
\begin{aligned}
a_1 &= 0, \\
a_{2j} &= n\left\{ e^{2j\pi i/n} - 1 \right\}, \\
a_{2j+1} &= n\left\{ e^{-2j\pi i/n} - 1 \right\},
\end{aligned}
\qquad j = 1, 2, \cdots, \frac{n-1}{2}. \tag{11}
$$

The method for determining the constants c_k is a standard one. Because the relation (7) is conceived of as an identity in D, it must continue to hold when D is varied in any way. If we first multiply it by $D - a_k$ and then let D approach a_k as a limit, every term in the sum except that for which $j = k$ approaches zero. Thus we infer that

$$\lim_{D \to a_k} \frac{D - a_k}{\left(1 + \dfrac{D}{n}\right)^n - 1} = c_k.$$

Hereby c_k is given as the limiting value of an indeterminate fraction of the 0/0 type. The method to evaluate this is to replace the numerator and the denominator of the fraction by their derivatives with respect to D before proceeding to the limit. Thus

$$c_k = \lim_{D \to a_k} \frac{1}{\left(1 + \dfrac{D}{n}\right)^{n-1}} = \frac{1}{\left(1 + \dfrac{a_k}{n}\right)^{n-1}} \cdot$$

Since a_k is a root of the equation (10), we can write the result

$$c_k = 1 + \frac{a_k}{n} \cdot \tag{12}$$

With this, Euler's exploratory base is completely laid. The values (11) and (12) make the equations (9) specific, and in terms of their solutions the function $u(x)$ fulfilling the equation (6) is given by the formula (8). He is ready now to let n become infinite, and to observe the resultant metamorphosis of the solution into one for the equation (5).

For each k

$$\lim_{n \to \infty} e^{2k\pi i/n} = e^0 = 1.$$

The formulas (11) for a_{2j} and a_{2j+1} are therefore indeterminate of the $\infty \cdot 0$ type as $n \to \infty$. On setting $n = 1/s$ we may evaluate their limits thus

$$
\begin{aligned}
\lim_{n \to \infty} a_{2j} &= \lim_{s \to 0} \frac{e^{2j\pi i s} - 1}{s} = 2j\pi i, \\
\lim_{n \to \infty} a_{2j+1} &= \lim_{s \to 0} \frac{e^{-2j\pi i s} - 1}{s} = -\,2j\pi i.
\end{aligned} \tag{13}
$$

From (12), therefore, $\lim c_k = 1$, and thus the formulas (8) and (9) transform into

$$y(x) = \sum_{j=1}^{\infty} y_j(x), \tag{14}$$

with

$$\begin{aligned} \{D - 2j\pi i\} y_{2j} &= f(x), \\ \{D + 2j\pi i\} y_{2j+1} &= f(x). \end{aligned} \tag{15}$$

We can rewrite (14) as

$$y(x) = y_1(x) + \sum_{j=1}^{\infty} \{y_{2j}(x) + y_{2j+1}(x)\}, \tag{16}$$

and integrate the equations (15) to obtain

$$y_{2j}(x) = \int_a^x f(t) e^{2j\pi i(x-t)} dt,$$

$$y_{2j+1}(x) = \int_a^x f(t) e^{-2j\pi i(x-t)} dt,$$

where a is any convenient constant.

Then, since Euler's relation (2) shows that

$$e^{2j\pi i(x-t)} + e^{-2j\pi i(x-t)} = 2 \cos 2j\pi(x - t),$$

we find that the formula (16) shapes up to

$$y(x) = \int_a^x f(t) dt + \sum_{j=1}^{\infty} 2 \int_a^x f(t) \cos 2j\pi(x - t) dt. \tag{17}$$

This is the elegant formula that Euler has derived for a solution of the difference equation (3).

Imaginative the deduction certainly is. As a proof of its conclusion we find it deficient, especially in a point of reasoning which was generally overlooked in Euler's time. The issue lodges in the passage from a finite n to the infinite, and was clearly recognized only about a century later. In the present instance it is involved in the passage from the equations (8) and (9) to the corresponding ones (14) and (15). This passage was based upon the limiting relations (13).

The correctness of the evaluations (13) for any specific integer j is unquestionable. This means that the values a_{2j} and a_{2j+1} can be made to approximate with greater and greater accuracy $2j\pi i$ and $-2j\pi i$ respectively, and hence that $u_j(x)$ approximates $y_j(x)$ more and more closely, as n is taken larger and larger. But by taking n so, more and more terms are brought into the sum (8). To make these terms good approximations in their turn, more terms again must be

brought in, and the process is thus unending. It is therefore not possible to conclude that because any term $u_j(x)$ can be made to approach $y_j(x)$ as a limit, therefore the whole sum of the $u_j(x)$ approaches the infinite series of the $y_j(x)$.

The oversight of this important logical point mars many mathematical works of the eighteenth century. This does not signify that the works are valueless or uninteresting. In many cases the reasoning can be made rigorous, and even aside from that the deductions often show clearly the character of important presumptive results. Inconclusive, or even erroneous, reasoning by no means implies a wrong conclusion. A quotation on this point is appropos [2]. In his *The Development of Mathematics*, E. T. Bell says:

"This raises an extremely interesting question: how did the master analysts of the eighteenth century—the Bernoullis, Euler, Lagrange, Laplace—contrive to get consistently right results in by far the greater part of their work in both pure and applied mathematics . . . [when what they] mistook for valid reasoning . . . is now universally regarded as unsound? No short answer is possible; but history shows that frequently the essential usable part of a mathematical doctrine is grasped intuitively long before any rational basis is provided for the doctrine itself. The creative mathematicians between Newton and Cauchy obtained mostly correct results . . . because . . . they had instinctively apprehended the self-consistent part of their mathematics."

He continues then, lest we rate our own sophistication too highly:

"Just as no short answer can dispose of our predecessor's good fortune, so none can dispose of ours. Like them, we consistently get meaningful results, although we realize that there is much obscurity in the foundations of our own analysis."

Criticisms of Euler's deduction, other than the one we have enlarged upon, have by no means been wanting. Such were made even contemporaneously by d'Alembert. Nevertheless the formula (17) is valid when the function $f(x)$ is possessed of certain properties. The ways by which that has been most generally established have taken the formula itself as their point of departure, ignoring how it was come by, and shown that it does give a solution of Euler's problem.

It has been observed [3]—and we include this observation by way of concluding this presentation—that the formula (17) lies only a step from one of the greatest deductions of mathematical analysis. Euler did not take this step, most probably because he had no motivation for doing so. It remained for Fourier to do so more than half a century after the date of Euler's work, namely in the early nineteenth century.

If in the formula (17) we replace x by $x+1$, and observe that the factor $\cos 2j\pi(x-t)$ is not thereby changed, we see that

$$y(x+1) = \int_a^{x+1} f(t)dt + \sum_{j=1}^{\infty} 2\int_a^{x+1} f(t) \cos 2j\pi(x - t)dt,$$

and hence that

$$(18) \qquad y(x+1) - y(x) = \int_x^{x+1} f(t)dt + \sum_{j=1}^{\infty} 2 \int_x^{x+1} f(t) \cos 2j\pi(x-t)dt.$$

By (3) the left-hand member of this equation is $f(x)$. In the right-hand member each integral has an integrand that is periodic with the period 1, and an interval of integration that is precisely such a period in length. The interval of integration can therefore be replaced by any other one of the same length, in particular by that from 0 to 1. Thus one finds that

$$f(x) = \int_0^1 f(t)dt + \sum_{j=1}^{\infty} 2 \int_0^1 f(t) \cos 2j\pi(x-t)dt.$$

When the cosine factors in this are expanded by the trigonometric addition formula, the result reads

$$(19) \qquad f(x) = \frac{A_0}{2} + \sum_{j=1}^{\infty} \{A_j \cos 2j\pi x + B_j \sin 2j\pi x\},$$

with

$$(20) \qquad \begin{aligned} A_j &= 2\int_0^1 f(t) \cos 2j\pi t dt, \\ B_j &= 2\int_0^1 f(t) \sin 2j\pi t dt. \end{aligned}$$

This is the Fourier expansion of the arbitrary function $f(x)$.

References

1. Euleri, L., De serierum determinatione seu nova methodus inveniendi terminos generalis serierum, Opera Omnia, Series Prima, xiv, p. 510.

2. Bell, E. T., The Development of Mathematics, New York, 1940.

3. Burkhardt, H., Entwickelungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik. Jahresbericht der Deutschen Mathematiker-Vereinigung, 10, 2, 1, Leipzig, 1908, p. 50.

QUANTUM MECHANICS AND HILBERT SPACE

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1. Introduction. This expository article is written in the spirit of the various "What is a ———?" articles that have appeared in this MONTHLY from time to time. It is an attempt to give mathematicians who have not studied quantum mechanics some idea of what it is about and to give those who have studied it only from the point of view of the physicists some idea of how it may be formulated† in a mathematically precise fashion. We shall avoid the difficulties that still plague the advanced portions of the subject by dealing only with nonrelativistic quantum mechanics.

2. The nature of quantum mechanics. The basic difference between classical mechanics and quantum mechanics is that the $2n$-tuple of real numbers which characterizes the state of a system in classical mechanics is replaced in quantum mechanics by a family of probability measures on the real line. The $2n$ first-order differential equations in $2n$-space which describe how the state of the classical system changes in time are replaced by a differential equation in an infinite-dimensional space which describes how these probability measures change with time. Under certain circumstances the probability measures are so highly concentrated (*i.e.*, assign a probability so close to one to such a short interval) that they behave like numbers. The resulting system of numbers then changes approximately according to the laws of classical mechanics. In this sense classical mechanics is a limiting case of quantum mechanics.

The passage from classical mechanics to the more exact quantum mechanics becomes necessary when one can no longer ignore the way in which different measurements interfere with one another—for example, when dealing with the submicroscopic world. Even in dealing with familiar large-scale phenomena this passage would become necessary if one were to make measurements to a much higher degree of accuracy than is now possible.

The reason that the above-mentioned interference forces a retreat from numbers to probability measures may be explained as follows. The act of measuring the value of an observable A may change the system in such a way that a measurement of B immediately following the measurement of A will not agree with a measurement of B immediately preceding the measurement of A. Moreover the change in the result of measuring B will be of an unpredictable nature. As analysis of the situation shows, this circumstance precludes the possibility of assigning any meaning to the statement "A and B have the values a and b at the time t." On the other hand it does not preclude the possibility of assigning

† While the formulation given here is based essentially on that to be found in von Neumann's classic book, Mathematische Grundlagen der Quantenmechanik, it differs from it in a number of respects. It is the writer's personal way of looking at the foundations of quantum mechanics and cannot be claimed to be the standard mathematical view if indeed there is such. For another approach the reader is referred to Chapter I of some recent University of Chicago notes by Irving Segal.

a meaning to the statement "At the time t measurements of A and B are statistically distributed with probability measures† α and β. This is because a statistical distribution is determined by making measurements on a sample containing a large number of replicas of the system under consideration and one may use distinct samples in determining α and β.

3. General mathematical formulation. Let $\mathcal{O}$ denote the set of all "observables" associated with a given physical system. By a *state* of our system we shall mean a *possible* simultaneous set of statistical distributions for the "observables" in $\mathcal{O}$. We denote the set of all states by $\mathcal{S}$. It will not be necessary for us to analyze the notions of state and observable. From a purely mathematical point of view $\mathcal{O}$ is an abstract set and each member of $\mathcal{S}$ is a function defined on $\mathcal{O}$ and having values in the set $\mathfrak{M}$ of all probability measures on the line. The system $\mathcal{O}$, $\mathcal{S}$ determines completely what we may call the statics of the system. To determine the dynamics we must specify how members of $\mathcal{S}$ change with time; that is we must determine for each real t the one-to-one transformation L_t of $\mathcal{S}$ onto $\mathcal{S}$ such that $L_t(\phi)$ is the state of the system a time t after it was in the state ϕ. Of course the family L_t need not be given directly in integrated form, but may be determined only by giving some sort of differential equation in t.

It should be noted that our formulation is still broad enough to include classical mechanics. In that case $\mathcal{O}$ is the set of all sufficiently regular real-valued functions defined on classical phase space $\mathcal{P}$ and the general member of $\mathcal{S}$ is defined by a probability measure on $\mathcal{P}$. The function from $\mathcal{O}$ to $\mathfrak{M}$ defined by the probability measure α in $\mathcal{P}$ takes the function f on $\mathcal{P}$ into the probability measure $E \rightarrow \alpha(f^{-1}(E))$. The L_t are obtained by integrating the classical equations of motion on $\mathcal{P}$ and using the resulting one parameter group of one-to-one transformations of $\mathcal{P}$ onto itself to induce a corresponding group of one-to-one transformations of $\mathcal{S}$ onto itself. Actually, of course, classical mechanics is usually concerned only with the special states defined by those probability measures which are concentrated in a single point of $\mathcal{P}$. Indeed the motion of these determines the motion of all of the others and it is for this reason that classical mechanics‡ need not concern itself with measures as such.

Given two states of a physical system one can consider a third state in which the system is in the first state with probability p and in the second with probability $1-p$. This leads us to make the following assumption about $\mathcal{S}$. If ϕ_1 and ϕ_2 are in $\mathcal{S}$ and $0 \leqq p \leqq 1$ then $p\phi_1+(1-p)\phi_2$ is also in $\mathcal{S}$. A state ϕ which can be so constructed from two other states is said to be a *mixed state.* A state which cannot be so constructed is said to be a *pure state.* When $\mathcal{O}$ and $\mathcal{S}$ are as in the preceding paragraph the pure states are obviously just those defined by probability measures concentrated in single points of $\mathcal{P}$. Physically speaking, a pure

† See the foundations of probability, by P. R. Halmos, this MONTHLY vol. 51, 1944, pp. 493-510.

‡ The word "need" here must be interpreted in a suitably restricted sense. Of course, it may still be quite important to discuss the motion of measures which are not concentrated in points—for example, in classical statistical mechanics.

state is one in which one cannot concentrate the probability measure for any observable without spreading out the probability measure for another. The characteristic features of quantum mechanics arise from the fact that the pure states are not trivially pure as they are in classical mechanics—for each pure state there are many observables with highly unconcentrated probability measures.

If f is a real-valued function of a real variable and A is an observable, there is an obvious physical sense in which one can construct an observable $f(A)$. A measurement of $f(A)$ is simply a measurement of A, giving the value a say, followed by a computation of $f(a)$. This leads us to make the following further assumption about the mathematical system $\mathcal{O}$, $\mathcal{S}$. For each $A\in\mathcal{O}$ and each real Borel function f on the real line there exists $B\in\mathcal{O}$ such that for every Borel set E and each $\phi\in\mathcal{S}$ $\phi(B)(E)=\phi(A)(f^{-1}(E))$. If we assume, as we shall, that no two distinct members of $\mathcal{O}$ are carried into the same probability measure by all members of $\mathcal{S}$, then B is uniquely determined by f and A. We denote it by $f(A)$.

4. The transition from observables to questions. Let us call an observable A a *question* if for every state ϕ the probability measure $\phi(A)$ is concentrated in the two points 0 and 1. It is not difficult to prove that A is a question if and only if $A^2=A$. If A is any observable and E is any Borel subset of the real line let ψ_E be the characteristic function of E; that is the function which is one for $x\in E$ and zero for x not in E. Then $\psi_E(A)$ is always a question which we shall denote by Q_E^A. The family of questions Q_E^A obtained by holding A fixed and letting E vary through the Borel subsets of the real line clearly determines the observable A. When Q is a question and ϕ is a state the probability measure $\phi(Q)$ is completely determined by its value at the set consisting of one alone. We shall call this value $m_\phi(Q)$. Since $m_\phi(Q_E^A)=m_\phi(\psi_E(A))=\phi(A)(\psi_E^{-1}(\{1\}))$ $=\phi(A)(E)$ we see that the state ϕ is completely determined by the values of m_ϕ at the questions in $\mathcal{O}$.

Let us denote the set of all questions in $\mathcal{O}$ by q. We partially order q by setting $Q_1\leqq Q_2$ whenever $m_\phi(Q_1)\leqq m_\varphi(Q_2)$ for all states ϕ. If Q_1 and Q_2 are questions such that $Q_1\leqq 1-Q_2$, we say that Q_1 and Q_2 are mutually exclusive and write $Q_1\perp Q_2$. If $Q_1, Q_2, \cdots$ are questions such that $Q_i\perp Q_j$ for $i\neq j$ there clearly exists at most one question Q such that $m_\phi(Q)=m_\phi(Q_1)+m_\phi(Q_2)+\cdots$ for all $\phi\in\mathcal{S}$. We assume that Q always exists and denote it by $Q_1+Q_2+\cdots$. It is not difficult to show† that Q is the least question R such that $R\geqq Q_j$ for all j.

If A is an observable then the mapping $E\rightarrow Q_E^A$ clearly has the following properties: (1) $E\cap F=0$ implies $Q_E^A\perp Q_F^A$, (2) $Q_{\{0\}}^A=0$ and $Q_{[-\infty,\infty]}^A=1$,‡ (3) $E=E_1\cup E_2\cup\cdots$ where $E_i\cap E_j=0$ for $i\neq j$ implies that $Q_E^A=Q_{E_1}^A+Q_{E_2}^A+\cdots$. If $E\rightarrow Q_E$ is any mapping of Borel sets in the line into questions which has the three properties just listed we shall call it a *question-valued measure*. Let $\bar{\mathcal{O}}$ denote the set of all question-valued measures. For each $\phi\in\mathcal{S}$ and $Q\in\bar{\mathcal{O}}$ let

† We are indebted to R. V. Kadison for this remark. In our original manuscript the fact in question appears as an added assumption.

‡ 1 is the unit constant observable, that is the observable A such that $\phi(A)$ is concentrated in the point 1 for all states ϕ.

$\phi(Q)(E)=m_\phi(Q_E)$. If we identify the members A of $\mathcal{O}$ with their associated question-valued measures Q^A so that we may consider $\mathcal{O}$ as a subset of $\bar{\mathcal{O}}$, then the preceding definition amounts to an extension of each ϕ from $\mathcal{O}$ to $\bar{\mathcal{O}}$. The system $\bar{\mathcal{O}}$, $\mathcal{S}$ then satisfies all axioms already laid down for $\mathcal{O}$, $\mathcal{S}$, and in addition the following: Every question-valued measure is the question-valued measure associated with some observable. We shall assume that we have added the observables in $\bar{\mathcal{O}}-\mathcal{O}$ to our system, that is, that our system $\mathcal{O}$, $\mathcal{S}$ satisfies the axiom just enunciated. Thus $\mathcal{O}$ and $\mathcal{S}$ can be reconstructed as soon as we are given q and the m_ϕ. It is clear that we could have taken q and the m_ϕ as our starting-point instead of $\mathcal{O}$ and $\mathcal{S}$.

The m_ϕ clearly have the following properties: (1) If $Q_1, Q_2, \cdots$ are questions such that $Q_i \perp Q_j$ for $i \neq j$ then $m_\phi(Q_1+Q_2+\cdots)=m_\phi(Q_1)+m_\phi(Q_2)+\cdots$, (2) $m_\phi(1)=1$. We shall call any function from the questions to the positive real numbers which has these two properties a *measure on the questions*. It is tempting to add the assumption that every measure on the questions is the m_ϕ for some state ϕ. Like the corresponding assumption for question-valued measures it can be attained by enlarging the system. However, the physical significance of states and observables differ in such a way that one is much more willing to make the first enlargement than the second. There is a sense in which one can construct observables from questions, but we have to be content with the states provided by nature. If we did add this assumption (and for all one knows to the contrary it holds for the system $\mathcal{O}$, $\mathcal{S}$ actually in use for quantum mechanics) it would follow that the pair $\mathcal{O}$, $\mathcal{S}$ is completely determined by the partially ordered set q of all questions and the "complementation operation" $Q \to 1-Q$. In any event it is natural to call this complemented partially ordered set the *logic*† of our system. The statics of the system is determined by its logic and a certain convex set of measures on the questions.

5. The logic of quantum mechanics. We have not yet added enough axioms to exclude classical mechanics. If we interpret "sufficiently regular" in our description of the $\mathcal{O}$, $\mathcal{S}$ for classical mechanics to mean being a Borel function, then the questions are in a natural one-to-one correspondence with the Borel subsets of phase space. The logic of classical mechanics is a Boolean algebra—the Boolean algebra of all Borel subsets of phase space. We see in particular that in the case of classical mechanics our questionable axiom is indeed satisfied. Every measure on the questions corresponds to a state. It is interesting to note that any two classical mechanical systems have isomorphic logics. This follows from a known set-theoretical result according to which there exists a one-to-one Borel set-preserving map between any two separable complete metric spaces.

We single out the system of quantum mechanics not by adding more general axioms but by stating explicitly what its logic is and which measures on the

† *Cf.* The logic of quantum mechanics, by G. Birkhoff and J. von Neumann, Ann. Math., vol. 37, 1936, pp. 823–843.

questions are associated with states. As in the case of classical mechanics, all quantum mechanical systems have (to within isomorphism) the same logic. This logic is the partially ordered set of all closed subspaces of a separable infinite dimensional Hilbert space (see the next section for a definition of Hilbert space), complementation being passage from a subspace to its orthogonal complement.

It would, of course, be very interesting if this structure for q and $Q \to 1-Q$ could be deduced from a set of physically meaningful and plausible axioms. Such a deduction is not at present available. It has simply been found that physical systems behave as if q had the indicated structure. For us it is an axiom. To mathematicians familiar with lattice theory this axiom is not so artificial and arbitrary as it may seem at first sight to others. The lattice of all subspaces of a vector space is one of the principal examples of a partially ordered set which is not a Boolean algebra but nevertheless has many regularity properties. Moreover, one can show that if the lattice of all closed subspaces of a Banach space has an order inverting involution with the properties of $Q \to 1-Q$, then this Banach space has an equivalent norm under which it is a Hilbert space. We shall postpone our description of $\mathcal{S}$ until after the exposition of basic Hilbert space theory to which we propose to devote the next two sections.

6. Definition and elementary properties of Hilbert space. A Hilbert space $\mathcal{H}$ is a vector space over the complex numbers in which there is given a complex-valued function of two variables (ϕ, ψ) such that: (1) For fixed ψ, (ϕ, ψ) is a linear function of ϕ, (2) $(\phi, \psi) = \overline{(\psi, \phi)}$, (3) $(\phi, \phi) > 0$ unless $\phi = 0$. (4) Under $\rho(\phi, \psi) = \sqrt{(\phi-\psi, \phi-\psi)}$, $\mathcal{H}$ is a complete metric space. The complex number (ϕ, ψ) is called the inner product of ϕ and ψ, and $\sqrt{(\phi, \phi)}$ is called the norm of ϕ and written $\|\phi\|$.

The two most familiar examples of Hilbert spaces are the following: (I) $\mathcal{H}$ is the set of all sequences $c_1, c_2, \cdots$ of complex numbers such that $|c_1|^2 + |c_2|^2 + \cdots < \infty$, and the inner product of $c_1, c_2, \cdots$ with $d_1, d_2, \cdots$ is $c_1\bar{d}_1 + c_2\bar{d}_2 + \cdots$, (II) $\mathcal{H}$ is the set of all complex-valued Lebesgue measurable square-integrable functions on the interval $[a, b]$ $(-\infty \leqq a < b \leqq \infty)$, functions equal almost everywhere being identified, with $(f, g) = \int_a^b f(x)\bar{g}(x)dx$. On modifying (II) by passing from $[a, b]$ to a general measure space one gets a class of examples including both (I) and (II). It is important to note that except for differences in "dimension" all Hilbert spaces are abstractly equivalent—the dimension of a Hilbert space being defined as the maximum number of elements in a set $\{\phi_\alpha\}$ such that $(\phi_\alpha, \phi_\beta) = 0$ for $\alpha \neq \beta$ and $(\phi_\alpha, \phi_\alpha) = 1$. Actually any two such sets which are maximal in the sense that they cannot be enlarged and which lie in the same Hilbert space have the same cardinal number. Such a maximal set is said to be an orthonormal basis for the space. If $\mathcal{H}$ has dimension $\aleph_0$, as do examples (I) and (II) above, and $\phi_1, \phi_2, \cdots$ is an orthonormal basis, then every ϕ in $\mathcal{H}$ is uniquely representable in the form $c_1\phi_1 + c_2\phi_2 + \cdots$, where $|c_1|^2 + |c_2|^2 + \cdots < \infty$. The mapping $\phi \to \{c_1, c_2 \cdots\}$ is then an inner product preserving linear map of $\mathcal{H}$ on example (I). Practically the same considerations lead to a correspond-

ing proof for higher dimensional spaces. A separable Hilbert space always has dimension $\aleph_0$ or less.

A closed subspace of a Hilbert space is by definition a linear subspace which is closed in the metric. Every closed subspace is a Hilbert space in its own right. If M is a closed subspace then $M^\perp$, the orthogonal complement of M, is by definition the set of all ψ such that $(\phi, \psi)=0$ for all $\phi\in M$. It can be shown that $M^{\perp\perp}=M$ for all M, and that every $\phi\in\mathfrak{H}$ is uniquely of the form $\phi_1+\phi_2$, where $\phi_1\in M$ and $\phi_2\in M^\perp$.

7. Linear operators in Hilbert space. Let $\mathfrak{H}$ be a Hilbert space. Let T be a linear function mapping $\mathfrak{H}$ into $\mathfrak{H}$. T is continuous if and only if it is bounded, in the sense that $\|T(\phi)\|/\|\phi\|$ is bounded for $\phi\neq 0$. The least upper bound of these numbers $\|T(\phi)\|/\|\phi\|$ is called the norm $\|T\|$ of T. For each bounded linear operator T there is a unique bounded linear operator T^* such that $(T(\phi), \psi) = (\phi, T^*(\psi))$ for all ϕ and ψ in $\mathfrak{H}$. T^* is called the adjoint of T. If $T=T^*$, T is said to be *self-adjoint.* If $T^*T=TT^*=I$ (where I is the identity operator) then T is said to be *unitary.* T is unitary if and only if it is one-to-one, has all of $\mathfrak{H}$ for its range and preserves the inner product. The unitary operators are thus the automorphisms of $\mathfrak{H}$. Let M be a closed subspace of $\mathfrak{H}$. For each $\phi\in\mathfrak{H}$ let $P_M(\phi)$ denote the unique $\psi\in\mathfrak{H}$ such that $\phi-\psi$ is in $M^\perp$; that is, let P_M be the "projection" of ϕ on M. Then P_M is a bounded self-adjoint linear operator and $P_M^2=P_M$. Conversely if T is any self-adjoint bounded linear operator such that $T^2=T$, it is easy to show that $T=P_M$ where M is the set of all ϕ such that $T(\phi)=\phi$. For this reason the bounded self-adjoint linear operators T such that $T^2=T$ are called *projections.* The mapping $M\rightarrow P_M$ sets up a one-to-one correspondence between the closed subspaces and the projections such that for all M, $P_{M^\perp}=1-P_M$. Let T be any bounded self-adjoint linear operator. Then $(T(\phi), \psi)$ as a function of ϕ and ψ has properties (1) and (2) listed in the definition of an inner product. Moreover it also has the property: $|(T(\phi), \psi)|\leqq\|T\|\cdot\|\phi\|\,\|\psi\|$. We shall call any function $[\phi, \psi]$ which has properties (1) and (2), and is such that $|[\phi,\psi]|\leqq K\|\phi\|\,\|\psi\|$ for some positive real number K, a bounded Hermitian bilinear form. It is a theorem that every bounded Hermitian bilinear form is of the form $(T(\phi), \psi)$ for a uniquely determined self-adjoint bounded linear operator T.

8. Quantum statics. Let $\mathfrak{H}$ be the Hilbert space whose lattice of closed linear subspaces is the logic of our quantum-mechanical system $\mathfrak{O}$, $\mathfrak{S}$ so that we may identify questions in $\mathfrak{O}$ with closed subspaces of $\mathfrak{H}$. Let ϕ be any element of $\mathfrak{H}$ such that $\|\phi\|=1$. Then $M\rightarrow(P_M(\phi), \phi)$ is readily verified to be a measure on the questions. More generally if $\phi_1, \phi_2, \cdots$ is any sequence of members of $\mathfrak{H}$ all having norm one and $\gamma_1, \gamma_2, \cdots$ is any sequence of positive real numbers such that $\gamma_1+\gamma_2+\cdots=1$ then $\gamma_1(P_M(\phi_1)\cdot\phi_1)+\gamma_2(P_M(\phi_2), \phi_2)+\cdots$ converges for all M and as a function of M is a measure on the questions. We complete our axioms concerning quantum statics by assuming that the measures on the ques-

tions which define states are just those of the form† $M \to \gamma_1(P_M(\phi_1), \phi_1) + \gamma_2(P_M(\phi_2), \phi_2) + \cdots$. It is easy to show that a state is pure if and only if it is defined by a single element of $\mathcal{H}$; that is if and only if the associated measure on the questions is of the form $M \to (P_M(\phi), \phi)$. It is also easy to show that ϕ_1 and ϕ_2 define the same pure state if and only if $\phi_1 = e^{i\theta}\phi_2$ for some real θ.

If A is any observable and $E \to Q_E^A$ is the corresponding question-valued measure we may construct what we shall call a "projection-valued measure" by replacing each closed subspace Q_E^A by the projection upon it. We leave the formulation of the definition of projection-valued measure to the reader. We will denote the projection-valued measure thus associated with the observable A by P^A and the projection which it assigns to the Borel set E by P_E^A. Obviously the mapping $A \to P^A$ sets up a one-to-one correspondence between all observables on the one hand and all projection-valued measures on the other. The probability distribution of the observable A in the pure state defined by the Hilbert-space element ϕ is of course just $E \to (P_E^A(\phi), \phi)$—a formula which sums up the whole of quantum statics. In the sequel we shall speak only of pure states and we shall find it convenient to identify them with the Hilbert-space elements which define them.

9. Observables and operators. Let A be an observable which is *bounded* in the sense that $P_E = I$ for some bounded subset E of the real line. Then for all ϕ and ψ in $\mathcal{H}$, $E \to (P_E^A(\phi), \psi)$ is a countably additive complex-valued set function $\alpha_{\phi,\psi}^A$ with respect to which the Stieltjes integral $\int_{-\infty}^{\infty} x d\alpha_{\phi,\psi}^A(x)$ exists. The resulting function of ϕ and ψ is easily seen to be a bounded Hermitian bilinear form. Hence there exists a bounded self-adjoint linear operator T^A uniquely determined by A such that for all ϕ and ψ in $\mathcal{H}$, $(T^A(\phi), \psi) = \int_{-\infty}^{\infty} x d\alpha_{\phi,\psi}^A(x)$. Thus every bounded observable A is associated with a unique bounded self-adjoint linear operator. It follows from the celebrated spectral theorem that the converse is true. Every bounded self-adjoint linear operator is derivable in the manner just described from a uniquely determined bounded projection-valued measure, and hence is of the form T^A for a uniquely determined bounded observable A. $A \to T^A$ thus sets up a one-to-one correspondence between the set of all bounded observables in $\mathcal{O}$ and the set of all bounded self-adjoint linear operators. This correspondence may be extended to one between all observables in $\mathcal{O}$ and all self-adjoint linear operators by suitably defining the notion of unbounded self-adjoint linear operator. Let T be linear but let its domain be (perhaps) only a dense linear subspace of $\mathcal{H}$, and let us make no assumptions about its boundedness or continuity. The adjoint of T is then defined as follows: ψ is in the domain

† As indicated earlier, no other measures on the questions are known at present. Indeed, since these words were written, A. M. Gleason has made considerable progress toward showing that there are none. Using his results one can replace our assumption about the specific form of the allowable measures by one which asserts, simply, that these measures are not too badly "non-measurable."

of T^* if and only if $(T(\phi), \psi)$ is continuous as a function of ϕ for all ϕ in the domain of T. $T^*(\psi)$ is then the unique element of $\mathcal{H}$ such that $(T(\phi), \psi) = (\phi, T^*(\psi))$ for all ϕ in the domain of T. T is said to be self-adjoint whenever $T = T^*$ (equality including having the same domain). When T is bounded and everywhere defined this definition coincides with the one already given. The construction of an unbounded self-adjoint operator from an unbounded projection-valued measure is analogous to the corresponding construction in the bounded case but is complicated by the fact that the relevant bilinear form is not everywhere defined. We shall not give details here. We exploit the one-to-one correspondence between observables and self-adjoint linear operators which we have just described by identifying every observable with its corresponding operator.

Let A be a self-adjoint linear operator in $\mathcal{H}$ and let P^A be the corresponding projection valued measure. Let G denote the union of all open sets E for which $P_E^A = 0$. Then G itself is an open set and $P_G^A = 0$. The closed complement of the open set G is called the *spectrum* S_A of A. The significance of S_A for A regarded as an observable is obviously the following. In every state the probability measure of A is concentrated in S_A and for every open interval F on the real line which intersects S_A there exists a state in which the probability measure of A is concentrated in F. In this sense S_A is the set of possible values for the observable A. Every closed subset of the real line is the spectrum of some self-adjoint linear operator and hence the set of possible values of some observable.

The set of all $\lambda \in S_A$ such that $P_{\{\lambda\}}^A \neq 0$, where $\{\lambda\}$ denotes the set whose only element is λ, is called the *point spectrum* of A. If ϕ is an element of norm one in the range of $P_{\{\lambda\}}^A$ then it is a state in which the observable A has the value λ with probability one. Conversely if ϕ is a pure state in which A has the value λ with probability one then ϕ is in the range of $P_{\{\lambda\}}^A$. The range of $P_{\{\lambda\}}^A$ coincides with the set of all ϕ for which $A(\phi) = \lambda\phi$; that is with the set of all *proper vectors* of A belonging to the *proper value* λ. The point spectrum of A is always at most denumerable. A is said to have a *pure point spectrum* if $P_D^A = I$ where D is the point spectrum; that is if in every state the probability measure of A is concentrated in D. A has a pure point spectrum if and only if there exists a basis of proper vectors. One of the more spectacular ways in which quantum mechanics differs from classical mechanics is in the occurrence in quantum mechanics of observables with a sizable point spectrum whose classical analogues are continuous functions on phase space.

The identification of observables with self-adjoint operators together with the operation $a \rightarrow f(A)$ already defined for observables leads at once to a definition of $f(A)$ whenever f is a Borel function and A is a self-adjoint operator. This definition can of course be formulated without reference to quantum mechanics. When f is a polynomial it agrees with the usual algebraic definition of a polynomial function of an operator. If A_1 and A_2 are bounded self-adjoint linear operators it can be shown that $A_1A_2 = A_2A_1$ if and only if $P_E^{A_1}P_F^{A_2} = P_F^{A_2}P_E^{A_1}$ for all E and F. When A_1 and A_2 are unbounded and consequently not everywhere defined this condition on the projection-valued measures is taken as the defini-

tion of the commutativity of the operators. With this definition it can be shown that A_1 and A_2 commute if and only if there exists a third self-adjoint operator A_3 and Borel functions f_1 and f_2 such that $A_1=f_1(A_3)$ and $A_2=f_2(A_3)$. There is thus an obvious sense in which observables which commute when considered as operators are "simultaneously measurable". The circumstance which prevents our assigning an obvious physical sense to $f(A_1, A_2)$, where f is a Borel function of two variables and A_1 and A_2 are observables, is the possible lack of "simultaneous measurability" of A_1 and A_2. It can be shown that whenever A_1 and A_2 are self-adjoint linear operators for which there is a mapping $f\rightarrow f(A_1, A_2)$ having reasonable properties, then A_1 and A_2 commute. In this sense commutativity of A_1 and A_2 considered as operators is equivalent to simultaneous observability of A_1 and A_2 considered as observables.

10. Quantum dynamics. We have yet to study the nature of the one-parameter group of transformations $t\rightarrow L_t$ of $\mathcal{S}$ onto $\mathcal{S}$ which describes the way in which our system $\mathcal{O}$, $\mathcal{S}$ changes with time. Our basic dynamical assumption is that each L_t is defined by a unitary transformation U_t of $\mathcal{H}$ onto $\mathcal{H}$, and that L_t depends upon t in such a manner that $|(U_t(\phi), \psi)|$ is continuous in t for all ϕ and ψ in $\mathcal{H}$. Each U_t is determined by L_t up to multiplication by a complex number of modulus one. It turns out that one can choose the arbitrary constants so that $U_{t_1+t_2}=U_{t_1}U_{t_2}$ for all t_1 and t_2, and so that $(U_t(\phi), \psi)$ is itself continuous. Moreover, the fact that L_t is of the given form may be deduced from quite general hypotheses about the action of the L_t on $\mathcal{S}$. The continuous one-parameter group $t\rightarrow U_t$ is, of course, not quite uniquely determined by the L_t. If c is any real number then $t\rightarrow U_t$ and $t\rightarrow e^{-itc}U_t$ come from the same L_t's and define the same dynamics. This however is the extent of the ambiguity.

Let H be any self-adjoint linear operator. We define e^{-itH} for all real t as $\cos(tH)-i\sin(tH)$. Since $\sin x$ and $\cos x$ are bounded functions, e^{-itH} is a bounded linear operator even when H is not bounded. As a matter of fact, it is unitary, and $t\rightarrow e^{-itH}$ is a continuous one-parameter unitary group of operators. According to a fundamental theorem of M. H. Stone, every continuous one-parameter group of unitary operators is of this form, with a uniquely determined self-adjoint H. Thus $H\rightarrow U_t$, where $U_t=e^{-itH}$, sets up a one-to-one correspondence between the self-adjoint operators and the continuous one-parameter unitary groups. If $t\rightarrow U_t$ determines the dynamics of our system and $U_t=e^{-itH}$, then the self-adjoint operator H also determines the dynamics of our system. Since $e^{-itc}e^{-itH}=e^{-it(c+H)}$, the ambiguity in U produces a certain ambiguity in H as well. H is determined only up to an additive constant operator by the L_t. We shall ignore this ambiguity and suppose a particular H chosen once and for all. We shall call H the *dynamical operator* of the system.

If ϕ is in the domain of H and is of norm one then the variable pure state $e^{-itH}(\phi)=\phi_t$ satisfies the differential equation: $d\phi_t/dt=-iH(\phi_t)$. This is the abstract form of what is called Schrödinger's equation, and plays the role in quantum mechanics played by Hamilton's equations in classical mechanics. It

is a first-order differential equation whose solutions are the trajectories of the pure states.

Let ϕ be a proper vector of H having norm one and corresponding proper value λ. Then $e^{-itH}(\phi)=e^{-it\lambda}\phi$, and thus is the same state for all time t. In other words, the state defined by ϕ is constant in time. It is what is called a *stationary state*. Conversely every stationary state ϕ is a proper vector of H. The existence of pure states which are stationary is a feature of quantum mechanics which has no counterpart in classical mechanics. Of course, a given H need not have any proper vectors, but those which arise most commonly in practice have a good many.

In classical mechanics certain functions on phase space are constant on the trajectories and are called integrals. It is natural to call a quantum mechanical observable an *integral* if its probability-measure in every state ϕ is constant in time. An easy calculation shows that the observable A is an integral if and only if it commutes with H when regarded as an operator. Now H itself is a self-adjoint operator and as such is an observable; moreover this observable is an integral. It clearly plays a central role in the theory. As we shall see, it is a constant multiple of an observable which is the quantum-mechanical analogue of the energy integral of classical mechanics. Thus the stationary states of a quantum-mechanical system are just those in which the energy has a definite value with probability one and each stationary state is associated with a definite value of the energy.

11. The quantum mechanics of n *"interacting particles"*. The discussion so far has been rather general and abstract. Except in the case of the energy we have given no indications whatever concerning the physical meaning of the observable identified with a particular operator. Moreover, we have not explained how one constructs the quantum-mechanical refinement of a single classical system. We now remedy these deficiencies by exhibiting and discussing a quantum-mechanical system which in the limiting case of highly concentrated probabilities behaves like a system of n classical mass particles moving under mutual central forces.

Consider the Hilbert space of all square-summable functions on Euclidean $3n$ space R^{3n}. Let D denote the formal differential operator

$$f \to \sum_{j=1}^{n} -\frac{1}{\mu_j}\left(\frac{\partial^2 f}{\partial x_j^2}+\frac{\partial^2 f}{\partial y_j^2}+\frac{\partial^2 f}{\partial z_j^2}\right)+Wf,$$

where W is a real function on R^{3n} and the μ_j are positive real numbers. For a wide class of choices of W there is a unique self-adjoint operator H_D whose restriction to the twice differentiable functions coincides with D. We consider the quantum-mechanical system whose Hilbert space is $\mathfrak{L}^2(R^{3n})$ and whose dynamical operator is H_D. For each real-valued Borel function g on R^{3n} let A_g be the self-adjoint operator $f\to fg$—the domain when g is unbounded being defined in the obvious way. The projection-valued measure associated with A_g can be shown to

take the set E into the projection A_ψ where ψ is the characteristic function of $g^{-1}(E)$. Thus the probability measure for A_g in the state f is

$$E \to \int_{g^{-1}(E)} \cdots \int |f|^2 dx_1 \cdots dz_n.$$

Let f be a state such that the mean value $\overline{A}_{x_j} = \int \cdots \int x_j |f|^2 dx_1 \cdots dz_n$ of A_{x_j} exists. A simple calculation shows that

$$\frac{d^2}{dt^2}(\overline{A}_{x_j}) = -\frac{2}{\mu_j}\int \cdots \int |f|^2 \left(\frac{\partial W}{\partial x_j}\right) dx_1 \cdots dz_n = -\frac{2}{\mu_j}\overline{A}_{\frac{\partial W}{\partial x_j}}$$

whenever the relevant derivatives exist. There are of course similar results for the rates of change of the mean values of the A_{y_j} and the A_{z_j}. When f is zero outside of a small neighborhood of a point in R^{3n} the mean values in these formulas are approximately the coordinates of this point and (assuming W to have continuous partial derivatives) the values of the $\partial W/\partial x_j$, $\partial W/\partial y_j$, $\partial W/\partial z_j$ at this point. In other words, when f varies in such a manner that the probability measures of the A_{x_j}, A_{y_j} and A_{z_j} in that state remain highly concentrated, so that the set of them may be represented approximately by a point in R^{3n}, then this point moves so as to satisfy the differential equations

$$\frac{\mu_j}{2}\frac{d^2x_j}{dt^2} = -\frac{\partial W}{\partial x_j}, \quad \frac{\mu_j}{2}\frac{d^2y_j}{dt^2} = -\frac{\partial W}{\partial y_i}, \quad \frac{\mu_j}{2}\frac{d^2z_j}{dt^2} = -\frac{\partial W}{\partial z_j}.$$

These are simply the classical equations of motion for a system of n mass-particles of masses $\frac{1}{2}\mu_j$ and potential energy function W. They are also the classical equations of motion for such a system where the masses are $\frac{1}{2}c\mu_j$, the potential energy function is cW, and c is any positive real number. In classical mechanics masses are determined by the equations of motion only up to a multiplicative constant which is then fixed by arbitrarily choosing a unit of mass. If quantum mechanics had been thought of at the outset one would presumably have made use of the natural unit of mass provided by Schrödinger's equation and defined $\frac{1}{2}\mu_j$ as the mass of the j-th particle. As it is, the mass m_j of the j-th particle is some constant multiple of $\frac{1}{2}\mu_j$. This constant is the same for all systems (for any given units of mass, length and time) and is generally denoted by $\hbar$. $2\pi\hbar$ is the fundamental constant introduced into the old quantum theory by Max Planck and called Planck's constant. Planck's constant is usually denoted by h. In terms of the m_j, the classical potential energy $V = W/\hbar$, and $\hbar$, Schrödinger's equation for our system takes the standard form:

$$-\frac{1}{i}\frac{\partial f}{\partial t} = -\frac{\hbar}{2}\sum_{j=1}^{n}\frac{1}{m_j}\left(\frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2} + \frac{\partial^2 f}{\partial z_j^2}\right) + \frac{Vf}{\hbar}.$$

Henceforth, we shall suppose V to be the potential energy arising from mutual central forces between the classical particles so that, in particular, there are linear and angular momentum integrals for the limiting classical system.

In $g \to A_g$ we have a natural one-to-one correspondence between those observables in classical mechanics which depend upon the coordinates, but not the velocities, and certain quantum-mechanical observables. It turns out *not* to be possible to extend this correspondence to one between all classical-mechanical observables and all quantum-mechanical observables. Indeed since classical mechanics is only a limiting case of quantum mechanics there is no reason to expect such a correspondence. Two different quantum-mechanical observables can well coincide in the classical limit. On the other hand there are, as it turns out, well defined quantum analogues for certain of the more important velocity dependent classical-mechanical observables. For example, the formal differential operator $(\hbar/im_j)\partial/\partial x_j$ defines† an observable whose mean value coincides with the time derivative of the mean value of A_{x_j} in every state for which these things have a meaning. In this sense it is the analogue of the classical-mechanical observable known as the x component of the velocity of the j-th particle. To remind ourselves of the analogy we give it the same name in quantum mechanics and do likewise for $(\hbar/im_j)\partial/\partial y_j$ and $(\hbar/im_j)\partial/\partial z_j$. The analogy may be pursued further. The observable corresponding to

$$\sum_{j=1}^{n} m_j\left(\frac{1}{i}\,\frac{\hbar}{m_j}\,\frac{\partial}{\partial x_j}\right) = \frac{\hbar}{i}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}$$

is an integral just as is the corresponding sum in classical mechanics. For obvious reasons it is called the x component of the total momentum and the observable corresponding to the term $(\hbar/i)\partial/\partial x_j$ is called the x component of the momentum of the j-th particle. The y and z components of momentum are defined analogously. In a similar manner one is led to define angular momentum observables and to the discovery that the total angular momentum about any axis is an integral. The angular momentum observables are interesting in that they all have pure point spectra; the spectrum in each case being the set of all integral multiples of $\hbar$. Finally, if one takes the classical energy integral $\sum_{j=1}^{n} \frac{1}{2} m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) + V$ and tries to construct a corresponding quantum-mechanical observable by substituting the analogues already discovered for the $\dot{x}_j$, $\dot{y}_j$, $\dot{z}_j$ and V, one is led to the formal differential operator

$$f \to -\frac{\hbar^2}{2}\sum_{j=1}^{n}\frac{1}{m_j}\left(\frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2} + \frac{\partial^2 f}{\partial z_j^2}\right) + Vf$$

which is just $\hbar$ times the differential operator D with which we started. For this reason the observable $\hbar H_D$ is called the energy observable. As we have already remarked it is obviously an integral. The fact that one has natural analogues in quantum mechanics for the energy and momentum observables

† The correspondence between formal differential operators and self adjoint operators on Hilbert space is rather complicated in general. However, for the members of a wide class of first-order operators, including those with which we shall deal, there is a canonical way of passing to a corresponding self-adjoint operator.

lies deeper than the above formal considerations indicate. It can be traced back to the relationship of the observables in question to certain one parameter groups of automorphisms of the system.

12. The quantum mechanics of the atom. The quantum-mechanical model of an atom is not precisely what one would expect from the considerations of the preceding paragraph and the classical picture of an atom as a nucleus surrounded by n electrons. In order to produce results in reasonable agreement with experiment the model resulting from this picture must be modified in three respects. (We simplify the discussion by replacing the (relatively heavy) nucleus by a fixed force field so that an n electron atom leads to an n-body problem.) First $\mathcal{L}^2(R^{3n})$ must be replaced by the Hilbert space of all square summable functions from R^{3n} to a 2^n-dimensional Hilbert space. Second this new Hilbert space must be replaced by the subspace consisting of those functions f such that $f(T_{ij}(p)) = -f(p)$ for all p in R^{3n} and all i and j with $i \neq j$. Here T_{ij} is the map of R^{3n} onto itself which takes each point into the point obtained by interchanging the coordinates of i-th and j-th particles. Thirdly the restriction of H_D to this subspace must be changed by adding a "small perturbation" which we shall not describe explicitly. The first change has the effect of adding certain new observables—the so-called spin angular moments. The second change eliminates a good many observables—all of those in which the n electrons are not treated symmetrically. The fact that there are observables corresponding to $x_1+x_2+\cdots+x_n$, $x_1^2+x_2^2+\cdots+x_n^2$ *etc.*, but none corresponding to x_1, $x_2, \cdots, x_n$ individually, is interpreted as indicating a new sense in which an electron is not a classical particle. An electron is supposed to have no individuality. It is supposed to be meaningless to make a distinction between a pair of electrons and the same pair with the electrons interchanged. The situation is analogous to that of a pair of identical kinks traveling along in a stretched string. It is possible to have motion in which the kinks seem to have changed places although in fact the state of the string is just as before.

With the indicated modifications, the quantum-mechanical refinement of the classical picture of the atom leads to a model which explains atomic phenomena, including chemistry, in a truly remarkable manner.†

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† *Added in proof.* Since this article was sent to the printer, Professor Gleason has improved the result referred to in the footnote on page 51. A proof that there are no measures on the questions other than those associated with states will appear in the Journal of Mathematics and Mechanics.

MULTIDERIVATIVES AND MULTI-INTEGRALS†

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1. Introduction. The derivative of a function f at a place, a, is defined‡ as the limit of $[f(a')-f(a)]/(a'-a)$ as $a' \to a$. Traditionally, this difference quotient is interpreted geometrically by P^*P'/PP^*, that is, the ratio of the height to the base in the right triangle PP^*P' in Figure 1.

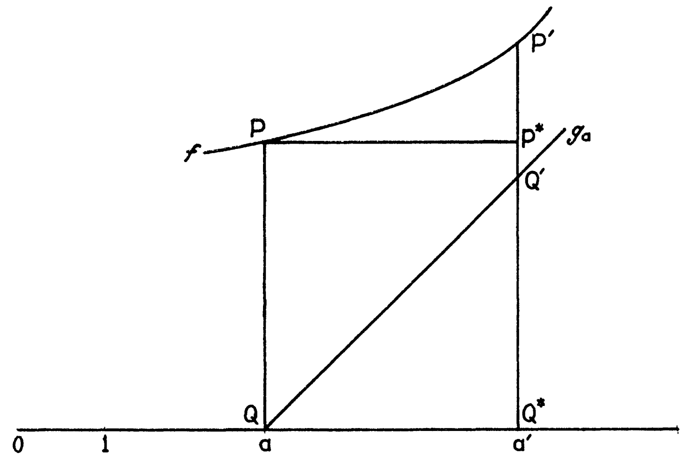

FIG. 1

There is, however, another possible interpretation. Figure 1 shows the line g_a of slope 1 through the point Q below P. Since Q^*Q', the altitude of g_a above Q^*, equals $QQ^*(=PP^*)$, the difference quotient may also be interpreted by P^*P'/Q^*Q', that is, the ratio of the difference between altitudes of f to the difference between altitudes of g_a—altitudes at two neighbor places: a and a′. The line g_a is the graph of a linear function (likewise denoted by g_a) which assumes for any x the value $g_a(x)=x-a$. This linear polynomial serves, as it were, as a gauge function with which one compares f in the neighborhood of a. The derivative of f is obtained by letting a′ approach a.

This other interpretation lends itself to far-reaching extensions, recently developed by S. S. Shü and myself. Since our joint note§ is very condensed and rather difficult to read, in Sections 2–5 of the present paper the principal ideas are expounded in detail, stressing motivations and including examples. The last sections contain new material.

† The work on this paper was done as part of Project DA-11-022-ORD-1494 sponsored by the Office of Ordnance Research.

‡ Symbols of functions and function variables are italicized, while numerical variables are printed in roman type.

§ Generalized derivatives and expansions, Proc. Nat. Acad. Sci., vol. 41, 1955, pp. 591–595.

The simplest of the extensions mentioned above results from the comparison of a function f with a *quadratic* gauge function in the neighborhood of *two* places, a and b. More specifically, what serves as the gauge function is the polynomial $g_{a,b}$, assuming for any x the value $g_{a,b}(x) = (x-a)(x-b)$. What will be compared are the difference between chords of f and the difference between chords of $g_{a,b}$—chords between two neighbor pairs of places: between a and b, and between a′ and b′ (see Fig. 2). Any chord is the graph of a linear polynomial. Under certain conditions, also the chord difference quotient (*i.e.*, the ratio of the differences between the chords) is a linear polynomial such that, as a′ and b′ approach a and b, respectively, there exists a linear limit polynomial. For any a and b, this limit polynomial may be called the *bi*derivative of f at a and b. I will here denote it by ${}_2\mathbf{D}f(a, b)$; its value for x by ${}_2\mathbf{D}f(a, b; x)$. For any a and b, the biderivative ${}_2\mathbf{D}f(a, b)$ is a linear function; that is, to say, there exist two numbers $P(a, b)$ and $Q(a, b)$ such that

$$ {}_2\mathbf{D}f(a, b; x) = P(a, b)x + Q(a, b) \text{ for any x.} $$

The numbers $P(a, b)$ and $Q(a, b)$ of course depend on f and, in general in a nonlinear way, on a and b.

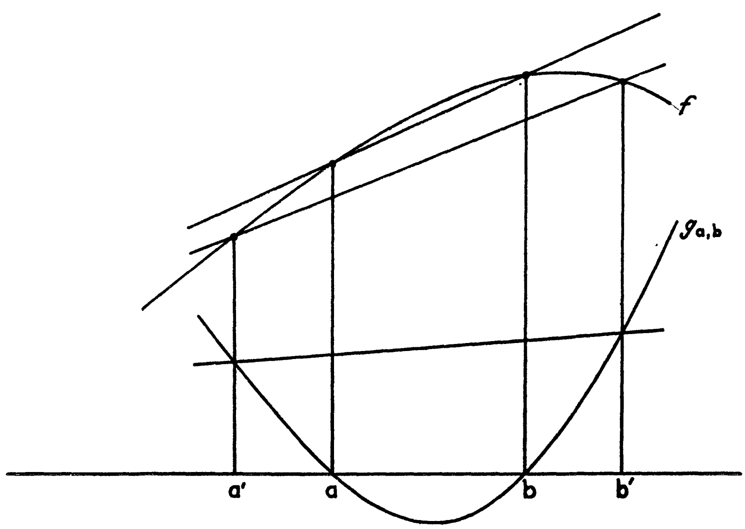

FIG. 2

More generally, ${}_n\mathbf{D}f(a_1, \cdots, a_n)$, the n-derivative of f at n places, results from the comparison of f with the gauge polynomial $g_{a_1,\cdots,a_n}$ of degree n whose value for x is $(x-a_1)\cdot \cdots \cdot(x-a_n)$. The value that this n-derivative assumes for x will be denoted by ${}_n\mathbf{D}f(a_1, \cdots, a_n; x)$. It is a polynomial of degree $n-1$ in x.

The case $n=1$ can be brought in line with this general concept by defining a *uni*derivative ${}_1\mathbf{D}f(a)$ of f at a, namely, as the polynomial of degree 0 (that is,

as the constant function) assuming for any x the value ${}_1\mathbf{D}f(\mathrm{a};\mathrm{x})=f'(\mathrm{a})$.

In Sections 2 and 3 it will appear that the prefixed subscript n, introduced in the present paper in denoting the n-derivative, has the advantage of making possible discussions of the operator ${}_n\mathbf{D}$ itself and of the result ${}_n\mathbf{D}f$ of its application to a function f, instead of confining the study to the polynomials ${}_n\mathbf{D}f(\mathrm{a}_1, \cdots, \mathrm{a}_n)$.

2. The basic definition. The idea outlined in the introduction will now be developed in detail in the case $\mathrm{n}=2$. Extensive use will be made of interpolation polynomials. For any function h and any two numbers c and d ($\neq$c) belonging to the domain of h, the linear polynomial assuming the values $h(\mathrm{c})$ and $h(\mathrm{d})$ for c and d, respectively, will be denoted by $h[\mathrm{c}, \mathrm{d}]$; its value for x, by $h[\mathrm{c}, \mathrm{d}; \mathrm{x}]$. If c belongs to the domain of h', then $h[\mathrm{c}, \mathrm{c}]$ will denote the linear polynomial having at c the value $h(\mathrm{c})$ and the derivative $h'(\mathrm{c})$. The graph of $h[\mathrm{c}, \mathrm{d}]$ is the chord of the curve h between c and d if $\mathrm{d}\neq\mathrm{c}$, and the tangent to h at c, if $\mathrm{d}=\mathrm{c}$.

Now let f be a given function and let a and b be two different numbers belonging to the domain of f. The quadratic polynomial called $g_{\mathrm{a},\mathrm{b}}$ in the introduction will now, for the sake of brevity, be denoted by g. Thus $g(\mathrm{x})=(\mathrm{x}-\mathrm{a})(\mathrm{x}-\mathrm{b})$ for any x. For any two numbers a′ and b′

$$g[\mathrm{a}', \mathrm{b}'; \mathrm{x}] = (\mathrm{a}' + \mathrm{b}' - \mathrm{a} - \mathrm{b})\mathrm{x} + (\mathrm{ab} - \mathrm{a}'\mathrm{b}'). \tag{1}$$

In particular, $g[\mathrm{a}, \mathrm{b}; \mathrm{x}]=0$ for any x.

The following quotient of two linear polynomials

$$\frac{f[\mathrm{a}', \mathrm{b}'; \mathrm{x}] - f[\mathrm{a}, \mathrm{b}; \mathrm{x}]}{g[\mathrm{a}', \mathrm{b}'; \mathrm{x}] - g[\mathrm{a}, \mathrm{b}; \mathrm{x}]} \tag{2}$$

is not in general linear in x nor does (2) have a limit as $\mathrm{a}'\to\mathrm{a}$ and $\mathrm{b}'\to\mathrm{b}$. Most fortunately, however, both difficulties disappear if the numbers a′ and b′ are chosen in such a way that

$$\mathrm{a}' + \mathrm{b}' - \mathrm{a} - \mathrm{b} = 0. \tag{3}$$

In this case, $g[\mathrm{a}', \mathrm{b}']$ is the constant function of value $\mathrm{ab}-\mathrm{a}'\mathrm{b}'$; that is to say, the chord of the parabola in Figure 2 is horizontal, while (2) becomes

$$\frac{f[\mathrm{a}', \mathrm{b}'; \mathrm{x}] - f[\mathrm{a}, \mathrm{b}; \mathrm{x}]}{\mathrm{ab} - \mathrm{a}'\mathrm{b}'} \tag{4}$$

which is linear in x. In order to verify (under very natural assumptions about f) that (4) possesses a (linear) limit function as $\mathrm{a}'\to\mathrm{a}$ and $\mathrm{b}'\to\mathrm{b}$, it is convenient to make use of (3) by setting

$$\mathrm{b}' - \mathrm{b} = \mathrm{u} = \mathrm{a} - \mathrm{a}'.$$

This makes (4) equal to

$$(5)\quad \begin{aligned}&\frac{1}{u(b-a+u)}\left\{\frac{f(b+u)-f(a-u)}{b-a+2u}-\frac{f(b)-f(a)}{b-a}\right\}x\\ &+\frac{1}{u(b-a+u)}\left\{\frac{(b+u)f(a-u)-(a-u)f(b+u)}{b-a-2u}-\frac{bf(a)-af(b)}{b-a}\right\}.\end{aligned}$$

The limit of this quotient as u→0 can be determined by l'Hospital's Rule under the assumptions that $f'(a)$ and $f'(b)$ exist. (These are the natural assumptions about f alluded to previously.) One readily finds that the limit of (5) as u→0 is

$$(6)\quad \begin{aligned}&\frac{1}{(b-a)^2}\left\{f'(a)+f'(b)-2\frac{f(b)-f(a)}{b-a}\right\}x\\ &\qquad+\frac{1}{(b-a)^2}\left\{(a+b)\frac{f(b)-f(a)}{b-a}-bf'(a)-af'(b)\right\}.\end{aligned}$$

For any a and b ($\neq$a), the linear function assuming for any x the value (6) will be called the biderivative of f at (a, b) and will be designated by ${}_2\mathbf{D}f(a, b)$, while (6) will be denoted by ${}_2\mathbf{D}f(a, b; x)$. Clearly, ${}_2\mathbf{D}f(a, b) = {}_2\mathbf{D}f(b, a)$. For any a and b, the two coefficients of the linear expression (6), which have been called $P(a, b)$ and $Q(a, b)$ in the introduction, may be written as follows:

$$P(a, b) = \frac{2}{(b-a)^2}\left\{\frac{f'(a)+f'(b)}{2}-\frac{f(b)-f(a)}{b-a}\right\}$$

and

$$Q(a, b) = \frac{a+b}{(b-a)^2}\left\{\frac{f(b)-f(a)}{b-a}-\frac{af'(b)+bf'(a)}{a+b}\right\}.$$

Thus the graph of ${}_2\mathbf{D}f(a, b)$ is a straight line whose slope is >0 or $=0$ or <0 depending upon whether the slope of the chord of f between a and b is greater than, equal to, or less than the average of the slopes of f at a and b.

The case b = a can be treated similarly. If $f'''(a)$ exists, then by repeated application of l'Hospital's Rule one finds that the limit of (4) as u→0 is

$$(7)\quad \frac{f'''(a)}{6}x+\frac{f''(a)}{2}-\frac{af'''(a)}{6} = {}_2\mathbf{D}f(a, a; x).$$

Summarizing one can say:

The symbol	read	denotes
${}_2\mathbf{D}$	the biderivative	an operator
${}_2\mathbf{D}f$	the biderivative of f	a 2-parameter family of linear functions
${}_2\mathbf{D}f(a, b)$	the biderivative of f at (a, b)	a linear function
${}_2\mathbf{D}f(a, b; x)$	the biderivative of f at (a, b) evaluated at x	a number

The last number exists for any function f and any two numbers a and b such that $f'(a)$ and $f'(b)$ exist, if $b \neq a$, and $f'''(a)$ exists, if $b = a$.

More generally, if A is any (unordered) system of n numbers (that is, a finite set of numbers, each endowed with a multiplicity $\geqq 1$, the sum of all multiplicities being equal to n), then the n-derivative of f at A is a polynomial ${}_n\mathbf{D}f(A)$ of degree $n-1$. It is defined as the limit of difference quotients

$$\frac{f[A'] - f[A]}{g[A'] - g[A]} \quad \text{as } A' \to A,$$

where the gauge function g is a polynomial of degree n, namely, the polynomial whose value for x is the product of the factors $x-a$ for all a in the system A. In order to make the difference quotient a polynomial of degree $n-1$ and to assure the existence of a limit as $A' \to A$ (under differentiability assumptions concerning f), one must choose A' in such a way that $g[A']$ is a constant polynomial. This means, in generalization of (3), that the sums of the numbers in A' and in A must be equal; but also the sums of the products of the pairs in A' and in A must be equal; and so on, for all elementary symmetric functions except the last, that is, the products of the numbers in A' and in A. Then the denominator of the difference quotient, that is, $g[A'] - g[A]$, is the constant function whose value is the difference between these products.

While the introductory geometrical interpretation of multiderivatives was confined to real numbers and real functions, clearly all algebraic and analytic discussions can be extended verbatim to complex numbers and complex functions.

3. Examples and simple theorems. For the n-th power function, assuming for any x the value x^n, the coefficients in (6), that is, $P(a, b)$ and $Q(a, b)$, can be shown to be polynomials in a and b of degrees $n-3$ and $n-2$, respectively. More specifically, one finds

$f(x)$	${}_2\mathbf{D}f(a; b; x)$
x	0
x^2	1
x^3	$x + (a + b)$
x^4	$2(a + b)x + (a^2 + b^2)$
x^5	$(3a^2 + 4ab + 3b^2)x + (a^3 - a^2b - ab^2 + b^3)$
x^6	$(4a^3 + 6a^2b + 6ab^2 + 4b^3)x + (a^4 - 2a^3b - 3a^2b^2 - 2ab^3 + b^4)$
x^n	$\sum_{k=0}^{n-3} (k + 1)(n - k - 2)a^{n-k-3}b^k x + \sum_{k=0}^{n-2} [(k + 1)^2 - kn]a^{n-k-2}b^k.$

These formulae remain valid if $b = a$, in which case they yield

$${}_2\mathbf{D}f(a, a; x) = \frac{n(n - 1)(n - 2)}{6} a^{n-3}x - \frac{n(n - 1)(n - 5)}{6} a^{n-2},$$

in agreement with (7).

If

$$f(\mathrm{x}) = \begin{cases} 0 \text{ for } \mathrm{x} < 0 \\ 1 \text{ for } \mathrm{x} > 0 \end{cases},$$

then

$$_2\mathbf{D}f(\mathrm{ab}; \mathrm{x}) = \frac{2}{(\mathrm{b} - \mathrm{a})^3}\,\mathrm{x} + \frac{\mathrm{a} + \mathrm{b}}{(\mathrm{b} - \mathrm{a})^3}\,.$$

If $f(\mathrm{x}) = \mathrm{x}$ and $\mathrm{a} < 0 < \mathrm{b}$, then

$$_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{x}) = -\frac{2(\mathrm{a} + \mathrm{b})}{(\mathrm{b} - \mathrm{a})^3}\,\mathrm{x} + \frac{2(\mathrm{a}^2 + \mathrm{b}^2)}{(\mathrm{b} - \mathrm{a})^3}\,.$$

The operator $_n\mathbf{D}$ associates a polynomial of degree $\mathrm{n}-1$ with any function and any system of n numbers such that the function has certain differentiability properties at the numbers in the system. Clearly, for any integer n, the operator $_n\mathbf{D}$ is *linear* in the following sense:

$$_n\mathbf{D}(f_1 + f_2) = {_n\mathbf{D}}f_1 + {_n\mathbf{D}}f_2 \quad \text{and} \quad {_n\mathbf{D}}(cf) = c\,{_n\mathbf{D}}f.$$

For instance, in the case $\mathrm{n} = 2$ the former formula has the following meaning: If $f = f_1 + f_2$, then for any two numbers a and b for which the biderivatives of f_1 and f_2 exist and for any number x:

$$_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{x}) = {_2\mathbf{D}}f_1(\mathrm{a}, \mathrm{b}; \mathrm{x}) + {_2\mathbf{D}}f_2(\mathrm{a}, \mathrm{b}; \mathrm{x}).$$

The biderivative of any linear function is the constant function of value 0, wherefore, in the theory of biderivatives, the linear functions play a role similar to that of the constant functions in the classical theory of derivatives (or in the theory of uniderivatives). If l is a linear function, then

$$_2\mathbf{D}(f + l) = {_2\mathbf{D}}f \quad \text{for any } f.$$

On the other hand, as the reader can easily verify, $_2\mathbf{D}(l \cdot f)$ is not in general equal to $l \cdot {_2\mathbf{D}}f$. There is, however, a generalization to multiderivatives of the following classical statement: If $f(\mathrm{x}) = h(\mathrm{x}) \cdot (\mathrm{x} - \mathrm{a})$ for every x, then $f'(\mathrm{a}) = h(\mathrm{a})$. The extension to biderivatives reads:

If $f(\mathrm{x}) = h(\mathrm{x})(\mathrm{x} - \mathrm{a})(\mathrm{x} - \mathrm{b})$ for every x, then $_2\mathbf{D}f(\mathrm{a}, \mathrm{b}) = h[\mathrm{a}, \mathrm{b}]$; that is to say, $_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{x}) = h[\mathrm{a}, \mathrm{b}; \mathrm{x}]$ for every x.

4. Multiderivatives of higher order. In our joint paper, Shü and I have introduced multiderivatives of second and higher order by recursive definitions. Higher order multiderivatives can also be introduced by direct limiting processes just as $\mathrm{f}''(\mathrm{a})$ can be defined as the limit of

$$\frac{f(\mathrm{a}') - f(\mathrm{a}) - f'(\mathrm{a}) \cdot (\mathrm{a}' - \mathrm{a})}{(\mathrm{a}' - \mathrm{a})^2/2!} \quad \text{as} \quad \mathrm{a}' \to \mathrm{a}.$$

For instance, the second order biderivative of f at (a, b) may be defined as the limit of

$$\frac{f[a', b'] - f[a, b] - {}_2\mathbf{D}f(a, b)\cdot g[a', b']}{g^2[a', b']/2!}$$

as (a′, b′) approaches (a, b) in such a way that $b'-b=a-a'=u$ and, therefore, $g[a', b']=ab-a'b'=u(b-a+u)$. If $b\neq a$, the repeated application of l'Hospital's Rule yields

$$\begin{aligned}{}_2\mathbf{D}^2f(a, b; x) = &\left\{\frac{f''(b) - f''(a)}{(b-a)^3} - 6\,\frac{f'(b)+f'(a)}{(b-a)^4} + 12\,\frac{f(b)-f(a)}{(b-a}\right\} x \\ &+ \left\{\frac{bf''(a) - af''(b)}{(b-a)^3} + 2\,\frac{(a+2b)f'(a) + (b+2a)f'(b)}{(b-a)^4}\right. \\ &\left. - 6\,\frac{(a+b)(f(b)-f(a))}{(b-a)^5}\right\}\end{aligned}$$

provided that $f''(a)$ and $f''(b)$ exist. For instance, the second order biderivative of all polynomials of a degree $\leqq 3$ is the constant function of value 0.

$f(x)$	$\frac{1}{2}\,{}_2\mathbf{D}^2f(a, b; x)$
x^3	0
x^4	1
x^5	$x+2a+2b$
x^6	$(3a+3b)x+3a^2+3ab+3b^2$
x^7	$(6a^2+9ab+6b^2)x+4a^3+3a^2b+3ab^2+4b$
x^8	$(10a^3+18a^2b+18ab^2+10b^3)x+5a^4+2a^3b+2ab^3+5b^4$
x^n	$\sum_{k=0}^{n-5}\frac{(k+2)(k+1)}{2}\cdot\frac{(n-k-3)(n-k-4)}{2}a^{n-k-5}b^kx+\sum_{k=0}^{n-4}[(k+2)^2-kn]\frac{(k+1)(n-k-3)}{2}a^{n-k-4}b^k$

The preceding expression for $\frac{1}{2}\,{}_2\mathbf{D}^2f(a, b; x)$ has been derived by induction. At my suggestion, Dr. Berthold Schweizer has computed the m-th order derivative of the n-th power function. If $f(x)=x^n$ and one sets

$$\frac{1}{m!}\,{}_2\mathbf{D}^m f(a, b; x) = P_n^{(m)}(a, b)x + Q_n^{(m)}(a, b),$$

then Schweizer found

$$P_n^{(m)}(a, b) = \sum_{k=0}^{n-2m-1}\binom{k+m}{m}\binom{n-m-k-1}{m}a^{n-2m-k-1}b^k$$

and

$$Q_n^{(m)}(a, b) = P_{n+1}^{(m)}(a, b) - (a+b)\cdot P_n^{(m)}(a, b).$$

The principal application of $f^{(n)}(a)$ (that is, the classical n-th order derivative of f at a) is to expansions of $f(x)$ in powers of $x-a$. For, according to Taylor,

$$f(x) = f(a) + f'(a)\cdot(x - a) + \tfrac{1}{2}f''(a)\cdot(x - a)^2 + \cdots.$$

In the realm of complex functions, Cauchy expanded $f(z)$ in powers of $z-a$ with coefficients like those in Taylor's expansion; that is to say, the coefficient of $(z-a)^m$ in Cauchy's series is $(1/m!)f^{(m)}(a)$. Moreover, Cauchy identified $f^{(m)}(a)$ and, therefore, that coefficient of $(z-a)^m$ with a line integral along any rectifiable path located within the domain of analyticity of f and encircling the point, a, once in the counterclockwise sense.

Just 100 years ago, Jacobi initiated the study of the expansion of $f(z)$ in powers of $(z-a)(z-b)$ and, more generally, in powers of a polynomial p of higher degree and evaluated the coefficients.† Just 50 years ago, Kienast‡ identified the coefficients with line integrals extending those of Cauchy. But only the new concept of multiderivatives of any order yields expansions with coefficients like those in Taylor's series. Setting $f[a, b] = {}_2\mathbf{D}^0 f(a, b)$, Shü and I have proved:

$$\begin{aligned} f(z) = {}& {}_2\mathbf{D}^0 f(a, b; z) + {}_2\mathbf{D} f(a, b; z)(z - a)(z - b) \\ & + \tfrac{1}{2}\,{}_2\mathbf{D}^2 f(a, b; z)[(z - a)(z - b)]^2 + \cdots. \end{aligned}$$

More generally, in expanding f in powers of a polynomial p of degree n, we identified the coefficient of the m-th power of p with the n-derivative of m-th order of f at the system A_p of the zeros of p divided by m!

$$f(z) = {}_n\mathbf{D}^0 f(A_p; z) + {}_n\mathbf{D} f(A_p; z)\cdot p(z) + \tfrac{1}{2}\,{}_n\mathbf{D}^2 f(A_p; z)\cdot p^2(z) + \cdots.$$

5. Multiderivatives with respect to general gauge functions. In Section 2, the biderivative of f at (a, b) has been defined as the limit of the quotients (2) as $(a', b') \to (a, b)$ subject to the condition (3). The denominator in (2) is the difference of interpolation polynomials for the function g, which at x assumes the value $(x-a)(x-b)$. This gauge function g is, as it were, adjusted to the pair (a, b) at which the biderivative of f is to be defined. Its graph is the parabola with zeros at a and b. But the same denominator in (2) would result if g were replaced by the function j^2 assuming for any x the value $j^2(x) = x^2$. Indeed,

$$j^2[a', b'] - j^2[a, b] = g[a', b'] - g[a, b].$$

† Crelle's Journal, vol. 53, 1856, pp. 103–126. *Cf.* also C. G. J. Jacobi, Gesammelte Werke, vol. 6, pp. 203–230.

‡ A. Kienast, Inaugural Dissertation, University of Zürich, 1906. *Cf.* also P. Montel, Leçons sur les Séries de polynomes à une Variable Complexe, Paris, 1910, pp. 45–54.

B. Schweizer obtained the expressions for $P_n^{(m)}(a, b)$ and $Q_n^{(m)}(a, b)$ by using Kienast's form of the coefficients in the expansion of the n-th power function in powers of the polynomial assuming for any x the value $(x-a)(x-b)$. This computation of the m-th order biderivative is based on the expansion theorem given *l.c.*§ p. 58.

Yet j^2 is in no way adjusted to the particular pair (a, b), wherefore j^2 may serve as the gauge function in defining the biderivative of f at any pair (a, b).

This situation suggested to Shü and myself the possibility of an important generalization. For any differentiable function g, we define a biderivative of f with respect to g, namely, as the limit of the difference quotients (2) as (a′, b′) →(a, b) subject to the condition

$$g[a', b'; x] - g[a, b; x] = \text{const.}$$

In view of the last restriction, a linear limit function exists. I will here denote its value for x by ${}_2\mathbf{D}_g f(a, b; x)$; the linear function itself, by ${}_2\mathbf{D}_g f(a, b)$; the two-parameter family of linear functions in this way associated with f, by ${}_2\mathbf{D}_g f$; and the operator, by ${}_2\mathbf{D}_g$.

The operator ${}_2\mathbf{D}$, discussed in Section 2, clearly is the special case obtained if $g=j^2$. In a formula, ${}_2\mathbf{D}={}_2\mathbf{D}_{j^2}$, just as the ordinary uniderivative ${}_1\mathbf{D}$ is the uniderivative relative to the identity function j assuming for any x the value $j(x)=x$.

In our joint paper, Shü and I have shown that the m-th order biderivatives of f with respect to g are essentially the coefficients of the m-th power of g in an expansion of f in powers of g. Such expansions considerably generalize those of Jacobi, which are in powers of a polynomial.

6. The most general biderivative of f at (a, b). In the preceding definition of ${}_2\mathbf{D}_g f$, one function g has served as the gauge function for any pair of numbers (a, b). The present section is devoted to the most general biderivative of f at one particular pair (a, b). In Section 2, a gauge parabola has been used, and ${}_2\mathbf{D}f(a, b; x)$ has been defined as the limit of the expression (5) as $u\to 0$. The limit, obtained by l'Hospital's Rule, remains unchanged if, in the denominator of (5), $u(b-a+u)$ is replaced by $(b-a)u$. The denominator $u(b-a+u)$ is the altitude at $a-u$ and $b+u$ of the gauge parabola, whereas $(b-a)u$ is the altitude at these two places of the V-shaped polygon consisting of the tangents to the parabola at a and at b, respectively. This polygon (which has vertical symmetry) is the graph of the function assuming the value

$$(a-b)(x-a) \text{ for any } x \leqq \tfrac{1}{2}(a+b),$$
$$(b-a)(x-b) \text{ for any } x \geqq \tfrac{1}{2}(a+b).$$

This situation suggests the definition of a more general biderivative of f at (a, b) by comparing f with any V-shaped polygon, namely, for any two unequal numbers p and q, with the polygon $v_{p,q}$ having the altitude

$$p(x-a) \text{ for any } x \leqq \frac{bq-ap}{q-p},$$
$$q(x-b) \text{ for any } x \geqq \frac{q-p}{bq-ap}.$$

The polygon $v_{p,q}$ (which is vertically skew, unless $p=-q$) has the altitude t for $b+t/q$ and $a+t/p$. Setting $1/q-1/p=r$, one has to replace (5) by

$$\frac{1}{t}\left[\frac{f(b+t/q)-f(a+t/p)}{b-a+rt}-\frac{f(b)-f(a)}{b-a}\right]x$$
$$+\frac{1}{t}\left[\frac{(b+t/q)f(a+t/p)-(a+t/p)f(b+t/q)}{b-a+rt}-\frac{bf(a)-af(b)}{b-a}\right].$$

Its limit as $t\to 0$ is the biderivative of f with respect to $v_{p,q}$, which assumes at x the following value:

$$\frac{1}{(b-a)}\left\{\left[\frac{f'(b)}{q}-\frac{f'(a)}{p}-\left(\frac{1}{q}-\frac{1}{p}\right)\frac{f(b)-f(a)}{b-a}\right]x\right.$$
$$\left.+\left[\left(\frac{b}{p}f'(a)\right)-\frac{a}{q}f'(b)\right)+\left(\frac{a}{q}-\frac{b}{p}\right)\frac{f'(b)-f'(a)}{b-a}\right]\right\}.$$

It is easily seen that by varying p and q one can obtain the biderivative at (a, b) of f with regard to any gauge function g. All one has to do is to set

$$p=g'(a)-\frac{g(b)-g(a)}{b-a}\quad\text{and}\quad q=g'(b)-\frac{g(b)-g(a)}{b-a}\cdot$$

7. A geometric representation of the biderivative. The uniderivative, ${}_1\mathbf{D}f$, is a one-parameter family of constant functions (see Section 1). Geometrically, the function corresponding to a particular value, a, of the parameter may be represented by the horizontal line of altitude $f'(a)$ carrying, as it were, a bead at the abscissa, a, connecting the line visibly with the corresponding value of the parameter. The graph of the classical derivative function, f', is, as it were, a string through all those beads. Knowing this string f' one can, of course, reconstruct the uniderivative ${}_1\mathbf{D}f$ by attaching a horizontal line to each bead.

The biderivative ${}_2\mathbf{D}f$ is a two-parameter family of linear functions. But the representation by a two-parameter family of straight lines, each carrying two beads, would not be very transparent. The question arises whether the biderivative is not capable of a simpler geometric representation. At first, one might hope to find a curve through all the beads such that each straight line of the family is a chord of that curve joining two beads. Unfortunately, however, except in trivial cases such a curve does not exist.

The chords of any curve h form a two-parameter family. The altitude at x of the line corresponding to the parameters (a, b) is

$$M(a, b)x+N(a, b),$$

where

$$M(a, b)=\frac{h(b)-h(a)}{b-a}\quad\text{and}\quad N(a, b)=\frac{bh(a)-ah(b)}{b-a}.$$

The functions M and N satisfy the following conditions:

1) They are symmetric; that is to say, $M(\mathrm{a}, \mathrm{b}) = M(\mathrm{b}, \mathrm{a})$ and $N(\mathrm{a}, \mathrm{b}) = N(\mathrm{b}, \mathrm{a})$;
2) $\mathrm{a}(\partial M/\partial \mathrm{a})(\mathrm{a}, \mathrm{b}) + (\partial N/\partial \mathrm{b})(\mathrm{a}, \mathrm{b}) = 0$ and $\mathrm{b}(\partial M/\partial \mathrm{b})(\mathrm{a}, \mathrm{b}) + (\partial N/\partial \mathrm{a})(\mathrm{a}, \mathrm{b}) = 0$.

In view of the symmetry of M and N, the two conditions 2) are equivalent.

Conversely, if two functions M and N satisfy conditions 1) and 2), then there exists a curve whose chord family is described by M and N. However, if ${}_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{x}) = P(\mathrm{a}, \mathrm{b})\mathrm{x} + Q(\mathrm{a}, \mathrm{b})$, then one readily shows that, except in the case where f is a quadratic function, P and Q do not satisfy condition 2). Incidentally, this fact also proves the insolubility of bidifferential equations of the form

$$ {}_2\mathbf{D}f = {}_2\mathbf{D}^0 f; $$

that is to say,

$$ {}_2\mathbf{D}f(\mathrm{a}, \mathrm{b}) = f[\mathrm{a}, \mathrm{b}] \quad \text{or} \quad {}_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{x}) = f[\mathrm{a}, \mathrm{b}; \mathrm{x}] $$

for any a, b, and x.

Of course, one might represent the biderivative ${}_2\mathbf{D}f$ by two surfaces, the graphs of the functions P and Q. But there exists a more interesting interpretation by a single surface, which I will denote by ${}_2f'$. It is the surface having, at any (a, b), the height

$$ {}_2f'(\mathrm{a}, \mathrm{b}) = {}_2\mathbf{D}f(\mathrm{a}, \mathrm{b}; \mathrm{a}). $$

This single surface obviously permits the reconstruction of the two-parameter family of straight lines ${}_2\mathbf{D}f$. All one has to do in order to construct the line corresponding to the parameter values (a, b) is to join in the cartesian plane the points

$$ (\mathrm{a}, {}_2f'(\mathrm{a}, \mathrm{b})) \quad \text{and } (\mathrm{b}, {}_2f'(\mathrm{b}, \mathrm{a})). $$

In particular, for the power functions one finds:

$f(\mathrm{x})$	${}_2f'(a, b)$
x	0
x^2	1
x^3	$2\mathrm{a} + \mathrm{b}$
x^4	$3\mathrm{a}^2 + 2\mathrm{ab} + \mathrm{b}^2$
. . .	. . . .
x^n	$\sum_{\mathrm{k}=0}^{\mathrm{n}-2} (\mathrm{n} - \mathrm{k} - 1)a^{\mathrm{n}-\mathrm{k}-2}b^{\mathrm{k}}$

If $f''(\mathrm{a})$ exists, one can easily show that

$$ \lim_{\mathrm{b}\to\mathrm{a}} {}_2f'(\mathrm{a}, \mathrm{b}) = \mathrm{f}''(\mathrm{a}), $$

which, according to (7) equals ${}_2\mathbf{D}f(a, a; a)$ if ${}_2\mathbf{D}f(a, a)$ exists. For instance,

if $f(x) = x^n$, then ${}_2f'(a, a) = \frac{1}{2}n(n-1)a^{n-2}$;
if $f(x) = |x|$, then ${}_2f'(a, b) = 2b/(b-a)^2$, which of course has no limit as $b \to a$.

Clearly, also ${}_2\mathbf{D}^n f$ can be represented by a two-place-function or surface ${}_2f^{(n)}$. Similarly, n-derivatives can be represented by n-place-functions.

8. Multi-integration. The classical integration process associates with any continuous function g a two-place function assuming for any pair of numbers, a, a′, the value $\int_a^{a'} g(t)dt$—briefly, $\int_a^{a'} g$. If, in particular, $g = f'$, then $\int_a^{a'} g = f(a') - f(a)$.

In order to make it possible for the integrand to be a uniderivative (rather than a classical derivative function), one might replace g by a one-parameter family of constant functions. The question then arises whether there are extensions of this concept in the direction of multi-integration.

The situation will be illustrated in the case of the bi-integral of a given one-parameter family of linear functions. Let

$$M(u, v)x + N(u, v)$$

be the value for x of the linear function corresponding to the parameter pair (u, v). Can one define a process associating with this family of linear functions a 4-place function whose value for the two pairs (a, b) and (a′, b′) deserves the name of a bi-integral—more precisely, the name of the value of the bi-integral of the family from (a, b) to (a′, b′)? A proper symbol for that value would be

$${}_2\int_{(a,b)}^{(a',b')} (Mx + N).$$

In particular, does $M(u, v)x + N(u, v) = {}_2\mathbf{D}f(u, v; x)$ imply

$${}_2\int_{(a,b)}^{(a',b')} (Mx + N) = f[a', b'] - f[a, b]\,?$$

Such a process indeed exists provided that $a' + b' = a + b$. Set $u = b' - b = a - a'$; choose a positive integer n, and set up the product sum

$$\sum_{k=1}^{n}\left[M\left(a - k\frac{u}{n}, b + k\frac{u}{n}\right)x + N\left(a - k\frac{u}{n} b + k\frac{u}{n}\right)\right]\left(b - a + 2k\frac{u}{n}\right)\frac{u}{n}\cdot$$

The limit of this expression as $n \to \infty$ clearly is

$$\int_0^1 [M(a - ut, b + ut)x + N(a - ut, b + ut)](b - a + 2ut)u\,dt.$$

For instance, if $M(v, w) = 2(v + w)$ and $N(v, w) = v^2 + w^2$ for every v and w, then M and N determine the biderivative of the 4-th power function, j^4; that is to say,

$$M(\mathrm{v},\ \mathrm{w})\mathrm{x} + N(\mathrm{v},\ \mathrm{w}) = {}_2\mathrm{D}j^4(\mathrm{v},\ \mathrm{w};\ \mathrm{x}) \text{ for any v, w, and x;}$$

and, correspondingly,

$$_2\int_{(\mathrm{a},\mathrm{b})}^{(\mathrm{a}-\mathrm{u},\mathrm{b}+\mathrm{u})} (M\mathrm{x} + \mathrm{N}) = j^4[\mathrm{a} - \mathrm{u},\ \mathrm{b} + \mathrm{u}] - \frac{j^4}{4}\ [\mathrm{a},\ \mathrm{b}] \text{ for every a, b, and u.}$$

The addition of any linear function to j would, of course, not change either the biderivative or the difference of the chords. In this respect, linear functions play the same role in bi-integration that constant functions play in uni-integration.

While the reciprocity between biderivation and bi-integration is restricted to what might be called the symmetric case (that is, to a bi-integration from (a, b) to a pair (a − u, b + u)), a linear bi-integration operator exists for the transition from any pair (a, b) to any pair (a′, b′). This operator associates a four-place function with every one-parameter family of linear functions.

GENERALIZED JACOBI EXPANSIONS AND CORRESPONDING DERIVATIVES*

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1. Introduction. It is natural to ask whether by suitably generalizing the definition of derivative, it may be possible to formulate as successive derivatives the coefficients A_r in a Jacobi expansion

$$f(x) = A_0(x) + A_1(x)P(x) + \cdots A_r(x)P(x)^r + \cdots, \tag{1}$$

where P denotes a polynomial of degree n and the coefficients A_r are polynomials of lower degree. Several years ago the writer noticed a property of this kind. If we write $P(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$, we can regard the coefficients A_r (for given f) as functions of the $n+1$ parameters $x, a_0, \cdots, a_{n-1}$; then if f is regular at the zeros of P, $k!A_k$ is the k-th partial derivative of A_0 with respect to a_0, and various relations connect derivatives with respect to $a_1, \cdots, a_{n-1}$. As part of a project involving the writer, Menger and Shü† recently defined a derivative of a function at a system of points in such a way that the coefficients $k!A_k$ are indeed successive derivatives of f at the system of zeros of P. In their definition, the interpolation polynomial of f at a one-parameter family of n-tuples is used somewhat as in our Section 3, and their derivatives are defined as polynomials whose coefficients are functions, and are accordingly two-place functions, or even $(n+1)$-place functions if the n points are regarded as arbitrary, rather than one-place functions like the elementary derivative.

In the present article the Menger-Shü derivative is somewhat generalized, and is then reconstituted as a one-place function under fairly general conditions. It becomes then remarkably similar to the elementary derivative.

A method of obtaining a large variety of expansions, both new and old, is described at the end of Section 3.

Properties of the generalized derivative are briefly developed in Section 4, and a useful expression analogous to a divided difference formula is found. Analogues of Rolle's and mean-value theorems are discussed in Section 5. Applications to the solution of certain types of differential-functional equations are indicated in Section 6. As an interesting example, but independently of the general theory, a very simple way of developing the theory of Jacobi series is given in Section 2. The simplicity is induced here, first by separating from f the discussion of the mapping associated with P, and then reducing the study to that of certain coefficient functions rather than of f itself. The same idea lies behind the general algorithm of Section 3.

The author wishes to express his gratitude to both Professors Menger and Shü for their inspiration and help in connection with the project and this paper. References to the literature will be found in their work.†

* This work was sponsored by the Office of Ordnance Research under Project TB 2-0001(888).

† K. Menger and S. S. Shü, Generalized derivatives and expansions, Proc. Nat. Acad. Sci., vol. 41, 1955, pp. 591–595. Further references to the literature will be found herein.

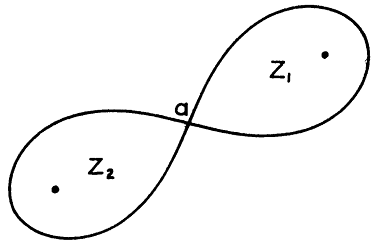
2. An example: expansion in powers of a polynomial. It will be instructive to precede the general theory of Section 3 with an interesting special case. The treatment is independent of the general theory and really simple.

Consider the mapping between two complex planes defined by

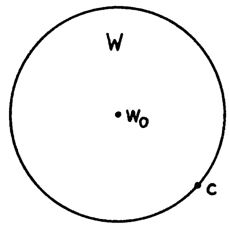
$$w = z^2 - 2az + b, \tag{1}$$

where a and b are given complex numbers. Set $c = b - a^2$. If w_0 is any point different from c in the w-plane, the interior W of the circle with center w_0 and passing through c is mapped into the two half interiors Z_1 and Z_2 of a lemniscate with vertex at the point a in the z-plane.

z-plane



w-plane



Equation (1) has two solution-functions $z_1 = \phi_1(w)$ and $z_2 = \phi_2(w)$ which are regular in W and map it one-to-one onto Z_1 and Z_2 respectively.

Consider now a function f regular in both Z_1 and Z_2. We allow the possibility that, in the sense of analytic continuation, f may be a different analytic function in the two parts. With f are associated two functions a_0 and a_1 defined on W as follows:

$$f(z) = a_0(w) + a_1(w)z \tag{2}$$

under $z = z_1 = \phi_1(w)$ and $z = z_2 = \phi_2(w)$, w in W.

On solving for a_0 and a_1, one finds

$$a_0(w) = \frac{z_1 f(z_2) - z_2 f(z_1)}{z_1 - z_2}, \qquad a_1(w) = \frac{f(z_1) - f(z_2)}{z_1 - z_2}. \tag{3}$$

Since f is regular in Z_1 and Z_2, and since $\phi_1(w) \neq \phi_2(w)$ in W, it is evident from (3) that a_0 and a_1 are regular in W. Hence we can expand

$$a_j(w) = \sum_{n=0}^{\infty} c_{nj}(w - w_0)^n \qquad (j = 0, 1), \tag{4}$$

in series convergent in W. This gives at once the expansion

$$f(z) = \sum_{n=0}^{\infty} (c_{n0} + zc_{n1})(h(z) - h(z_0))^n, \tag{5}$$

valid in Z_1 and Z_2. Here $h(z)=z^2-2az+b$ and $h(z_0)=w_0$.

Consider now the expression

$$a_0'(w) + za_1'(w) \tag{6}$$

obtained from (2) by differentiating the coefficient functions. The one-place function f^* defined by

$$f^*(z) = a_0'(h(z)) + za_1'(h(z)) \tag{7}$$

bears the same relation to (6) (under (1)) as f does to the right member of (2). We will call the function f^* the h-derivative of f, and designate it by D_hf. It has in Z_1 and Z_2 the expansion

$$\sum_{n=1}^{\infty} n(c_{n0} + zc_{n1})(h(z) - h(z_0))^{n-1}. \tag{8}$$

We may call the linear function $c_{00}+zc_{01}$ the *linear value of f at z_0*, or perhaps better, *at z_0 and $2a-z_0$*. (Here z_0 and $2a-z_0$ are *conjugate* points, for which h has the same value.) It follows that the linear function $c_{10}+zc_{11}$ is the linear value of D_hf; and that $n!(c_{n0}+zc_{n1})$ is the linear value at z_0 and $2a-z_0$ of D_h^nf, where

$$D_h^nf = D_h(D_h^{n-1}f) \qquad (n = 2, 3, \cdots).$$

If the two parts of f are not analytic continuations of one another, then c must be a singularity (and, if a path around c is possible, a branch point) of a_0 or a_1. If f is given as regular in a neighborhood of z_0 and $2a-z_0$, smaller than Z_1 and Z_2, a circle of some radius less than $|c-w_0|$ can be used, and an expansion of f obtained in a corresponding region.

The ideas of this section extend in an obvious manner to the expansion of analytic functions in powers of given polynomials of any degree.

3. The interpolation value and bifunction of a function. Let $W, Z_1, \cdots, Z_n$ be subsets of a ring R. Let W be mapped one-to-one onto Z_i by a function p_i $(i=1, \cdots, n)$. We may denote a sequence of corresponding elements by $w, z_1, \cdots, z_n$, so that $z_i=p_i(w)$. We shall, unless otherwise indicated, assume that for every given w the n elements $z_1, \cdots, z_n$ are distinct.

Write Z for the set of all elements in $Z_1, \cdots, Z_n$. By an *interpolation basis* is meant a sequence of n functions $q_1, \cdots, q_n$ defined on Z to R, such that for every function f on Z to R there exist for every given w uniquely determined elements $a_1(w), \cdots, a_n(w)$ of R such that

$$f(z_i) = a_1(w)\cdot q_1(z_i) + \cdots + a_n(w)\cdot q_n(z_i) \qquad (i = 1, \cdots, n). \tag{1}$$

In this manner n coefficient functions $a_1, \cdots, a_n$ on W to R are associated with f.

A two-place function of the form $b_1q_1+ \cdots +b_nq_n$, where b_i are functions

"of w", and the q_i "of z", will be called a *bifunction*. The bifunction $a_1q_1+ \cdots +a_nq_n$ defined by (1) will be termed the *interpolation function of f* relative to the basis. For a given value of w, the function "of z"

$$a_1(w)\cdot q_1 + \cdots + a_n(w)\cdot q_n \tag{2}$$

will be called the *interpolation value* at w, or the *w-value* of f. Such w-values will have derivative zero in the sense to be defined in Section 4, and will play the role of constants.

Examples easily show that, in general, it is not true conversely that every bifunction is the interpolation function of some function g on Z to R. It is natural to ask whether the p's and q's can be selected so that every bifunction will be an interpolation function. We shall find a necessary and sufficient condition for this to be the case, assuming however that q_1 is the constant function 1, and that $z_1, \cdots, z_n$ are distinct for each w. This condition is that there exists a function h on Z to W such that

$$h(p_i(w)) = w \qquad (w \text{ in } W;\ i = 1, \cdots n). \tag{3}$$

In other words the inverse mapping is also one-valued; if the Z's overlap, common points come from the same w. A function h satisfying (3) will be called a *gauge function.*

The necessity of the existence of a gauge function follows from the fact that $w\cdot 1+0\cdot q_2+ \cdots +0\cdot q_n$ must be the interpolation function of some function, and if we denote this function by h we have (3). The sufficiency follows from the observation that the one-place function obtained from the bifunction $b_1q_1+ \cdots +b_nq_n$ by substituting h for w, that is, the function defined by

$$b_1(h(z))\cdot q_1(z) + \cdots + b_n(h(z))\cdot q_n(z) \qquad (z \text{ in } Z), \tag{4}$$

has the bifunction $b_1q_1+ \cdots +b_nq_n$ as its interpolation function. It should be remarked in passing that while this argument is incomplete when equalities are allowed among $z_1, \cdots, z_n$, it can be extended so that the construction in (4) applies to certain other cases, with a suitable modification of the definition of interpolation function.

We thus have a remarkable situation in which there is a unique association between one-place functions f on Z to R and bifunctions. Any linear operator acting on the coefficient functions $b_1, \cdots, b_n$ induces a corresponding linear operator on the one-place functions f; and conversely. For example, if in appropriate cases $b_1, \cdots, b_n$ have a derivative at w, then there is defined a corresponding "derivative" of f at z (where z may be identified with any of $z_1, \cdots, z_n$). Again, if the coefficient functions can be expanded in a series of any kind (for example, a power series or trigonometric series), or can be represented by certain types of integrals, then there is a corresponding expansion or integral representation of f; and conversely. To establish the validity of such expressions it is only necessary to determine those properties of f which cause the coefficient functions to possess the corresponding expansions. For example

in Section 2, the analyticity of the coefficient functions at points w for which the associated z's are distinct followed from the analyticity at the points z of f.

In cases of equality among $z_1, \cdots, z_n$ for given w, the n equations (1) are no longer independent. In appropriate cases one may adopt a convention like the familiar one of having not only the values of f and $a_1(w)q_1 + \cdots + a_n(w)q_n$ agree on $z_1, \cdots, z_n$, but also their derivatives up to order $k-1$ at points of multiplicity k. Menger and Shü have given conditions under which the coefficient functions have certain continuity or differentiability properties, when $q_i(z) = z^{i-1}$.

4. Definition, evaluation and properties of a generalized derivative. We now assume that R is the field of real or complex numbers, and that the sets W, $Z_1, \cdots, Z_n$ are either simply-connected open regions, or open intervals or arcs. We assume $q_1, \cdots, q_n$ to be continuously differentiable on Z up to order n. By definition of interpolation basis, if $z_1, \cdots, z_n$ are distinct for a given w, the determinant $\Delta = |q_j(z_i)|$ cannot vanish.

If the mapping functions $p_1, \cdots, p_n$ are continuous at a point w of W, then the application of Cramer's rule to (1) of Section 3, with z_i replaced by $p_i(w)$ $(i=1, \cdots, n)$, expresses $a_j(w)$ as a quotient of two determinants each of which is continuous in w if f is continuous at each of $z_1, \cdots, z_n$. Hence if f is continuous, the coefficient functions a_j are continuous at points w for which $z_1, \cdots, z_n$ are distinct. Also at such points, if the mapping functions $p_1, \cdots, p_n$ are differentiable up to order k and f is differentiable up to order k, and the q's are likewise differentiable, then the coefficient functions are differentiable up to order k. When equalities occur among $z_1, \cdots, z_n$, the situation can be more complicated, and it is usually necessary to assume continuity or differentiability of $p_1, \cdots, p_n$ of a higher order than that desired for the coefficient functions.

The bifunction defined by

$$a_1'(w)q_1(z) + \cdots + a_n'(w)q_n(z) \qquad (w \text{ in } W, z \text{ in } Z), \tag{1}$$

will, when there is a gauge function, be the interpolation function of the one-place function defined for each z of Z by

$$a_1'(h(z))q_1(z) + \cdots + a_n'(h(z))q_n(z). \tag{2}$$

The one-place function so defined will be called the h-derivative of f relative to the system p_i, q_i; or briefly, the *h-derivative of f*, and denoted by D_hf.

If f is a function of h, say $\psi(h)$, the corresponding bifunction is $\psi(w)\cdot 1 + 0\cdot q_2 + \cdots$, and hence the h-derivative is $\psi'(h)$. Similarly, the h-derivative of a product in which one of the factors is a function of h is given by a formula analogous to that in the elementary calculus:

$$D_h(f\cdot g(h)) = f\cdot g'(h) + D_hf\cdot g(h). \tag{3}$$

If f has the form $c_1q_1 + \cdots + c_nq_n$, where the coefficients $c_1, \cdots, c_n$ are constants, then $D_hf = 0$. Such functions may be called h-constants. The h-derivative

of a sum of terms is the sum of their h-derivatives. If A is an h-constant, the h-derivative of $A\cdot h^k$ is kAh^{k-1}. (Our discussion here is brief; conditions under which these formulae hold are obvious.)

We will now derive an explicit formula for D_hf in the case where $z_1, \cdots, z_n$ are distinct and f is differentiable at the points $z_1, \cdots, z_n$, and where $h'(z)$ does not vanish at these n points. The expression obtained will be in terms of $z_1, \cdots, z_n$, but becomes a function "of z" by identifying z with any one of these n quantities, and then through the mapping process expressing the others in terms of that one.

From (1) of Section 3 we have, for every w in W,

$$f(p_i(w)) = a_1(w)q_1(p_i(w)) + \cdots + a_n(w)q_n(p_i(w)) \qquad (i = 1, \cdots, n). \tag{4}$$

Differentiating with respect to w, and noting that $h(p_i(w))=w$ gives $h'(p_i) = 1/p_i'$, we have (assuming existence of all derivatives listed) that

$$a_1' \cdot q_1(p_i) + \cdots + a_n' \cdot q_n(p_i) = \left(f'(p_i) - \sum_j a_j \cdot q_j'(p_i)\right)/h'(p_i). \tag{5}$$

Hence the left member is the value at $p_i(=p_i(w))$ of

$$\left(f' - \sum a_j \cdot q_j'\right)/h'. \tag{6}$$

Since the value of (6) at p_i is given by the left member of (5) we can use (6) to evaluate the one-place function D_hf. We transform (6) by substituting $a_j(h(z)) = \Delta_j/\Delta$, obtained by applying Cramer's rule to the equations

$$\sum_j a_j(h(z)) \cdot q_j(z_i) = f(z_i) \qquad (i = 1, \cdots, n), \tag{7}$$

where $z_i = p_i(h(z))$. Here Δ is the determinant $|q_j(z_i)|$, and Δ_k is obtained by replacing the column of elements $q_k(z_i)$ by $f(z_i)$, $k=1, \cdots, n$. Hence by (6), the derivative D_hf has the form

$$\left\{\Delta \cdot f'(z) - \sum_j \Delta_j q_j'(z)\right\} \Big/ (\Delta \cdot h'(z)),$$

and this is seen to be the expansion of

$$\begin{vmatrix} f'(z) & 0 & q_2'(z) & q_3'(z) & \cdots & q_n'(z) \\ f(z_1) & 1 & q_2(z_1) & q_3(z_1) & \cdots & q_n(z_1) \\ f(z_2) & 1 & q_2(z_2) & q_3(z_2) & \cdots & q_n(z_2) \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ f(z_n) & 1 & q_2(z_n) & q_3(z_n) & \cdots & q_n(z_n) \end{vmatrix} \cdot \frac{1}{\Delta \cdot h'(z)}, \tag{8}$$

where as noted above, z can be identified with any of $z_1, \cdots, z_n$, and the other

conjugates expressed in terms of that one. In the case $q_j(z)=z^{j-1}$, the entire expression (8) is the divided difference of f at the points $z, z_1, \cdots, z_n$, where z coincides with one of the z_i. Presumably the usual conventions for divided differences will extend the validity of this expression for $D_h f$ when equalities occur among the n conjugates of z.

Formula (8) suggests a "rate" interpretation of $D_h f$: it may be described as a weighted measure of the difference between the local rate of change of f at z, and the average rate of change between z and its conjugate points. If $n=2$, the expression (8) reduces to

$$\frac{1}{h'(z)}\left\{f'(z) - q_2'(z)\cdot\frac{f(z^*)-f(z)}{q_2(z^*)-q_2(z)}\right\}, \tag{9}$$

where z^* is the conjugate of z. If $q_2(z)=z$, this becomes

$$\frac{1}{h'(z)}\left\{f'(z) - \frac{f(z^*)-f(z)}{z^*-z}\right\}. \tag{10}$$

If here $h(z)=z^2-rz+s$, then $z^*=r-z$ and $h'(z)=z-z^*$. Graphically this can be described as the one-place function obtained from the graph of $y=f(x)$ in the following manner. Take the difference between the chords at $z+k$ and z^*-k, and at z and z^*, divide by k, take the limit as k tends to zero, thus getting a straight line; divide by $z-z^*$, getting another straight line. Retain of this line only the two points with abscissae z and z^*: the locus of all such point pairs is the graph of $D_h f$.

Notice that the denominator $\Delta\cdot h'(z)$ in (8) is the value of the numerator with f replaced by h.

5. A remark on analogues of Rolle's or mean-value theorems. Consider the case $n=2$, $h(x)=x^2-rx$, r real. On the graph of $y=f(x)$ let the points with abscissae x_1, $a-x_1$, x_2, $a-x_2$ be collinear; in other words, let the linear values of f at x_1 and x_2 be the same. Let f be continuous at the four points and suppose f' exists between x_1 and x_2, and between $r-x_1$, $r-x_2$. By (10) of Section 4, $D_h f$ exists between x_1 and x_2. A direct analogue of Rolle's theorem would require the conclusion that for some point ξ between x_1 and x_2, the linear value of $D_h f$ at ξ should vanish; in other words, the points with abscissae ξ and $a-\xi$ should have a common tangent. Brief examination of a figure will show that in general this is not so, except in the rather special cases in which f is either symmetric or anti-symmetric about the point r; that is, for x between x_1 and x_2, either $f(r-x)=f(x)$, or $f(r-x)=-f(x)$. By writing f as the sum of $f_1(x)=\frac{1}{2}(f(x)+f(a-x))$ and $f_2(x)=\frac{1}{2}(f(x)-f(a-x))$, one can develop (exactly as in the elementary calculus) Taylor expansions with remainder terms. However, these remainder terms involve linear values of the n-th h-derivatives of f_1 and f_2 at different intermediate points.

However, closer examination shows that this type of development, while

interesting because of the analogy with elementary calculus, is really a subterfuge for the expansion of the coefficient functions $a_1(t)$ and $a_2(t)$, where

$$f(x) = a_1(t) + xa_2(t), \text{ under } t = h(x) - h(x_1);$$

and the symmetric and anti-symmetric cases are simply those in which $a_1(t)$ and $a_2(t)$ are linearly dependent. When Z_1 is a real interval, one can construct for the general situation of Sections 3, 4, expansions of the Taylor type with remainder terms involving values at n different intermediate points of derivatives of the coefficient functions. Nonetheless, expansions in which the remainders involve only one intermediate point are also possible in fairly general cases, and indeed, to get such expansions, we need only keep x constant in the expression $a_1(t)\cdot q_1(k)+ \cdots +a_n(t)\cdot q_n(k)$, apply Taylor's formula to this function of t, and replace k by x, t by $h(x)$, and the intermediate point τ provides a single intermediate point ξ such that $h(\xi) = \tau$.

6. Certain differential-functional equations. We mention briefly an application to the solution of certain types of differential equations involving values of the solution function at two or more points. If a function h can be found having equal values at such "conjugate points", and if the equation can be expressed in the form $D_h f=g$ for a suitable choice of p's and q's, then the equation can be solved by forming the bifunction of g, integrating the coefficient functions with respect to their argument, and then substituting $h(z)$ for that argument in the resulting expressions.

In some cases, where the conjugate points can be represented in terms of one another by the same analytic expression, one can even avoid using the parameter w, by using $dw=h'(z)dz$. For example, with (10) of Section 5 in mind, let us solve the equation

$$f'(x) - \frac{f(x) - f(1-x)}{2x-1} = x^2.$$

Dividing by $h'(x)=2x-1$, we choose functions $a_1(t)$, $a_2(t)$ such that

$$x^2/(2x-1) = a_1(t) + xa_2(t), \qquad (1-x)^2/(1-2x) = a_1(t) + (1-x)a_2(t).$$

Hence $a_2(t)=(2x^2-2x+1)/(2x-1)$, $2a_1(t)+a_2(t)=1$, and since $dt=(2x-1)dx$,

$$\begin{aligned}\int a_2(t)dt &= \int (2x^2-2x+1)/(2x-1)dx = \tfrac{1}{2}(x^2-x) + \tfrac{1}{4}\ln|2x-1| + c_1,\\ &= x^2 - x - 2\int a_1(t)dt - 2c_2,\end{aligned}$$

hence

$$\begin{aligned}f(x) &= \int a_1(t)dt + x\int a_2(t)dt\\ &= (2x^3-x^2-x)/4 + ((2x-1)/8)\ln|2x-1| + c_1x + c_2.\end{aligned}$$

A COVERING THEOREM FOR UNIVALENT FUNCTIONS

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Let U denote the class of normalized univalent functions $f(z)$ of the form

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots, \qquad |z| < 1.$$

An elementary yet interesting and important property shared by functions of U is stated in the Bieberbach coefficient theorem [1]:

If $f(z)$ belongs to U then

$$|a_2| \leqq 2;$$

moreover, $|a_2| = 2$ for a function $f(z)$ of U if and only if

$$f(z) = k_\alpha(z) = \frac{z}{(1 - e^{i\alpha}z)^2}, \qquad |z| < 1, \alpha \textit{ real.}$$

An immediate consequence of the above result is the well known Koebe-Bieberbach covering theorem [1, 2] (see, for example, [3, pp. 209 ff.]):

If $f(z)$ belongs to U and omits the value γ, then

$$|\gamma| \geqq 1/4;$$

moreover, $\gamma = -e^{-i\alpha}/4$, α real, is an omitted value for $f(z)$ of U if and only if $f(z) = k_\alpha(z)$.

This theorem shows that the map of $|z| < 1$ by a function of U always contains the open disc $|w| < 1/4$, and that there is no larger disc with center at $w = 0$ which is contained in every such map. For contrast with the result to be presented below, we emphasize the fact that any boundary point of the disc $|w| < 1/4$ corresponds to an omitted value for some function of U.

Let U_+ denote the subclass of U composed of the identity function, z, and those functions of U for which the first non-zero power series coefficient, a_m, $m \geqq 2$, satisfies $a_m > 0$. Since for real θ, $e^{i\theta}f(e^{-i\theta}z)$ belongs to U whenever $f(z)$ belongs to U, it follows that to each function of U there corresponds by such a transformation a uniquely determined function of U_+.

For $f(z)$ in U_+ and for fixed ϕ let $\rho_f(\phi)$ denote the distance, perhaps infinite, from $w = 0$, along the ray $\arg w = \phi$, to the nearest boundary point of the map of $|z| < 1$ by $w = f(z)$. Since the function $f(z) = z$ is in U_+,

$$\rho(\phi) = \operatorname*{glb}_{f \in U_+} \rho_f(\phi)$$

is finite and, because of the Koebe-Bieberbach theorem, satisfies

$$1/4 \leqq \rho(\phi) \leqq 1.$$

From the definition of $\rho(\phi)$ it is evident that if $\gamma=\rho e^{i\phi}$ is an omitted value for a function of U_+, then $\rho\geqq\rho(\phi)$. It is also evident that $\rho(-\phi)=\rho(\phi)$, since the function

$$\bar{f}(z) = z + \bar{a}_2 z^2 + \cdots + \bar{a}_n z^n + \cdots, \qquad |z| < 1,$$

belongs to U_+ whenever $f(z)$ belongs to U_+, and the values taken on by $\bar{f}(z)$ in $|z|<1$ are the conjugates of values taken on by $f(z)$ in $|z|<1$.

THEOREM. *If $f(z)$ belongs to U_+ and for $|z|<1$ omits the value $\gamma=\rho e^{i\phi}$ then $\rho\geqq\rho(\phi)$, and*

$$\rho(\phi) = 1/2, \qquad 0 \leqq |\phi| \leqq \pi/2,$$

$$\tfrac{1}{2}|\sin\phi| < \rho(\phi) \leqq \tfrac{1}{2}, \qquad \pi/2 < |\phi| \leqq 3\pi/4,$$

$$\frac{1}{4|\cos\phi|} < \rho(\phi) \leqq \tfrac{1}{2}, \qquad 3\pi/4 \leqq |\phi| < \pi,$$

$$\rho(+\pi) = 1/4.$$

Moreover, for $0\leqq|\phi|<\pi/2$, $\rho(\phi)e^{i\phi}$ is not an omitted value for any function of U_+, but for $\pi/2\leqq|\phi|\leqq\pi$, each value $\rho(\phi)e^{i\phi}$ is an omitted value for some function of U_+.

We start the proof by deducing a necessary condition for a function $f(z)$ of U_+ to omit the value $\gamma=\rho e^{i\phi}$. The function

$$g(z) = \frac{f(z)}{1-f(z)/\gamma} = z + \left(a_2 + \frac{1}{\gamma}\right)z^2 + \cdots$$

belongs to U and hence $|a_2+1/\gamma|\leqq 2$, or, in an equivalent statement,

$$(4 - a_2^2)\rho^2 - 2a_2\rho\cos\phi - 1 \geqq 0,$$

and equality can hold here only if $g(z)=k_\alpha(z)$ for suitable real α.

Geometric motivation for the continuation of the proof may be had by using equality in the above necessary condition and regarding a_2, $0\leqq a_2\leqq 2$, as a parameter. It is readily found that the interiors of the resulting circles have in common a region on whose boundary the value of ρ is precisely that given as a lower bound for $\rho(\phi)$ in the theorem.

We return to the proof of the theorem and suppose that $\rho\leqq 1/2$, $0\leqq|\phi|\leqq\pi/2$. Then $(4-a_2^2)\rho^2-2a_2\rho\cos\phi-1=-(1-4\rho^2)-a_2\rho(a_2\rho+2\cos\phi)\leqq 0$, and in view of the stated necessary condition it follows that $\rho=1/2$, $a_2=0$, $g(z)=k_\alpha(z)$ and

$$f(z) = \frac{z}{1-2(e^{i\alpha}-e^{-i\phi})z+e^{2i\alpha}z^2}.$$

We observe that $\alpha=-\phi$ since $a_2=0$, and conclude thereafter that $\phi=\pm\pi/2$ since $a_3=-e^{-2i\phi}$ must be positive. Thus we have shown that $\rho(\phi)\geqq 1/2$ for $0\leqq|\phi|\leqq\pi/2$ and that no function of U_+ omits the value $\frac{1}{2}e^{i\phi}$ for $0\leqq|\phi|<\pi/2$.

The function $f(z)$ obtained above is $z/(1-z^2)$, which omits the values $\pm i/2$, and, in particular, $\rho(\pm\pi/2)=1/2$.

Suppose next that $\rho\leqq|\sin\phi|/2$, $\pi/2<|\phi|\leqq 3\pi/4$. Then

$$(4-a_2^2)\rho^2-2a_2\rho\cos\phi-1=4\rho^2-\sin^2\phi-(a_2\rho+\cos\phi)^2\leqq 0,$$

from which it follows that $\rho=|\sin\phi|/2$, $a_2\rho+\cos\phi=0$, and hence that $a_2=-2\cos\phi/|\sin\phi|$. In addition, $g(z)=k_\alpha(z)$, or

$$f(z)=\frac{z}{1-2(e^{i\alpha}-e^{-i\phi}/|\sin\phi|)z+e^{2i\alpha}z^2},$$

and

$$a_2=2\left[\cos\alpha-\frac{\cos\phi}{|\sin\phi|}+i\left(\sin\alpha+\frac{\sin\phi}{\sin|\phi|}\right)\right].$$

Comparison of the values of a_2 gives $\phi=-\alpha=\pm\pi/2$, which is not in the permissible range of ϕ. Thus no function of U_+ omits $\rho e^{i\phi}$ for $\rho\leqq|\sin\phi|/2$, $\pi/2<|\phi|\leqq 3\pi/4$, and $\rho(\phi)\geqq|\sin\phi|/2$ in this range.

Now suppose that $\rho\leqq 1/4|\cos\phi|$, $3\pi/4\leqq|\phi|\leqq\pi$. Since $\cos^2\phi\geqq 1/2$,

$$(4-a_2^2)\rho^2-2a_2\rho\cos\phi-1\leqq-\frac{(2-a_2)(8\cos^2\phi-2-a_2)}{16\cos^2\phi}\leqq 0,$$

and it follows from the necessary condition that $a_2=2$, $f(z)=z/(1-z)^2$, and $\gamma=-1/4$. Thus $\rho(\pm\pi)=1/4$, $\rho(\phi)\geqq 1/4|\cos\phi|$ for $3\pi/4\leqq|\phi|<\pi$, and no function of U_+ omits $\rho e^{i\phi}$ for $\rho\leqq 1/4|\cos\phi|$, $3\pi/4\leqq|\phi|<\pi$.

Next we show that for any β, $0<|\beta|<\pi/2$, there exists a function of U_+ which omits the value $te^{i\beta}$, where t, which is necessarily greater than $1/2$, is arbitrarily close to $1/2$. This will complete the proof that $\rho(\phi)=1/2$, $0\leqq|\phi|\leqq\pi/2$.

The function $f(z)=z/(1+e^{-2i\beta}z^2)$ belongs to U and omits the values $e^{i\beta}/2$, $e^{i(\beta+\pi)}/2$. It is readily seen that for $|\lambda|<1$,

$$h(z,\lambda)=\frac{1}{(1-\lambda\bar\lambda)f'(\lambda)}\left[f\left(\frac{z+\lambda}{\bar\lambda z+1}\right)-f(\lambda)\right]$$

belongs to U, and by computation it is found that

$$h(z,\lambda)=\frac{z+\dfrac{\bar\lambda-e^{-2i\beta}\lambda}{1-e^{-2i\beta}\lambda^2}}{1+2\dfrac{\bar\lambda+e^{-2i\beta}\lambda}{1+e^{-2i\beta}\lambda^2}z+\dfrac{\bar\lambda^2+e^{-2i\beta}}{1+e^{-2i\beta}\lambda^2}z^2}$$
$$=z+A(\lambda)e^{i\alpha(\lambda)}z^2+\cdots,$$

where

$$A(\lambda)e^{i\alpha(\lambda)} = e^{-i\beta}\left[\frac{e^{i\beta}\bar{\lambda} - e^{-i\beta}\lambda}{1 - e^{-2i\beta}\lambda^2} - 2\,\frac{e^{i\beta}\bar{\lambda} + e^{-i\beta}\lambda}{1 + e^{-2i\beta}\lambda^2}\right].$$

If λ is chosen so that $A(\lambda) > 0$, then the function

$$f(z, \lambda) = e^{i\alpha(\lambda)}h[e^{-i\alpha(\lambda)}z, \lambda] = z + A(\lambda)z^2 + \cdots$$

belongs to U_+. Another computation shows that the omitted values of $f(z, \lambda)$ corresponding to the omitted values $\pm e^{i\beta}/2$ of $f(z)$ are

$$\frac{e^{i\beta}}{2}\cdot e^{i\alpha(\lambda)}\,\frac{(1 + e^{-2i\beta}\lambda^2)(1 - e^{-i\beta}\lambda)}{(1 - \lambda\bar{\lambda})(1 + e^{-i\beta}\lambda)}, \qquad -\frac{e^{i\beta}}{2}\cdot e^{i\alpha(\lambda)}\,\frac{(1 + e^{-2i\beta}\lambda^2)(1 + e^{-i\beta}\lambda)}{(1 - \lambda\bar{\lambda})(1 - e^{-i\beta}\lambda)}.$$

For $\lambda = ce^{i\gamma}$ power series expansion gives

$$X = A(\lambda)e^{i\alpha(\lambda)}\,\frac{(1 + e^{-2i\beta}\lambda^2)(1 - e^{-i\beta}\lambda)}{1 + e^{-i\beta}\lambda} = c[-e^{-i\gamma} - 3e^{i(\gamma-2\beta)} + O(c)],$$

and it is readily found that a value γ_0 of γ for which

$$\operatorname{Im}\,[-e^{-i\gamma} - 3e^{i(\gamma-2\beta)}] = \sin\gamma - 3\sin(\gamma - 2\beta) = 0$$

is given by

$$\sin\gamma_0 = -\frac{3\sin 2\beta}{\sqrt{10 - 6\cos 2\beta}}, \qquad \cos\gamma_0 = \frac{1 - 3\cos 2\beta}{\sqrt{10 - 6\cos 2\beta}}.$$

For $\gamma = \gamma_0$, Re $[-e^{-i\gamma} - 3e^{i(\gamma-2\beta)}] > 0$ and, because of the continuity of the function X, for $c > 0$, sufficiently small, there is a closed interval about γ_0 in which Re $X > 0$. For γ in this closed interval Im $[-e^{-i\gamma} - 3e^{i(\gamma-2\beta)}]$ changes sign and hence, for $c > 0$, sufficiently small, there is a value γ_1 for which Im $X = 0$. Thus for each $c > 0$, sufficiently small, there is a γ_1 for which $\lambda = ce^{i\gamma_1}$ gives $A(\lambda) > 0$ and

$$\arg e^{i\alpha(\lambda)}\,\frac{(1 + e^{-2i\beta}\lambda^2)(1 - e^{-i\beta}\lambda)}{1 + e^{-i\beta}\lambda} = 0.$$

That is, for $\lambda = ce^{i\gamma_1}$

$$\frac{e^{i\beta}}{2}\left|\frac{(1 + e^{-2i\beta}\lambda^2)(1 - e^{-i\beta}\lambda)}{(1 - \lambda\bar{\lambda})(1 + e^{-i\beta}\lambda)}\right| = te^{i\beta}$$

is an omitted value for $f(z, \lambda)$. For any β, $0 \leqq |\beta| \leqq \pi$ it is clear that $t \to 1/2$ as $c \to 0$ and it follows that $\rho(\phi) \leqq 1/2$, $0 \leqq |\phi| \leqq \pi$.

There remains to be proved the statement that each value $\rho(\phi)e^{i\phi}$, $\pi/2 \leqq |\phi| \leqq \pi$, is an omitted value for some function of U_+. It is sufficient to consider the interval $\pi/2 < |\phi| < \pi$ since $\rho(\pm\pi/2) = 1/2$, $\rho(\pm\pi) = 1/4$, and the values $\pm i/2$, $-1/4$, are omitted by the functions $z/(1-z^2)$, $z/(1-z)^2$, respectively. Also, we

need consider only the case $1/4<\rho(\phi)<1/2$, $\pi/2<|\phi|<\pi$ since the function $f(z, \lambda)$ obtained by putting $\beta=-\pi/2$, $\lambda=c$, $0<c<1$, is

$$\left(z+\frac{2c}{1+c^2}z^2\right)\Big/(1-z^2),$$

and this function of U_+ omits the values

$$\frac{1}{2}\left(-\frac{2c}{1+c^2}\pm i\frac{1-c^2}{1+c^2}\right)$$

whose modulus is $1/2$.

From the definition of $\rho(\phi)$ we see that either there is a function of U_+ which omits $\rho(\phi)e^{i\phi}$, or else there is a sequence of functions $\{f_n(z)\}$ of U_+ for which $\rho_{f_n}(\phi)>\rho(\phi)$, $\rho_{f_n}(\phi)\to\rho(\phi)$. In the latter case, the sequence $\{f_n(z)\}$ has a limit function $F(z)$ in U since $U_+\subset U$ and U is a compact normal family; moreover there is a subsequence of $\{f_n(z)\}$ which converges uniformly to $F(z)$ in any closed subset of $|z|<1$. From the inequality $|a_2+1/\gamma|\leqq 2$ we get $a_2\geqq -2+1/|\gamma|$, and it follows that the coefficients a_2 for functions of the subsequence are ultimately bounded away from 0. This enables us to conclude that $F(z)$ belongs to U_+. The assumption that $F(z_0)=\rho(\phi)e^{i\phi}$ for some z_0 in $|z|<1$ together with the uniform convergence of the subsequence to $F(z)$ in a sufficiently small closed disc about z_0 leads to the conclusion that the functions of the subsequence do not omit values having $\rho(\phi)e^{i\phi}$ as a limit point, and it follows that $F(z)$ omits $\rho(\phi)e^{i\phi}$.

This completes the proof of the theorem since the possibility that $\rho(\phi)$ equals the stated lower bound for $\pi/2<|\phi|<\pi$ is now excluded.

It is worth noting that the region bounded by the curve $w=\rho(\phi)e^{i\phi}$ is not convex in the neighborhood of $w=-1/4$. This fact is an immediate consequence of the inequality $\rho(\phi)\cos\phi<-1/4$, $3\pi/4\leqq|\phi|<\pi$.

We remark in conclusion that the bound $\rho(\phi)\leqq 1/2$, $\pi/2<|\phi|<\pi$, can be improved by minimizing the modulus of an omitted value of $f(z, \lambda)$, subject to the condition that the argument of the omitted value shall be ϕ. This procedure has not been attempted since it appears to be very tedious and shows no promise of yielding the precise value of $\rho(\phi)$.

References

1. L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, K. Preuss. Akad. Wiss. Sitzungsberichte, 1916, pp. 940–955.

2. P. Koebe, Über die Uniformisierung belibiger analytischer Kurven, Nachr. Ges. Wiss. Göttingen, 1907, pp. 191–210.

3. Nehari, Conformal Mapping, McGraw-Hill, New York, 1952.

SOME CONVERGENCE PROBLEMS FOR CONTINUED FRACTIONS

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We discuss in this paper the problem of finding conditions for a continued fraction to converge in case it is given that certain subsequences of the sequence of approximants converge (Sec. 2 and Sec. 3). Use is made of formulas for cross-ratios of four approximants (Sec. 1). These formulas were used in [6] to obtain continued fractions f' whose sequences of approximants are rearrangements of the sequence of approximants of the continued fraction f. By applying known convergence criteria to f' we found new criteria for convergence of f. We are now able to find several interesting theorems in this way by means of convergence criteria discovered since [6] was written (Sec. 4 and Sec. 5).

1. Cross-ratios. Suppose a is the complex number sequence $\{a_p\}_{p=1}^{\infty}$, no term of which is 0, q a nonnegative integer, $a^q=\{a_{q+p}\}_{p=1}^{\infty}$ and $f_p^q=A_p^q/B_p^q$ the pth approximant of the continued fraction

$$f(a^q) = \frac{1}{1} + \frac{\overset{q}{a_1}}{1} + \frac{\overset{q}{a_2}}{1} + \cdots = \frac{1}{1} + \frac{a_{q+1}}{1} + \frac{a_{q+2}}{1} + \cdots,$$

so that

$$\begin{aligned} &A_0^q = 0,\ A_1^q = 1,\ A_{p+1}^q = A_p^q + a_{q+p}A_{p-1}^q, \\ &B_0^q = 1,\ B_1^q = 1,\ B_{p+1}^q = B_p^q + a_{q+p}B_{p-1}^q, \end{aligned} \qquad p = 1, 2, \cdots. \tag{1.1}$$

We omit the superscript q in case q is 0. An easy induction argument shows that $A_{p+1}^q=B_p^{q+1}$, p, $q=0, 1, \cdots$; and also, if $a_0=1$,

$$A_{p+k}B_p - A_pB_{p+k} = (-1)^p a_0a_1 \cdots a_pA_k^p, \qquad p, k = 0, 1, \cdots. \tag{1.2}$$

If $k=1$, (1.2) shows that $A_p\neq 0$ or $B_p\neq 0$, so that f_p is a point in the extended complex plane. If p and q are integers, $0\leqq p<q$, $B_p\neq 0$ and $B_q\neq 0$ then, by (1.2),

$$f_q - f_p = \frac{(-1)^p a_0a_1 \cdots a_pA_{q-p}^p}{B_pB_q}. \tag{1.3}$$

Therefore, if p, q, r and s are integers and $0\leqq p<q<r<s$,

$$[f_s, f_q, f_p, f_r] = \frac{(f_s - f_q)(f_p - f_r)}{(f_s - f_p)(f_q - f_r)} = \frac{A_{s-q}^q A_{r-p}^p}{A_{s-p}^p A_{r-q}^q}, \tag{1.4}$$

provided this cross-ratio exists. If $L=[f_p, f_q, f_r, f_s]$, then

$$\begin{aligned} &L = [f_q, f_p, f_s, f_r] = [f_r, f_s, f_p, f_q] = [f_s, f_r, f_q, f_p], \\ &\frac{1}{L} = [f_p, f_r, f_q, f_s] \quad \text{and} \quad 1 - L = [f_s, f_p, f_q, f_r]. \end{aligned} \tag{1.5}$$

Then, by (1.4),

$$[f_p, f_q, f_r, f_s] = 1 - \frac{A^p_{s-p} A^q_{r-q}}{A^p_{r-p} A^q_{s-q}}. \tag{1.6}$$

If, in particular, $q=p+1$, $r=p+2$ and $s=p+3$, this reduces to:

$$a_{p+2} = -[f_p, f_{p+1}, f_{p+2}, f_{p+3}], \qquad p = 0, 1, \cdots. \tag{1.7}$$

2. Continued fractions for which $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge. The statement that the sequence $\{x_p\}_{p=1}^{\infty}$ is absolutely convergent means the series $x_1 + \sum_{p=1}^{\infty} (x_{p+1} - x_p)$ is absolutely convergent.

THEOREM 2.1. *Each of the following conditions is sufficient for convergence of the continued fraction $f(a)$:*

(1) *$\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge and a has a bounded infinite subsequence* [7],

(2) *$\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge, one of these sequences converges absolutely, and the series $\sum |a_p|^{-1}$ diverges, and*

(3) *$\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge absolutely and the series $\sum |a_p|^{-1/2}$ diverges.*

Proof. This is an easy consequence of the formula (1.7), *i.e.*,

$$a_{p+2} \cdot (f_p - f_{p+2})(f_{p+1} - f_{p+3}) = -(f_p - f_{p+1})(f_{p+2} - f_{p+3}). \tag{2.1}$$

If $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge, a has a bounded infinite subsequence and $f(a)$ does not converge, there exists a positive number c and a positive integer N such that $|(f_p - f_{p+1})(f_{p+2} - f_{p+3})| > c$, if $p > N$, and an integer p greater than N such that $|a_{p+2}(f_p - f_{p+2})(f_{p+1} - f_{p+3})| < c$. This contradicts (2.1). If $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge, one absolutely, then $\sum |(f_p - f_{p+2})(f_{p+1} - f_{p+3})|$ converges and, by (2.1), $\sum |a_{p+2}|^{-1} |(f_p - f_{p+1})(f_{p+2} - f_{p+3})|$ converges. Since $\sum |a_{p+2}|^{-1}$ diverges, it follows that $f(a)$ converges. Finally, if $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge absolutely, we see, by (2.1), that $\sum |a_{p+2}|^{-1/2} |(f_p - f_{p+1})(f_{p+2} - f_{p+3})|^{1/2} \leqq \frac{1}{2} \sum \{ |f_p - f_{p+2}| + |f_{p+1} - f_{p+3}| \}$ so that $f(a)$ converges in case $\sum |a_{p+2}|^{-1/2}$ diverges.

In [2], R. E. Lane and the writer showed that, if $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge absolutely, then $f(a)$ converges only in case the series $\sum |b_p|$ diverges, where

$$b_1 = 1, \qquad b_{p+1} = 1/a_p b_p, \qquad p = 1, 2, \cdots. \tag{2.2}$$

David F. Dawson recently* obtained the following more comprehensive result:

If $\{b_p\}_{p=1}^{\infty}$ is a complex number sequence, the continued fraction

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots \tag{2.3}$$

converges in case

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* Unpublished.

(1) $\{f_{2p-1}\}$ *converges absolutely*, $\{f_{2p}\}$ *converges and* $\sum|b_{2p-1}|$ *diverges, or*
(2) $\{f_{2p-1}\}$ *converges*, $\{f_{2p}\}$ *converges absolutely and* $\sum|b_{2p}|$ *diverges.*

In [7] we gave an example of a divergent continued fraction $f(a)$ such that $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge and $\sum|a_p|^{-1/2}$ (and hence $\sum|b_p|$) diverges. For that example, $a_p=(-1)^p(p+1)^2$ so that $\sum|a_p|^{-1}$ converges. *There exists a divergent continued fraction* $f(a)$ *such that* $\{f_{2p-1}\}$ *and* $\{f_{2p}\}$ *converge and* $\sum|a_p|^{-1}$ *diverges.* In fact, if $f_0=0$,

$$f_{2p} = -1^{-1/2} + 2^{-1/2} - \cdots + (-1)^p p^{-1/2}, \qquad p = 1, 2, \cdots$$

and $f_{2p+1}=1+f_{2p}$, $p=0, 1, \cdots$, then $\{f_{2p-1}\}$ and $\{f_{2p}\}$ converge, $\{f_p\}$ diverges and, by (2.1), $a_{2p+2}=-(p+1)$, so that $\sum|a_p|^{-1}$ diverges. Note that the four sequences $\{f_{4p+1}\}$, $\{f_{4p+2}\}$, $\{f_{4p+3}\}$ and $\{f_{4p+4}\}$ are absolutely convergent.

3. The sets M_k. For each positive integer k, M_k denotes the set of all continued fractions $f(a)$ for which there exists a number k-tuple $L_1, \cdots, L_k$ such that $\lim_{p\to\infty} f_{kp+r}=L_r$, $r=1, \cdots, k$.

THEOREM 3.1. *If the continued fraction* $f(a)$ *belongs to* M_3 *and none of the sequences* $\{a_{3p}\}$, $\{a_{3p+1}\}$ *or* $\{a_{3p+2}\}$ *converges to the limit* -1, *then* $f(a)$ *converges.*

Proof. By (1.5) and (1.7),

$$(f_{p+1} - f_{p+3})(f_{p+2} - f_p) = \frac{1}{1 + a_{p+2}} (f_{p+1} - f_{p+2})(f_{p+3} - f_p).$$

If $f(a)$ belongs to M_3 and none of the sequences $\{a_{3p}\}$, $\{a_{3p+1}\}$ or $\{a_{3p+2}\}$ converges to the limit -1, it follows that each of the following statements is true: (1) $L_1=L_3$ or $L_2=L_3$, (2) $L_2=L_1$ or $L_3=L_1$ and (3) $L_3=L_2$ or $L_1=L_2$. Consequently, $L_1=L_2=L_3$ so that $f(a)$ is convergent.

The continued fraction $f(a)$ diverges if $a_p=-1$, $p=1, 2, 3, \cdots$. More generally, we have this theorem:

THEOREM 3.2. *The continued fraction* $f(a)$ *diverges in case any one of the following series converges*:

$$\begin{aligned}&\left|\frac{1}{a_0a_1}(1+a_1)\right| + \left|\frac{a_0a_1}{a_2}(1+a_3)\right| + \left|\frac{a_2}{a_0a_1\cdot a_3a_4}(1+a_4)\right| \\ &+ \left|\frac{a_0a_1\cdot a_3a_4}{a_2\cdot a_5}(1+a_6)\right| + \left|\frac{a_2\cdot a_5}{a_0a_1\cdot a_3a_4\cdot a_6a_7}(1+a_7)\right| \\ &+ \left|\frac{a_0a_1\cdot a_3a_4\cdot a_6a_7}{a_2\cdot a_5\cdot a_8}(1+a_9)\right| + \cdots,\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
(3.2)\qquad & \left|\frac{1}{a_0}(1+a_1)\right| + \left|\frac{a_0}{a_1a_2}(1+a_2)\right| + \left|\frac{a_1a_2}{a_0\cdot a_3}(1+a_4)\right| + \left|\frac{a_0\cdot a_3}{a_1a_2\cdot a_4a_5}(1+a_5)\right| \\
& + \left|\frac{a_1a_2\cdot a_4a_5}{a_0\cdot a_3\cdot a_6}(1+a_7)\right| \left|\frac{a_0\cdot a_3\cdot a_6}{a_1a_2\cdot a_4a_5\cdot a_7a_8}(1+a_8)\right| + \cdots,
\end{aligned}
$$

or

$$
\begin{aligned}
(3.3)\qquad & \left|\frac{1}{a_1}(1+a_2)\right| + \left|\frac{a_1}{a_2a_3}(1+a_3)\right| \quad \left|\frac{a_2a_3}{a_1\cdot a_4}(1+a_5)\right| + \left|\frac{a_1\cdot a_4}{a_2a_3\cdot a_5a_6}(1+a_6)\right| \\
& + \left|\frac{a_2a_3\cdot a_5a_6}{a_1\cdot a_4\cdot a_7}(1+a_8)\right| + \left|\frac{a_1\cdot a_4\cdot a_7}{a_2a_3\cdot a_5a_6\cdot a_8a_9}(1+a_9)\right| + \cdots.
\end{aligned}
$$

Proof. The continued fractions

$$
(3.4)\qquad 1 - \frac{a_1}{1+a_1+}\,\frac{a_2}{1+a_3-}\,\frac{a_3a_4}{1+a_4+}\,\frac{a_5}{1+a_6-}\,\frac{a_6a_7}{1+a_7+}\,\frac{a_8}{1+a_9-}\cdots,
$$

$$
(3.5)\qquad \frac{1}{1+a_1-}\,\frac{a_1a_2}{1+a_2+}\,\frac{a_3}{1+a_4-}\,\frac{a_4a_5}{1+a_5+}\,\frac{a_6}{1+a_7-}\,\frac{a_7a_8}{1+a_8+}\cdots
$$

$$
(3.6)\qquad \frac{1}{1+}\,\frac{a_1}{1+a_2-}\,\frac{a_2a_3}{1+a_3+}\,\frac{a_4}{1+a_5-}\,\frac{a_5a_6}{1+a_6+}\,\frac{a_7}{1+a_8-}\cdots
$$

have the sequences of approximants:

$$
\begin{gathered}
f_1, f_2, f_4, f_5, f_7, f_8, \cdots, f_{3p+1}, f_{3p+2}, \cdots, \\
f_0, f_2, f_3, f_5, f_6, f_8, \cdots, f_{3p}, f_{3p+2}, \cdots, \\
f_0, f_1, f_3, f_4, f_6, f_7, \cdots, f_{3p}, f_{3p+1}, \cdots,
\end{gathered}
$$

respectively. This can be proved by mathematical induction with the aid of the recurrence formulas (1.1) (with $q=0$). The convergence of any two of these continued fractions implies the convergence of the third; and $f(a)$ converges only in case these three continued fractions converge. Now, (3.4) may be thrown into the form

$$
1+\frac{1}{b_1+}\,\frac{1}{b_2+}\cdots,
$$

and diverges if $\sum|b_p|$ converges, *i.e.*, (3.4) diverges if the series (3.1) converges. Likewise, (3.5) diverges if the series (3.2) converges and (3.6) diverges if the series (3.3) converges.

Example. If $a_p=-p(p+2)/(p+1)^2$, $p=1, 2, 3, \cdots, f(a)$ diverges.

THEOREM 3.3. *Suppose k is an integer greater than* $1, f(a)$ *belongs to M_k, i and j are integers such that* $0\leqq i$, $0<j-i<k$ *and*

$$C_{ij}^k(p) = 1 - \frac{A_{k+j-i}^{kp+i} A_{k+i-j}^{kp+j}}{A_k^{kp+j} A_k^{kp+i}}, \qquad p = 0, 1, \cdots.$$

If the sequence $\{C_{ij}^k(p)\}_{p=0}^{\infty}$ *has a bounded infinite subsequence, then* $L_i = L_j$.

Proof. By (1.6), $[f_{kp+i}, f_{kp+j}, f_{k(p+1)+i}, f_{k(p+1)+j}] = C_{ij}^k(p)$ or

$$(f_{kp+i} - f_{kp+j})(f_{k(p+1)+i} - f_{k(p+1)+j}) = C_{ij}(p)\cdot(f_{kp+i} - f_{k(p+1)+i})(f_{kp+j} - f_{k(p+1)+j}).$$

Under the stated hypothesis, it follows that $L_i = L_j$.

4. Limaçons and cardioids. If the sequence of approximants of the continued fraction f' is some permutation of the sequence of approximants of $f(a)$, we may be able to deduce new convergence conditions for $f(a)$ by applying known conditions to f'. This is the idea used in [6] and further developed in [3].

We begin with the following theorem of [3]:

THEOREM 4.1. *If* $f_{2p}' = f_{2p+1}$ *and* $f_{2p+1}' = f_{2p}$, $p = 0, 1, \cdots$, *and*

$$(4.1)\quad a_1' = \frac{1 + a_2}{a_1},\ a_{2p}' = a_{2p}\ \textit{and}\ a_{2p+1}' = \frac{(1 + a_{2p})(1 + a_{2p+2})}{a_{2p+1}},\quad p = 1, 2, \cdots$$

then the sequence of approximants of $1 - f(a')$ *is* $\{f_p'\}_{p=0}^{\infty}$.

Proof. By equations (1.7) and (1.6), if p is a nonnegative integer, $a_{2p+2}' = -[f_{2p}', f_{2p+1}', f_{2p+2}', f_{2p+3}'] = -[f_{2p+1}, f_{2p}, f_{2p+3}, f_{2p+2}] = a_{2p+2}$; $a_{2p+3}' = -[f_{2p+1}', f_{2p+2}', f_{2p+3}', f_{2p+4}'] = -[f_{2p}, f_{2p+3}, f_{2p+2}, f_{2p+5}] = (1+a_{2p+2})(1+a_{2p+4})/a_{2p+3}$; and $1-(1+a_1')^{-1} = f_3$ or $a_1' = (1+a_2)/a_1$.

The known convergence theorem to be applied to $f(a')$ is [2]:

THEOREM 4.2. *If the sequence a satisfies the inequalities*

$$(4.2)\quad \begin{aligned} &|1 + a_1| > |a_1|, \quad |1 + a_1 + a_2| > |a_2|, \\ &|1 + a_p + a_{p+1}| \geqq |a_p| + |a_{p+1}|, \qquad p = 2, 3, \cdots, \end{aligned}$$

then the sequences $\{f_{2p+1}\}$ *and* $\{f_{2p+2}\}$ *are absolutely convergent and* $f(a)$ *converges only in case the series* $\sum|b_p|$ *defined by* (2.2) *is divergent.*

Note. If [8] $|a_p| \leqq 1/4$, $p=1, 2, \cdots$, then the inequalities (4.2) are satisfied and $\sum|b_p|$ diverges. Now, $|a_p'| \leqq 1/4$, $p=1, 2, \cdots$, provided $|a_{2p}| \leqq 1/4$ and $|a_{2p-1}| \geqq 25/4$, $p=1, 2, \cdots$. Hence, $f(a)$ converges if [3]:

$$(4.3)\qquad |a_{2p}| \leqq 1/4 \quad\text{and}\quad |a_{2p-1}| \geqq 25/4, \qquad p = 1, 2, \cdots.$$

THEOREM 4.3. *Suppose, for each positive integer* p, a_{2p} *is a complex number distinct from* 0 *such that* $\operatorname{Re} a_{2p} > -1/2$, $c_p = 1 + a_{2p}$, $r_p = |a_{2p}/(1+a_{2p})|$, *so that* $0 < r_p < 1$, *and* a_{2p-1} *is a complex number distinct from* 0 *such that*

$$|a_1 + c_1| > |c_1|, \qquad |a_1 + 1| > r_1|a_1|,$$

$$|a_{2p+1} + c_{p+1}| \geqq r_p |a_{2p+1}| + |c_{p+1}|, \quad |a_{2p+1} + c_p| \geqq r_{p+1}|a_{2p+1}| + |c_p|.$$

Then, the sequences $\{f_{2p+1}\}$ *and* $\{f_{2p+2}\}$ *converge absolutely and* $f(a)$ *converges only in case the series* $\sum |b_p|$ *defined by* (2.2) *is divergent.*

Proof. Under the hypothesis of the theorem, the sequence $\{a_p'\}$ defined by (4.1) satisfies the inequalities (4.2). Therefore, by Theorem 4.2, the sequences $\{f_{2p+1}\}$ and $\{f_{2p+2}\}$ are absolutely convergent; and $f(a')$ converges only if the series $\sum |b_p'|$ diverges, where $b_1' = 1$ and $b_{p+1}' = 1/a_p' b_p'$, $p = 1, 2, \cdots$. Thus, $f(a)$ converges only if $\sum |b_p'|$ diverges. It remains to be shown that the series $\sum |b_p'|$ diverges if the series $\sum |b_p|$ defined by (2.2) diverges.

Suppose $\sum |b_p|$ diverges and consider two cases according as $\sum |a_{2p}|^{-1}$ converges or diverges. Since $|a_{2p}|^{-1} = |b_{2p}' b_{2p+1}'|$, we see that $\sum |b_p'|$ diverges in the second case. In the first case, there exists a positive number r such that $r_1 r_2 \cdots r_p \to r$ as $p \to \infty$. Since $|b_{2p+2}'| = |b_2'/b_3| \cdot |b_{2p+3}| \cdot (r_1 r_2 \cdots r_p)(r_2 r_3 \cdots r_{p+1})$ and $|b_{2p+3}'| = |b_3'/b_2| \cdot |b_{2p+2}| / (r_1 r_2 \cdots r_p)(r_2 r_3 \cdots r_{p+1})$, it follows that $\sum |b_p'|$ diverges in the first case.

This completes a proof of Theorem 4.3.

Note. The only restriction on a_{2p} in this theorem is: $a_{2p} \neq 0$, $\operatorname{Re} a_{2p} > -1/2$; a_1 is exterior to each of two circles depending upon a_2; a_{2p+1} is not interior to the outer loops of either of two limaçons depending on a_{2p} and a_{2p+2}. The polar equations of these limaçons are:

$$\rho = \frac{2|c|}{1 - r^2}[r - \cos(\theta - \gamma)], \qquad c = |c|e^{i\gamma}, \qquad (c, r) = (c_{p+1}, r_p), (c_p, r_{p+1}).$$

These conditions are satisfied and $f(a)$ converges if (*cf.* (4.3))

$$(4.4) \qquad |a_{2p}| \leqq \frac{1}{4} \quad \text{and} \quad |a_{2p-1}| \geqq \frac{15}{4}, \qquad p = 1, 2, 3, \cdots.$$

Actually, $f(a)$ converges if $|a_{2p}| \leqq \frac{1}{4}$ and $|a_{2p-1}| \geqq 9/4$, $p = 1, 2, 3, \cdots$, [1].

Let P denote the parabolic disc consisting of all points z such that $|z| - \operatorname{Re} z \leqq \frac{1}{2}$. If the terms of the sequence a belong to P, the inequalities (4.2) are satisfied and therefore $f(a)$ converges only if the series $\sum |b_p|$ defined by (2.2) diverges. This is the *parabola theorem* [**4**]. If W is a point set containing P as proper subset, there exists a sequence a whose terms belong to W such that $\sum |b_p|$ diverges and $f(a)$ diverges [**5**]. However, if the terms of *a after the first* belong to P and $\operatorname{Re} a_1 > -1/2$, (4.2) holds and $f(a)$ converges only if $\sum |b_p|$ diverges.

THEOREM 4.4. *Suppose, for each positive integer* p,

$$|a_{2p}| - \operatorname{Re} a_{2p} \leqq \frac{1}{2},$$

$$\operatorname{Re}\left\{\frac{1 + a_2}{|1 + a_2|} \cdot \frac{1}{a_1}\right\} > -\frac{1}{2|1 + a_2|},$$

and

$$\left|\frac{1}{a_{2p+1}}\right| - \operatorname{Re}\left\{\frac{(1+a_{2p})(1+a_{2p+2})}{|(1+a_{2p})(1+a_{2p+2})|}\cdot\frac{1}{a_{2p+1}}\right\} \leqq \frac{1}{2|(1+a_{2p})(1+a_{2p+2})|}.$$

Then, the sequences $\{f_{2p+1}\}$ *and* $\{f_{2p+2}\}$ *converge absolutely and* $f(a)$ *converges only in case the series* $\sum|b_p|$ *defined by* (2.2) *diverges.*

Proof. The hypothesis implies that the terms of the sequence $\{a_p'\}$ defined by (4.1), after the first, belong to the parabolic disc P and $\operatorname{Re} a_1' > -\frac{1}{2}$. Thus, the inequalities (4.2) are satisfied by $\{a_p'\}$ so that the proof of Theorem 4.3 applies here.

In this theorem, the region of a_{2p+1} is a *cardioid plus its exterior.* In particular, $f(a)$ converges if

$$(4.5) \qquad |a_{2p}| \leqq \frac{1}{4} \quad \text{and} \quad |a_{2p-1}| \geqq \frac{9}{4}, \qquad p = 1, 2, 3, \cdots,$$

(*cf.* (4.3), (4.4) and [1]).

5. Other rearrangements. We conclude with three other examples of rearrangements and corresponding convergence theorems.

(1) If $f_{3p}'=f_{3p}$, $f_{3p+1}'=f_{3p+2}$ and $f_{3p+2}'=f_{3p+1}$, $p=0, 1, \cdots$, and

$$(5.1) \quad a_{3p+1}' = -\frac{a_{3p-1}}{1+a_{3p}+a_{3p+1}}, \quad a_{3p+2}' = \frac{1}{a_{3p+2}}, \quad a_{3p+3}' = -\frac{a_{3p+3}}{1+a_{3p+3}+a_{3p+4}},$$

$p=0, 1, \cdots$ (with $a_0=0$), then $f(a')/(1+a_1)$ has the sequence of approximants $\{f_p'\}_{p=0}^{\infty}$.

(2) If $f_{4p}'=f_{4p}$, $f_{4p+1}'=f_{4p+1}$, $f_{4p+2}'=f_{4p+3}$ and $f_{4p+3}'=f_{4p+2}$, $p=0, 1, \cdots$, and

$$(5.2) \quad \begin{aligned} a_{4p+1}' &= \frac{a_{4p+1}}{(1+a_{4p})(1+a_{4p+2})}, \quad a_{4p+2}' = -\frac{a_{4p+2}}{1+a_{4p+2}}, \\ a_{4p+3}' &= \frac{1}{a_{4p+3}} \quad \text{and} \quad a_{4p+4}' = -\frac{a_{4p+4}}{1+a_{4p+4}}, \qquad p = 0, 1, \cdots. \ (a_0 = 0), \end{aligned}$$

then $f(a')$ has the sequence of approximants $\{f_p'\}_{p=0}^{\infty}$.

(3) If $f_{5p}'=f_{5p}$, $f_{5p+1}'=f_{5p+1}$, $f_{5p+2}'=f_{5p+2}$, $f_{5p+3}'=f_{5p+4}$ and $f_{5p+4}'=f_{5p+3}$, $p=0, 1, \cdots$, and

$$(5.3) \quad \begin{aligned} a_{5p+1}' &= \frac{a_{5p+1}}{1+a_{5p}}, \quad a_{5p+2}' = \frac{a_{5p+2}}{1+a_{5p+3}}, \quad a_{5p+3}' = -\frac{a_{5p+3}}{1+a_{5p+3}}, \\ a_{5p+4}' &= \frac{1}{a_{5p+4}}, \quad a_{5p+4}' = -\frac{a_{5p+5}}{1+a_{5p+5}}, \qquad p = 0, 1, \cdots, (a_0 = 0), \end{aligned}$$

then $f(a')$ has the sequence of approximants $\{f_p'\}_{p=0}^{\infty}$.

We omit proofs analogous to that of Theorem 4.1.

If we apply Theorem 4.2 to $f(a')$, where a' is defined by (5.1), we obtain

THEOREM 5.1. *If* $|a_1|<1$, $|1+a_1+a_2|>|1+a_1|$ *and, for each positive integer* p,

$$|a_{3p}| + |a_{3p+1}| < 1,$$

$$\left|a_{3p+2}+\left(1+\frac{a_{3p+1}}{1+a_{3p}}\right)\right| \geqq \left|\frac{a_{3p+1}}{1+a_{3p}}\right|\cdot|a_{3p+2}| + \left|1+\frac{a_{3p+1}}{1+a_{3p}}\right| \quad \textit{and}$$

$$\left|a_{3p+2}+\left(1+\frac{a_{3p+3}}{1+a_{3p+4}}\right)\right| \geqq \left|\frac{a_{3p+3}}{1+a_{3p+4}}\right|\cdot|a_{3p+2}| + \left|1+\frac{a_{3p+3}}{1+a_{3p+4}}\right|,$$

then $f(a)$ *converges only in case the series* $\sum|b_p'|$ *diverges, where, in terms of* (5.1), $b_1'=1$ *and* $b_{p+1}'=1/b_p'a_p'$, $p=1, 2, 3, \cdots$.

In this case the boundaries of the regions are made up of arcs of circles and limaçons.

In the next two theorems, $1/P$ denotes the set to which z belongs only if $1/z$ belongs to the parabolic disc P defined in Section 4 and is a cardioid plus its exterior. The polar equation of this cardioid is $\rho=2(1-\cos\theta)$. Also, $-1-1/P$ denotes the set to which w belongs only if there is a point z in P such that $w=-1-1/z$ and $1/[-1-1/P]$ is the inversion in the unit circle of the region $-1-1/P$ and is a subset of the unit circular disc having -1 and $\frac{1}{3}$ on its boundary.

We apply the parabola theorem to $f(a')$, where a' is defined by (5.2), and obtain:

THEOREM 5.2. *If, for each nonnegative integer* p, a_{4p+3} *belongs to* $1/P$, a_{2p+2} *to* $1/[-1-1/P]$ *and (with* $a_0=0$*)*

$$|a_{4p+1}| - \operatorname{Re}\left\{\frac{|(1+a_{4p})(1+a_{4p+2})|}{(1+a_{4p})(1+a_{4p+2})}\cdot a_{4p-1}\right\} \leqq \frac{|(1+a_{4p})(1+a_{4p+2})|}{2},$$

then $f(a)$ *converges only in case the series* $\sum|b_p'|$ *diverges where, in terms of* a' *defined by* (5.2), $b_1'=1$ *and* $b_{p+1}'=1/b_p'a_p'$, $p=1, 2, \cdots$.

Finally, we apply the parabola theorem to $f(a')$, where a' is defined by (5.3), and get:

THEOREM 5.3. *If, for each nonnegative integer* p, a_{5p+4} *belongs to* $1/P$, a_{5p+3} *to* $1/[-1-1/P]$ *and, with* $a_0=0$,

$$|a_{5p+1}| - \operatorname{Re}\left\{\frac{|1+a_{5p}|}{1+a_{5p}}\cdot a_{5p+1}\right\} \leqq \frac{|1+a_{5p}|}{2}$$

and

$$|a_{5p+2}| - \operatorname{Re}\left\{\frac{|1+a_{5p+3}|}{1+a_{5p+3}}\cdot a_{5p+2}\right\} \leqq \frac{|1+a_{5p+3}|}{2},$$

then $f(a)$ *converges only in case the series* $\sum|b_p'|$ *diverges where, in terms of* (5.3), $b_1'=1$ *and* $b_{p+1}'=1/b_p'a_p'$, $p=1, 2, \cdots$.

Bibliography

1. Copp, George. Some convergence regions for continued fractions, Dissertation, The University of Texas, 1950.
2. Lane, R. E. and Wall, H. S. Continued fractions with absolutely convergent even and odd parts, Trans. Amer. Math. Soc., vol. 67, 1949, pp. 368–380.
3. Leighton, W. and Wall, H. S. On the transformation and convergence of continued fractions, Amer. J. of Math., vol. 58, 1936, pp. 267–281.
4. Scott, W. T. and Wall, H. S. A convergence theorem for continued fractions, Trans. Amer. Math. Soc., vol. 47, 1940, pp. 155–172.
5. Wall, B. M. Some convergence sets for a continued fraction, Dissertation, The University of Texas, 1956.
6. Wall, H. S. On continued fractions and cross-ratio groups of Cremona transformations, Bull. Amer. Math. Soc., vol. 40, 1934, pp. 587–592.
7. ———. Partially bounded continued fractions, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 1090–1093.
8. Worpitzky, J. Untersuchungen über die Entwickelung der monodronen und monogenen Funktionen durch Kettenbrüche, Friedrichs-Gymnasium und Realschule, Jahresbericht, Berlin, 1865, pp. 3–39.

SENSE AND ORIENTATION ON THE DISK

G. T. WHYBURN, University of Virginia

We shall be concerned with a simple and elementary approach to some fundamental results having to do with orientation and sense agreement for closed curves in a plane or on a disk. These theorems, although of basic and crucial importance in a really complete treatment of topological and mapping results in this setting, are frequently passed over in the early stages and treated, if at all, only after quite elaborate or sophisticated techniques have been developed. Thus they occur as special cases or as corollaries to deep theorems whose statements and meaning may be concealed from the beginner by their involvement in elaborate concepts and notation.

The critical issue in this topic is to define *sense* on a simple closed curve on a disk and *agreement in sense* for two such curves J_1 and J_2 so that each curve has exactly two senses, and then to show that if a given sense on J_1 agrees with say the "positive" sense on J_2 then it *cannot* also agree with the "negative" sense on J_2. All the results to be proven are well known, of course. The ones selected are of interest since they may be used to prove such results as the invariance of the

property of orientability for a 2-manifold, for example—both the invariance under subdivision and the topological invariance—making little use of subdivision techniques. A discussion of this topic, as well as a number of the theorems in the present paper, may be found treated by subdivision modification methods in Kerekjarto's book on topology.

DEFINITION. *The ordering (a, b, c) of three points a, b, c on a circle C of the complex plane is a positive ordering provided that when C plus its interior is regarded as a simplex with vertices a, b, c then (a, b, c) is a positive orientation of this simplex in the complex plane. Equivalently, (a, b, c) is a positive ordering of a, b, c provided that, assuming C has center at the origin and radius r, if $g(\theta)$ is the mapping $re^{i\theta}$ of the interval $(0, 2\pi)$ onto C and if $\theta_1<\theta_2<\theta_3$ are the least three values on $(0, 2\pi)$ in $g^{-1}(a+b+c)$, then $[g(\theta_1), g(\theta_2), g(\theta_3)]$ is an even permutation of the ordering (a, b, c).*

THEOREM 1. *Given concentric circles C and C' of radii r and r', $r'<r$, and three disjoint simple arcs aa', bb', cc' with $a+b+c\subset C$, $a'+b'+c'\subset C'$ and each lying except for its end points in the annular region between C and C'. Then if (a, b, c) is a positive ordering of the points a, b and c, (a', b', c') is a positive ordering of the points a', b' and c'.*

Proof. Suppose on the contrary that for some choice of the arcs aa', bb', cc' it were true that the ordering (a', b', c') is negative (*i.e.*, not positive). Then without loss of generality we may assume† that $r'=\frac{3}{4}r$. Further we may assume each of the arcs is of finite length so that the sum s of their lengths is finite. Also if σ is the g.l.b. of the length sum s for all possible sets of 3 arcs aa', bb', cc' as above with (a, b, c) a positive ordering but (a', b', c') a negative ordering, we may suppose our particular choice of arcs made so that their length sum s satisfies $s<\sigma+\frac{1}{4}r'$.

At least one of the three arcs ab, bc, ca of C is of length $\geqq 2\pi r/3$. We may suppose this true of the arc ca. Let a_1 be the midpoint of ca. Then the radius a_1o of C cannot intersect $aa'+bb'+cc'=T$. For suppose it does. Then let p be the first point of T on a_1o in the order a_1, o. Since the open arc bb' contains points inside the simple closed curve $J=abc$ (of C)$+aa'+cc'+a'b'c'$ (of C') but does not intersect J, it therefore lies entirely inside J. Accordingly, as a_1p-p is entirely outside J it follows that p cannot belong to the arc bb'. Thus p lies on

† To see this, note that we can apply the preliminary radial stretching transformation $(\rho, \theta)\to(\rho^*, \theta^*)$ given by

$$\rho^* = \frac{3r}{4r'}\rho, \qquad \theta^* = \theta, \text{ for } \qquad \rho \leqq r'$$

$$\rho^* = \frac{r\rho + 3r^2 - 4rr'}{4(r - r')}, \qquad \theta^* = \theta, \text{ for } \qquad \text{for } \rho \geqq r',$$

which will leave the origin and all points of C fixed and will not alter order relations of points on C' but maps C' into the circle with center O and radius $\frac{3}{4}r$.

either aa' or cc'. If $p \in aa'$, we replace the arc ap in aa' by the arc a_1p of a_1o; and this clearly diminishes the length sum s by more than $\frac{1}{4}r'$ because the length a_1p is $<\frac{1}{4}r$, whereas the length of ap (in aa') is $\geqq |a-a_1|-|a_1-p| \geqq r-\frac{1}{4}r = \frac{3}{4}r$. This makes the length sum of the new arcs a_1a', bb', cc' less than σ, which is impossible because (a_1, b, c) is still a positive ordering of the points a_1, b, c on C. Similarly if $p \in cc'$ we replace the arc cp of cc' by the arc a_1p of a_1o and reason as before; (a_1, a, b) is still a positive ordering of a_1, a, b so that the same contradiction is obtained. Hence T does not intersect the radius a_1o of C.

Now let c_1o be a radius of C with c_1 on the arc a_1c of C and so close to a_1o that no point of T is in the sector of C bounded by a_1o+c_1o+arc a_1c_1 of C. Let c_1c_1' and a_1a_1' be the segments of c_1o and a_1o respectively such that c_1' and a_1' are on C'. Consider the simple closed curve $K=$arc a_1abcc_1 (of C)$+c_1c_1'+$arc $c_1'a_1'$ of C' containing a', b' and $c'+a_1'a_1$. Since b and b' separate a and c on K and since the arcs aa', bb', cc' lie in $K+$ its interior but do not intersect, it follows that b and b' also separate a' and c' on K and that a and a' lie on one of the arcs of K from b to b' and c and c' on the other. Accordingly we must have the order $a_1'a'b'c'c_1'$ on the positively oriented arc which K has in common with C'. Thus (a', b', c') is a positive ordering of the points a', b', c' contrary to our supposition.

Definition. *Let A be a 2-cell with edge J and let J' be a simple closed curve on A lying interior to J. If a, b, c and a', b', c' are triples of distinct points on J and J' respectively, the sense $a'b'c'$ on J' given by the ordering (a', b', c') is said to agree with the sense abc on J given by the ordering (a, b, c) provided there exist disjoint simple arcs aa', bb', cc' lying except for their ends in the annular region of A bounded by J and J'.*

Theorem 2. *If the sense abc on J agrees with the sense $a'b'c'$ on J', it cannot agree with the sense $a'c'b'$. Thus abc agrees with the sense given by any even permutation of the ordering (a', b', c') but with no odd one.*

Proof. Let G denote the graph on A consisting of J, J' and the arcs aa', bb', cc' and let h be a homeomorphism mapping G into the complex plane so that J and J' map onto the circles $|z|=2$ and $|z|=1$ respectively with a, b, c going into $2, 2e^{2\pi i/3}, 2e^{4\pi i/3}$ respectively and a', b', c' into $1, e^{2\pi i/3}, e^{4\pi i/3}$ respectively and with the arcs aa', bb', cc' mapping onto the linear segments joining the images of their respective end points. Since each region on A complementary to G is bounded by a simple closed curve of G whose image bounds a region in $|z| \leqq 2$ complementary to $h(G)$, h can be extended to a homeomorphism of A onto $|z| \leqq 2$.

Now if, contrary to our theorem, it were possible to find disjoint arcs aa', bc', cb' on A, then $h(aa')$, $h(bc')$, $h(cb')$ would be disjoint arcs in $1 \leqq |z| \leqq 2$; and since $(2, e^{2\pi i/3}, e^{4\pi i/3}) = [h(a), h(b), h(c)]$ is a positive ordering on the circle $|z| = 2$, it would follow by Theorem 1 that $[h(a'), h(c'), h(b')]$ has to be a positive ordering on $|z|=1$. This is impossible because $h(a')=1$, $h(c')=e^{4\pi i/3}$, $h(b') = e^{2\pi i/3}$.

COROLLARY 1. *If A is a 2-cell with edge J and $h(A)=B$ is any homeomorphism, then if the sense $a'b'c'$ on a simple closed curve J' interior to J on A agrees with the sense abc on J, the sense $h(a')h(b')h(c')$ on $h(J')$ must agree with $h(a)h(b)h(c)$ on $h(J)$.*

COROLLARY 2. *Any homeomorphism of an annular ring onto itself which maps one edge onto itself in sense-preserving fashion must likewise map the other edge on itself in a sense-preserving fashion.*

COROLLARY 3. *Any homeomorphism $h(A)=A$ of a 2-cell onto itself which preserves sense on the edge J of A must likewise preserve sense for all simple closed curves interior to A, i.e., if J' is any such curve interior to A and $a'b'c'$ is a sense on J' agreeing with the sense abc on J then the sense $h(a')h(b')h(c')$ on $h(J)$ will necessarily also agree with the sense abc on J.*

Note. By a *curved* triangle (or 2-simplex) is meant a set homeomorphic with an ordinary straight triangle in which sides and vertices are designated on its edge. An orientation of such a triangle is determined by an ordering of its vertices as discussed above.

THEOREM 3. *If Δ_1 and Δ_2 are curved triangles with vertices x, y, z and x, z, w respectively lying inside a 2-cell Δ and having a side xz and only this side in common and if (x, y, z) and (x, z, w) are orientations of Δ_1 and Δ_2 respectively each of which agrees with the orientation (a', b', c') of Δ, then these orientations must agree with each other in that their ordering on the common side xz is opposite.*

Proof. Let x, y and z be joined to points of the edge J of Δ by disjoint simple arcs xa, yb and zc lying except for their ends in the annular region on Δ between J and the simple closed curve $xy+yz+zw+wx$. Then since the ordering xyz of the boundary of Δ_1 agrees with the ordering abc on J in sense, we can choose a, b and c as vertices of Δ and still have the given orientations of Δ_1 and Δ_2 agree with Δ. Now the arc $axzc$ divides the interior of Δ into two regions R_w and R_y containing w and y respectively. Also R_w+c contains a simple arc wc. Thus ax, $by+$side yz of Δ_1 and cw are disjoint arcs in the annulus bounded by J and the edge of Δ_2. Accordingly, the sense given by ordering (x, z, w) on the edge of Δ_2 agrees with the sense abc on J. Since xzw and $yzx(=xyz)$ are opposite on the common side xz, our result is established.

Dedication. This paper is dedicated to Lester R. Ford on the occasion of his seventieth birthday. In the past 30 years, throughout which it has been the author's privilege to know and be associated with Dr. Ford in various mathematical and personal endeavors, his stimulating personality, his keen and zestful perception in all phases of mathematics, and above all his unfailing kindly interest in his fellow man have greatly enriched the author's life and added much to his mathematical experience.

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ON VEBLEN-WEDDERBURN SYSTEMS

JAMES R. WESSON, Vanderbilt University

1. Introduction. In 1907, Veblen and Wedderburn showed that certain finite algebras could be used in order to coordinatize some finite projective planes [**4**], [**2**, p. 407]. Such an algebra was defined as a finite set of elements 0, $a, b, c, \cdots$ closed under two operations, addition $x+y$ and multiplication xy, and obeying the following:

(1) The elements form an Abelian group (with identity 0) under addition.
(2) For $a\neq 0$, the equation $xa=b$ has a unique solution x.
(3) For $a\neq 0$, the equation $ax=b$ has a unique solution x.
(4) $0a=a0=0$. (This postulate is redundant.)
(5) $(a+b)c=ac+bc$.

The postulate (5) replaces the left distributive law given by Veblen and Wedderburn; however, a new multiplication $\cdot$ defined by $x\cdot y=yx$ changes the above system into one satisfying the distributive law $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$. Therefore the two systems are anti-isomorphic [**3a**, p. 253].

2. Postulates. In the finite case, the system defined in Section 1 is equivalent to that defined by Hall [**3a**, p. 253]. The only additional postulate given by Hall is

(2.1) For $a\neq b$, the equation $xa=xb+c$ has a unique solution x,

and this condition is implied (in the finite case) by the other five. The proof of this statement follows.

First it will be shown that all nonzero elements of the additive group have the same order. Let $x_0=0$ and let $x_{i+1}=x_i+x$ for $i=0, 1, \cdots$. If a and b are two distinct nonzero elements, then $a=bc$, where $c\neq 0$, and from (5) we have $a_i=(bc)_i=b_ic$. Therefore, $a_i=0$ if and only if $b_i=0$. This leads to the following theorem [**3b**, p. 22].

THEOREM 2.1. *The system defined in Section* 1 *has exactly p^n elements, where p is a prime.*

LEMMA 2.1. *If $a\neq c$, the system of equations $ax+y=b$, $cx+y=d$ has a unique solution x, y.*

Proof. Let $u-v$ mean $u+(-v)$, where $-v$ is the additive inverse of v. Then the unique solution x, y of the lemma is given by $(a-c)x=b-d$, $y=b-ax=d-cx$. (Note that for any elements u and v, we have $-(uv)=(-u)v$.)

LEMMA 2.2. *If $a\neq b$, $x_1\neq x_2$, and $x_1a=x_1b+c$, then $x_2a\neq x_2b+c$.*

Proof. Suppose the contrary, and let $x_1a=x_1b+c=f$, $x_2a=x_2b+c=g$. Then the system of equations $x_1x+y=f$, $x_2x+y=g$ has two solutions, namely $x=a$, $y=0$, and $x=b$, $y=c$. This contradicts Lemma 2.1.

THEOREM 2.2. *If $a\neq b$, the equation $xa=xb+c$ has a unique solution x.*

Proof. The only nontrivial case is when $a\neq 0$ and $b\neq 0$. Let $x_1, x_2, \cdots$ be distinct elements, and let c_i $(i=1, 2, \cdots)$ be defined by $x_ia=x_ib+c_i$. Lemma 2.2 implies the distinctness of $c_1, c_2, \cdots$. Hence, for any c, the equation $xa=xb+c$ has exactly one solution x.

It is now clear that, in the finite case, the condition (2.1) need not be given the status of a postulate. Also, both of the postulates (4) may be dropped [**1**, p. 17].*

3. Veblen-Wedderburn systems with units. It will be shown how any system defined by the postulates of Section 1 can be changed into a system which satisfies the original postulates and also has a unit (multiplicative identity). That a unit can be introduced has been proved by Hall [**3a**, pp. 253–255]. The following differs only slightly from the approach used by Hall.

THEOREM 3.1. *Let S be a set which forms a system under the postulates of the first section, and let e be any nonzero element of S. For any elements x, y, define a new multiplication $\cdot$ by*

$$x\cdot y = xa, \quad \textit{where} \quad ea = y.$$

Then S forms a system satisfying the original postulates, and also $e\cdot y=y$ for all y.

Proof. Since addition is not affected, we need only to prove that postulates (2), (3), (4), and (5) are satisfied.

(4) $0\cdot y=0a=0$; $x\cdot 0=xa$, where $ea=0$, and therefore $a=0=x\cdot 0$.

(2) To establish postulate (2) it is proved that $x_1\neq x_2$ and $y\neq 0$ imply $x_1\cdot y\neq x_2\cdot y$. By definition

$$x_1\cdot y = x_1a \quad \text{where} \quad ea = y,$$
$$x_2\cdot y = x_2a \quad \text{where} \quad ea = y.$$

Therefore, $x_1a\neq x_2a$ and $x_1\cdot y\neq x_2\cdot y$.

(3) To establish condition (3) it is proved that $y_1\neq y_2$ and $x\neq 0$ imply $x\cdot y_1\neq x\cdot y_2$.

(5) By definition $(x_1+x_2)\cdot y=(x_1+x_2)a$, $x_1\cdot y=x_1a$, and $x_2\cdot y=x_2a$, where $ea=y$. The right distributive law for the old operations leads to the same law for the new operations.

Finally, $e\cdot y=ea$, where $ea=y$, and hence $e\cdot y=y$.

THEOREM 3.2. *Let S be a set which forms a system under the postulates of the first section, and let e be an element such that $ey=y$ for all y. For any elements x, y, define a new multiplication $\cdot$ by*

* Since $0b+ub=(0+u)b=ub=0+ub$, it follows that $0b=0$. Further $a0=0$. For if $a\neq 0$, and $ax=0$ is satisfied by $x=b\neq 0$, then the equation $yb=0$ is satisfied by $y=a\neq 0$. But this is impossible.

$$x \cdot y = ay \quad where \quad ae = x.$$

Then S forms a system satisfying the original postulates, and also $e \cdot x = x \cdot e = x$ for all x.

Proof. The proofs that postulates (4), (2), and (3) are satisfied are similar to the corresponding proofs under Theorem 3.1. For the proof of (5) let

$$x_1 \cdot y = a_1 y \quad \text{where} \quad a_1 e = x_1,$$

$$x_2 \cdot y = a_2 y \quad \text{where} \quad a_2 e = x_2,$$

$$(x_1 + x_2) \cdot y = ay \quad \text{where} \quad ae = x_1 + x_2.$$

Then $(a_1+a_2)e = a_1 e + a_2 e = x_1 + x_2 = ae$, and $a_1 + a_2 = a$. Therefore, $(x_1+x_2) \cdot y = ay = (a_1+a_2)y = a_1 y + a_2 y = x_1 \cdot y + x_2 \cdot y$. Finally, we need to show that $e \cdot x = x \cdot e = x$. By definition $e \cdot y = ay$ where $ae = e$. Therefore $a = e$ and $e \cdot y = y$. Also, $x \cdot e = ae$ where $ae = x$, and $x \cdot e = x$.

Marshall Hall, Jr. has proved that a finite projective plane which can be "represented by a Veblen-Wedderburn number system . . . may also be represented by a Veblen-Wedderburn number system with a unit" [**3a**, p. 258]. The operations given in Theorems 3.1 and 3.2 may be used in order to convert any system defined by Section 1 into a system with a unit.

A *finite Veblen-Wedderburn system* is defined as a set of elements 0, e, a, b, c, $\cdots$ subject to the following postulates:

(1), (2), (3), (4), and (5) from Section 1,

$$ex = xe = x \text{ for all } x. \tag{6}$$

Let $x_i = x + x + \cdots + x$ for i summands, $i = 1, 2, \cdots$. It is obvious that $x_i + x_j = x_{i+j}$, and it is not difficult to show that $(x_i)_j = x_{ij} = x_{ji}$. Also, $e_i e_j = (ee_j)_i = (e_j)_i = e_{ji} = e_{ij} = e_j e_i$.

4. Veblen-Wedderburn systems with a prime number of elements.

THEOREM 4.1. *A Veblen-Wedderburn system with a prime number of elements is a field.*

Proof. Since the additive order of each nonzero element is a prime p, the elements may be listed 0, e, e_2, e_3, $\cdots$, e_{p-1}, where $e_i = e + e + \cdots + e$ for i summands. Multiplication is associative and commutative since $e_i(e_j e_k) = e_i e_{jk}$ and $e_i e_j = e_j e_i = e_{ij}$.

5. The Veblen-Wedderburn system with eight elements. It will be proved that the Veblen-Wedderburn system with eight elements is a field. Remember that $x + x = 0$ for all x, and let 0, e, a be three distinct elements. It is easily seen that $e + a$ is a fourth element; further, aa is another element. For, if $aa = e$, then $(e+a)a = a + aa = e + a$ and $a = e$, a contradiction. If $aa = e + a$, then $(e+a)a = a + aa = a + e + a = e$. Then, if another element is called b, the entire eight ele-

ments may be called 0, e, a, $e+a$, b, $e+b$, $a+b$, $e+a+b$, and the element ba must be given by one of the following:

$$ba = e,$$
$$ba = e + b,$$
$$ba = a + b,$$
$$ba = e + a + b.$$

If $ba=e$, then $(a+b)a=aa+ba=e+a+e=a$.
If $ba=e+b$, then $(a+b)a=aa+ba=e+a+e+b=a+b$.
If $ba=a+b$, then $(e+a+b)a=e+a+b$.
If $ba=e+a+b$, then $(e+b)a=e+b$.

But since $a+b\neq e$ and $a\neq e$, each case is impossible. Hence $aa\neq e+a$. Therefore,

THEOREM 5.1. *The elements of a Veblen-Wedderburn system of order eight may be called* 0, e, a, $e+a$, aa, $e+aa$, $a+aa$, $e+a+aa$.

Taking the eight elements as given by the preceding theorem, and remembering that $x+x=0$ for all x, one may easily construct the addition table (Abelian Group of type 1, 1, 1). There are only two possible multiplication tables, and they are given below.

	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
e	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
a	a	aa	$a+aa$	$e+a$	e	$e+a+aa$	$e+aa$
$e+a$	$e+a$	$a+aa$	$e+aa$	$e+a+aa$	aa	e	a
aa	aa	$e+a$	$e+a+aa$	$a+aa$	a	$e+aa$	e
$e+aa$	$e+aa$	e	aa	a	$e+a+aa$	$e+a$	$a+aa$
$a+aa$	$a+aa$	$e+a+aa$	e	$e+aa$	$e+a$	a	aa
$e+a+aa$	$e+a+aa$	$e+aa$	a	e	$a+aa$	aa	$e+a$

	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
e	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
a	a	aa	$a+aa$	$e+aa$	$e+a+aa$	e	$e+a$
$e+a$	$e+a$	$a+aa$	$e+aa$	e	a	$e+a+aa$	aa
aa	aa	$e+aa$	e	$e+a+aa$	$e+a$	a	$a+aa$
$e+aa$	$e+aa$	$e+a+aa$	a	$e+a$	$a+aa$	aa	e
$a+aa$	$a+aa$	e	$e+a+aa$	a	aa	$e+a$	$e+aa$
$e+a+aa$	$e+a+aa$	$e+a$	aa	$a+aa$	e	$e+aa$	a

Either of the following two correspondences transforms the second table above into the first.*

$$(5.1)\quad \begin{array}{l} 0 \to 0, \quad e \to e, \quad a \to e + aa, \quad e + a \to aa, \quad aa \to e + a + aa, \\ \quad e + aa \to a + aa, \quad e + a + aa \to e + a, \quad a + aa \to a. \end{array}$$

$$(5.2)\quad \begin{array}{l} 0 \to 0, \quad e \to e, \quad a \to e + a + aa, \quad e + a \to a + aa, \quad aa \to e + a, \\ \quad e + aa \to a, \quad a + aa \to aa, \quad e + a + aa \to e + aa. \end{array}$$

The results of this section may be summarized.

THEOREM 5.2. *The Veblen-Wedderburn system of order eight is a field.*

References

1. R. H. Bruck, Recent advances in the foundations of Euclidean plane geometry, this MONTHLY, vol. 62, 1955, part II, pp. 2–17.

2. R. D. Carmichael, Groups of Finite Order, Boston, 1937.

3a. Marshall Hall, Jr. Projective planes. Trans. Amer. Math. Soc. vol. 54, no. 2, 1943, pp. 229–277.

b. ———, Finite projective planes, this MONTHLY, vol. 62, 1955, part II, pp. 18–24.

4. O. Veblen and J. H. M. Wedderburn, Non-Desarguesian and non-Pascalian geometries, Trans. Amer. Math. Soc., vol. 8, 1907, pp. 379–388.

* The only correspondences which transform the second table into the first are the two given above.

NEWTON'S METHOD AND MULTIPLE ROOTS

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Most calculus courses today cover the so-called "Newton's Method" for finding roots of equations. It is my purpose here to describe a rigorous, analytical proof of this proposition, using only results which are available in the calculus. In addition, I will generalize the method and consider the problem of multiple roots. This extension to multiple roots will then be seen to have its most fruitful application to the solution of polynomial equations. In conclusion I will give a well known theorem on rapidity of convergence.

THEOREM 1. *Let* $\phi(x)$, $\phi'(x)$, *and* $\phi''(x)$ *be continuous,* $f(x) = x - \phi(x)/\phi'(x)$' *and suppose that* $\phi'(x)\cdot\phi''(x) \neq 0$; *all in* $[a, b]$. *Suppose further that* $\phi(a)\cdot\phi(b) < 0$· *Then the equation* $\phi(x) = 0$ *has only one root in* $[a, b]$; *and if* $x_0 \in [a, b]$ *such that* $\phi(x_0)\cdot\phi''(x_0) > 0$, *the sequence* x_0, $x_1 = f(x_0), \cdots, x_{n+1} = f(x_n), \cdots$, *converges monotonically to this root* $x = p$. *On the other hand, if* $\phi(x_0)\cdot\phi''(x_0) < 0$, p *will be between* x_0 *and* x_1, *and if* $x_1 \in [a, b]$ *the sequence* $x_1, x_2, \cdots, x_n, \cdots$, *converges monotonically to* p.

Before proceeding with the proof, let us consider the geometric implications of the restrictions which have been placed on $\phi(x)$. First of all, since $\phi''(x)$ cannot change sign in (a, b), the curve $y=\phi(x)$ can have no inflection points there. Secondly, since $\phi'(x)\neq 0$ in (a, b) the curve $y=\phi(x)$ can have no horizontal tangents there. Finally, in view of the continuity of $\phi'(x)$, this last remark implies that the curve $y=\phi(x)$ has no maximum or minimum points in (a, b).

Proof of Theorem 1. First of all, since $\phi'(x)\cdot\phi''(x)\neq 0$ in $[a, b]$, it follows from Rolle's and Taylor's theorems that $\phi(x)=0$ has exactly one simple root in $[a, b]$.

Since $\phi''(x)$ is continuous and nonzero in the interval $[a, b]$ it will have constant sign there. Therefore, since $\phi(a)$ and $\phi(b)$ are of opposite sign, so also are $\phi(a)\cdot\phi''(a)$ and $\phi(b)\cdot\phi''(b)$. Furthermore, there being only one root of the equation $\phi(x)=0$ in the interval $[a, b]$, namely at $x=p$, the function $\phi(x)\cdot\phi''(x)$ holds the same sign in each of the intervals $[a, p)$ and $(p, b]$. Therefore the function $f'(x)=\{\phi(x)\phi''(x)\}/\{\phi'(x)\}^2$ has the same sign in each of the intervals $[a, p)$ and $(p, b]$ as $\phi(a)\cdot\phi''(a)$ and $\phi(b)\cdot\phi''(b)$, respectively, have. Now select the interval in which $f'(x)>0$; suppose this to be $(p, b]$, so that $\phi(b)\cdot\phi''(b)>0$.

Let $g(x)=f(x)-x$. Since $g'(p)=-1$ and $g'(x)$ is continuous in $[a, b]$, there exists $\epsilon>0$ such that $g'(x)<0$ in $[p, p+\epsilon)$. Now $g(p)=0$ hence $g(x)<0$, and from this $f(x)<x$ and consequently $\phi(x)/\phi'(x)>0$ all in $(p, p+\epsilon)$. Since $\phi(x)$ and $\phi'(x)$ are continuous and nonzero in $(p, b]$, then $\phi(x)/\phi'(x)$ has the same sign throughout $(p, b]$. Therefore we now have $\phi(x)/\phi'(x)>0$, and consequently $f(x)<x$ in $(p, b]$. But $f(p)=p$ and $f'(x)>0$ in $(p, b]$, hence $p<f(x)<x$ there.

Let the function $f(x)$ be the generating function for an iterative sequence $\{x_n\}$ so that $x_{n+1}=f(x_n)$, and let $x_0\in[a, b]$ be chosen such that $\phi(x_0)\cdot\phi''(x_0)>0$. Then $x_0\in(p, b]$ and $p<f(x_0)<x_0$, that is $p<x_1<x_0$. Thus it readily follows by induction that $p<x_{n+1}<x_n$ for $n=0, 1, \cdots$. Therefore, as a monotone decreasing sequence bounded below by p, $\{x_n\}$ is convergent to $q\geqq p$.

Now the sequences $\{x_n\}$ and $\{f(x_n)\}$ both converge to the same limit q. But $q\in(a, b)$ and $f(x)$ is continuous in $[a, b]$. Hence $q=\lim x_n=\lim f(x_n)=\lim_{x\to q} f(x)=f(q)$. Thus $x=q$ is a solution of the equation $x=f(x)$ and consequently of the equation $\phi(x)=0$, therefore $q=p$.

Suppose that $x_0\in[a, b]$ had been chosen such that $\phi(x_0)\cdot\phi''(x_0)<0$. Further suppose that $\phi(b)\cdot\phi''(b)>0$, so that $x_0\in[a, p)$. Thus $f'(x)<0$ in $[a, p)$, and since $f(p)=p$, we have $f(x)>p$ in $[a, p)$; hence our first iteration produces $x_1=f(x_0)>p$. On the other hand, if we had assumed $\phi(a)\cdot\phi''(a)>0$, we would have $x_0\in(p, b]$ and $x_1<p$. Thus, in case $x_0\in[a, b]$ is chosen such that $\phi(x_0)\cdot\phi''(x_0)<0$, then the first iteration provides $x_1=f(x_0)$ and $\phi(x_1)\cdot\phi''(x_1)>0$. Now if $x_1\in[a, b]$, the sequence $x_1, x_2, \cdots$ converges monotonically to the root p as desired.

A slight extension of our theorem is obtained as follows:

COROLLARY. *The restriction $\phi''(x)\neq 0$ in $[a, b]$ may be replaced by the weaker hypothesis that $\phi''(x)$ be either nonnegative or nonpositive in $[a, b]$.*

Proof. All of our proofs follow exactly as before with the exception that cer-

tain $<$ and $>$ signs are replaced by $\leqq$ and $\geqq$, respectively.

Up to this point we have had the restriction $\phi'(x)\neq 0$ in $[a, b]$, so that Newton's method would appear to be useful only in finding simple roots. However an important extension may be made which permits us to apply this method, under certain conditions, when multiple roots are to be located. With this purpose in mind we prove:

THEOREM 2. *Let $\phi(x)$ have continuous derivatives of orders $1, \cdots, n$ $(n\geqq 2)$ in $[a, b]$, with $\phi^{(n)}(x)\neq 0$ there. Suppose further that the equation $\phi(x)=0$ has a root of multiplicity n in $[a, b]$. Then if $x_0\in[a, b]$, the sequence $x_0, x_1=f(x_0), \cdots, x_{n+1}=f(x_n), \cdots,$ converges monotonically to the root $x=p$, where $f(x)=x-\phi(x)/\phi'(x)$.*

Before proceeding with the proof of this theorem we must derive certain expressions for $\phi(x)$, $\phi'(x)$, and $\phi''(x)$ which will be required there. The function $\phi(x)$ and its first n derivatives are defined and continuous in $[a, b]$. Hence, since p is a root of multiplicity n, it follows from Taylor's theorem that $\phi(p)=\phi'(p)=\cdots=\phi^{(n-1)}(p)=0$. Thus using Taylor's theorem again, we obtain $\phi(x)=\phi^{(n)}(\zeta_0)(x-p)^n/n!$, valid in $[a, b]$, with ζ_0 between x and p. In a similar manner, we obtain $\phi'(x) = \phi^{(n)}(\zeta_1)(x - p)^{n-1}/(n - 1)!$, and if $n \geqq 3$, $\phi''(x)=\phi^{(n)}(\zeta_2)(x-p)^{n-2}/(n-2)!$, both valid in $[a, b]$, and with ζ_1 and ζ_2 between x and p.

Proof of Theorem 2. First of all we wish to show that we have isolated in $[a, b]$ a single root, multiple or simple as the case may be, of each of the equations $\phi(x)=0, \phi'(x)=0, \cdots, \phi^{(n-1)}(x)=0$. Consider, for instance, the equation $\phi(x)=0$. Suppose that there exists $\xi_0\neq p$ in $[a, b]$ such that $\phi(\xi_0)=0$; then, according to Rolle's theorem, there exists $\xi_1\in[a, b]$, $\xi_1\neq p, \xi_0$, such that $\phi'(\xi_1)=0$. We now have $\phi'(p)=\phi'(\xi_1)=0$; and according to Rolle's theorem there exists $\xi_2\in[a, b]$, $\xi_2\neq p, \xi_0, \xi_1$, such that $\phi''(\xi_2)=0$. Continuing in this manner we finally arrive at the contradictory result that there exists $\xi_n\in[a, b]$ such that $\phi^{(n)}(\xi_n)=0$.

Let us now consider the behavior of $\phi(x)\cdot\phi''(x)$ in $[a, b]$. Since $\phi^{(n)}(x)$ is continuous and nonzero in $[a, b]$, we have either $\phi^{(n)}(x)>0$ or $\phi^{(n)}(x)<0$ there. Therefore, if $n\geqq 3$ we obtain $\phi(x)\cdot\phi''(x)=\phi^{(n)}(\zeta_0)\cdot\phi^{(n)}(\zeta_2)(x-p)^{2n-2}/[(n-2)!n!]>0$; and if $n=2$, $\phi(x)\cdot\phi''(x)=\phi''(\zeta_0)\cdot\phi''(x)(x-p)^2/2>0$; both inequalities being true whenever $x\in[a, b]$ and $x\neq p$. Thus we now have $\phi(x)\cdot\phi''(x)>0$, and consequently $f'(x)>0$ whenever $x\in[a, b]$ and $x\neq p$.

We next consider the function $f(x)$. Since $f(x)=x-\phi^{(n)}(\zeta_0)(x-p)/\phi^{(n)}(\zeta_1)n$, valid in $[a, b]$, then $\lim_{x\to p} f(x)=p$. But $f'(x)>0$ in $(p, b]$, hence $f(x)>p$ there.

Finally consider the function $g(x)=f(x)-x$. This function and its derivative are continuous in $[a, b]$, except at $x=p$, where neither is defined. For $n\geqq 3$, we have $g'(x)=\{(n-1)\cdot\phi^{(n)}(\zeta_0)\cdot\phi^{(n)}(\zeta_2)/n[\phi^{(n)}(\zeta_1)]^2\}-1$, valid in $[a, b]$. For $n=2$, we have $g'(x)=\{\phi''(\zeta_0)\cdot\phi''(x)/2[\phi''(\zeta_1)]^2\}-1$, valid in $[a, b]$. Hence, due to the continuity of $\phi^{(n)}(x)$, we obtain $\lim_{x\to p} g'(x) = -1/n$. Therefore there

exists $\epsilon > 0$ such that $g'(x) < 0$ for $x \in (p, p+\epsilon)$. But $g(x) = -\phi(x)/\phi'(x) = -\phi^{(n)}(\zeta_0)(x-p)/n\phi^{(n)}(\zeta_1)$, valid for $x \in [a, b]$; hence $\lim_{x \to p} g(x) = 0$. Thus we obtain $g(x) < 0$ and from this $f(x) < x$ and consequently $\phi(x)/\phi'(x) > 0$, all valid in $(p, p+\epsilon)$.

Now since $\phi(x)$ and $\phi'(x)$ are continuous and nonzero in $(p, b]$, then $\phi(x)/\phi'(x)$ has the same sign throughout $(p, b]$. Therefore $\phi(x)/\phi'(x) > 0$, and consequently $f(x) < x$ in $(p, b]$. But $f(p) = p$ and $f'(x) > 0$ in $(p, b]$, hence $p < f(x) < x$ there. The remainder of the proof is merely a repetition of that of Theorem 1.

This extension of Newton's method, although interesting theoretically, may be very difficult to apply to "practical" problems. The reason for this is the difficulty of isolating roots of any given multiplicity greater than one. However, in the case of polynomials, this difficulty is readily resolved.*

Returning to the sequences of Theorem 1, we first prove the

LEMMA. *Let $\phi(x)$, $\phi'(x)$, and $\phi''(x)$ be continuous in $[a, b]$. Suppose that $\phi'(x) \neq 0$ in $[a, b]$, and let $k =$ g.l.b. $|\phi'(x)|$ and $K =$ l.u.b. $|\phi''(x)|$ for $x \in [a, b]$. Further suppose that $\phi(a) \cdot \phi(b) < 0$ so that there exists exactly one simple root $x = p$ of the equation $\phi(x) = 0$ in $[a, b]$. Then $f(x) = x - \phi(x)/\phi'(x)$ and its derivative are continuous in $[a, b]$, and if $q \in [a, b]$, $|f'(x)| \leqq [k + K(b-a)] \cdot K \cdot |q-p|/k^2$ for $x \in [a, b]$ such that $|x-p| \leqq |q-p|$.*

Proof. Since $\phi'(x)$ is continuous on the closed interval $[a, b]$ there exists $\gamma \in [a, b]$ such that $|\phi'(\gamma)| = k$. According to the mean value theorem, whenever $x \in [a, b]$ we have $\phi'(x) - \phi'(\gamma) = \phi''(\xi) \cdot (x-\gamma)$ where $\xi \in [a, b]$. Therefore $|\phi'(x)| \leqq |\phi'(\gamma)| + K \cdot |x-\gamma|$ so that $|\phi'(x)| \leqq k + K(b-a)$ on $[a, b]$.

According to the mean value theorem, whenever $x \in [a, b]$ we obtain $\phi(x) - \phi(p) = \phi'(\zeta) \cdot (x-p)$ with $\zeta \in [a, b]$. Therefore for any $q \in [a, b]$ we obtain $|\phi(x)| \leqq [k + K(b-a)] \cdot |q-p|$ for all $x \in [a, b]$ such that $|x-p| \leqq |q-p|$.

Now concerning the rapidity of convergence of the iterative sequence $[x_n]$ in Theorem 1, we have

THEOREM 3. *Suppose that $\phi(x)$ in Theorem 1 is such that $K < 2k$ and that $[a, b]$ is chosen so that $(b-a) < (2-K/k)(k^2/K^2)$. Then if x_k represents the root $x = p$ with accuracy to m decimal places, the next iteration $x_{k+1} = f(x_k)$ will represent p with accuracy to at least $2m$ decimal places.*

Proof. By hypothesis $|x_k - p| < (.5) \cdot 10^{-m}$. But $(x_{k+1} - p) = f(x_k) - f(p) = f'(\xi) \cdot (x_k - p)$ where $\xi \in [a, b]$ and $|\xi - p| < |x_k - p|$. Therefore, using the lemma, $|x_{k+1} - p| < 2 \cdot |x_k - p|^2 < (.5) \cdot 10^{-2m}$ as desired.

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* See J. V. Uspensky's Theory of Equations, pp. 65–67.

MAJORANTS OF POLYNOMIAL DERIVATIVES

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Introduction. In this paper we consider the relationship between a polynomial and its derivative. S. Bernstein gave the theorem [4]:

If $P(x)$ is a polynomial of degree n, then

$$|P'|/\max|P| \leqq n/[(b-x)(x-a)]^{1/2}, \qquad a \leqq x \leqq b. \tag{1}$$

He also gave the theorem [1]:

If in the interval $a \leqq x \leqq b$, $P(x)$ is a monotonically increasing polynomial of degree $(2n+1)$, then $|P|/\max P' \geqq (1/2)(b-a)/(n+1)^2$, $a \leqq x \leqq b$.

There results, as a corollary:

If in the interval $a \leqq x \leqq b$, $P(x)$ is a monotonically increasing polynomial of degree $(2n+1)$, then

$$P'/|P| \leqq 2(n+1)^2/(b-a), \qquad a \leqq x \leqq b. \tag{2}$$

A. Markoff gave the theorem [5]:

If $P(x)$ is a polynomial of degree n then

$$|P'|/\max|P| \leqq 2n^2/(b-a), \qquad a \leqq x \leqq b. \tag{3}$$

In the neighborhood of $x=a$ and $x=b$, this is better than (1).

I. Schur imposed the condition that the polynomial vanish at the end points of the interval. He obtained the theorem [9]:

If $P(x)$ is a polynomial of degree $n \geqq 2$, with $P(a)=P(b)=0$, then

$$|P'|/\max|P| \leqq [2n\cot\pi/2n]/(b-a), \qquad a \leqq x \leqq b.$$

However, for large n, this is $O(n^2)$.

P. Erdös imposed the condition that the polynomial have only real zeros. He obtained [2] a majorant of order n, namely:

Let $P(x)$ be a polynomial of degree n. If P has only real zeros, none in (a, b), then

$$|P'|/\max|P| < \frac{en}{b-a}, \qquad a \leqq x \leqq b.$$

This is the best possible result.

Bernstein too found a majorant of order n, (except at the end points of the interval); and Erdös [2] found one of order $\sqrt{n}$, namely:

Let $P(x)$ be a real polynomial of degree n, with $|P| \leqq 1$ for $-1 \leqq x \leqq 1$, having no zero in the interior of the unit circle. Given any C between 0 and 1, there will

exist a positive N *depending on* C, *such that for* $n>N$, $|P'|<(4/c^2)\sqrt{n}$, $-1+C<x<1-C$.

The results (1), (2), (3) have been extended [8] in several directions, namely: higher derivatives, fractional derivatives, two independent variables, other functions, and other point sets. For example, S. Bernstein [1] gave the following extension to the complex variable:

Let $P(z)$ *be a real or complex polynomial of degree* n, *with* $|P|\leqq 1$ *for* $|z|\leqq 1$, z *complex. Then* $|P'|\leqq n$, $|z|\leqq 1$.

P. D. Lax [6] added the condition that $P(z)$ have no zeros inside the unit circle. Then $|P'|\leqq n/2$, $|z|\leqq 1$.

N. G. de Bruijn* gave the theorems:

Let $P(z)$ *and* $Q(z)$ *be polynomials of a complex variable* z; *such that when* $\operatorname{Re} z\geqq 0$, $Q\neq 0$ *and* $|P|\leqq|Q|$. *Then for* $\operatorname{Re} z\geqq 0$, $|P'|\leqq|Q'|$. Also:

Let $P(z)$ *and* $Q(z)$ *be polynomials of a complex variable* z; *such that the zeros of* Q *lie in a closed convex region, of finite or infinite extent in the plane; and such that the degree of* P *does not exceed the degree of* Q. *If* $|P|\leqq|Q|$ *for all points of the boundary, then also* $|P'|\leqq|Q'|$ *for all points of the boundary.*

In the present paper we shall derive four additional majorants of polynomial derivatives. Only in the case of the first two can we show that the constants are the best possible; that is, the inequality is invalid if the constant have a smaller value.

Positive coefficients. Consider the case of positive coefficients.

THEOREM I. *If the coefficients of a polynomial* $P(x)$ *of degree* n *are real and nonnegative, then* $P'/P\leqq nx^{-1}$, $x>0$. (*The case* $P\equiv x^n$ *shows that this is a best possible result.*) *If, moreover, the constant term is zero, then* $P'/P\geqq x^{-1}$, $x>0$. (*The case* $P\equiv x$ *shows that this is a best possible result.*)

Proof. Write $P(x)$ as $\sum_{j=0}^{n} a_jx^{n-j}$, $a_j\geqq 0$. Then

$$P'(x)=\sum_{j=0}^{n-1}(n-j)a_jx^{n-j-1}$$
$$=\frac{1}{x}\sum_{j=0}^{n-1}(n-j)a_jx^{n-j}\begin{cases}\leqq \dfrac{n}{x}\displaystyle\sum_{j=0}^{n-1}a_jx^{n-j}\leqq\dfrac{n}{x}P(x),\\[2ex] \geqq\dfrac{1}{x}\displaystyle\sum_{j=0}^{n-1}1\cdot a_jx^{n-j}=P(x)/x.\end{cases}$$

This completes the proof.

Bounded ratios of coefficients. The majorant of Theorem I can be halved, if

* Inequalities concerning polynomials in the complex domain, Nederl. Akad. Wetensch., Proc., vol. 50, 1947, pp. 1265–1272.

the conditions are tightened, as we now show:

THEOREM II. *Define* $P(x)=\sum_{j=0}^{n} c_j x^{n-j}$, $c_j>0$. *If* $0<c\leqq c_j/c_{j-1}$, $(j=1,\cdots,n)$, *then* $P'/P\leqq(1/2)nx^{-1}$, $0<x\leqq c$.

The constant is the best possible.

Proof. $xP'=\sum_{j=0}^{n-1}(n-j)c_jx^{n-j}=\sum_{j=0}^{n}(n-j)c_jx^{n-j}$. Now by Chebyshev's inequality [3], if $a_j\geqq 0$ and $b_j\geqq 0$ are monotonic sequences, one increasing and one decreasing, and if $p_j\geqq 0$, then $\sum p_ja_j\cdot\sum p_jb_j\geqq\sum p_ja_jb_j\cdot\sum p_j$.

When $p_j=1$, this becomes $\sum_{j=0}^{n} a_j\cdot\sum_{j=0}^{n} b_j\geqq(n+1)\sum_{j=0}^{n} a_jb_j$. Since $x\leqq c_j/c_{j-1}$, then $c_jx^{n-j}\geqq c_{j-1}x^{n-(j-1)}$. Thus c_jx^{n-j} *increases* as j increases. On the other hand, $n-j$ *decreases* as j increases.

Define $a_j=n-j$ and $b_j=c_jx^{n-j}$.

By Chebyshev's inequality,

$$\sum_{j=0}^{n}(n-j)\cdot\sum_{j=0}^{n}c_jx^{n-j}\geqq(n+1)\sum_{j=0}^{n}(n-j)c_jx^{n-j}.$$

Therefore, $\sum_{j=0}^{n}(n-j)\cdot\sum_{j=0}^{n}c_jx^{n-j}\geqq(n+1)xP'$. This is $(1/2)n(n+1)P\geqq(n+1)xP'$.

It remains to consider the constant $n/2$. When $P=x^n+\cdots+x+1$, at $x=1$,

$$xP'/P=\frac{n+(n-1)+\cdots+2+1}{\sum_{j=0}^{n}1}=n/2$$

Hence $n/2$ is the best constant.

Bounded coefficients. Consider now the case of bounded coefficients.

THEOREM III. *Define* $P(x)=x^n+\sum_{j=1}^{n}c_jx^{n-j}$. *If* $|c_j|\leqq c-1$, $c>1$, $(j=1,\cdots,n)$, *then* $|P'/P|\leqq n(x-(1/2)c)^2/[x(x-c)^2]$, $x>c$.

Proof. There exist constants a_j, $(j=1,\cdots,n)$, real or complex, such that $P(x)=\prod_{j=1}^{n}(x+a_j)$. Thus the zeros of $P(x)$ are $-a_i$, $(i=1,\cdots,n)$.

Suppose for some i, that $|a_i|>1$. Then for this i, we have

$$\begin{aligned}|a_i|^n&\leqq\sum_{j=1}^{n}|c_j|\cdot|a_i|^{n-j}\leqq(c-1)\sum_{j=1}^{n}|a_i|^{n-j}\\&=(c-1)(|a_i|^n-1)/(|a_i|-1)<(c-1)|a_i|^n/(|a_i|-1).\end{aligned}$$

Divide by $|a_i|^n$, which is >1. There results $1<(c-1)/(|a_i|-1)$, and so $|a_i|<c$.

As a consequence, either $|a_i|\leqq 1$ or $|a_i|<c$, for all i, $(i=1,\cdots,n)$. Hence in any case, $|a_i|<c$ for all i. Since $x>c$, then $x>|a_i|$, $(i=1,\cdots,n)$. Now $\log P=\sum_{j=1}^{n}\log(x+a_j)$. Differentiate.

Thus $P'/P=\sum_{j=1}^{n}(x+a_j)^{-1}$. Then

$$|P'/P| \leqq \sum_{j=1}^{n} |x + a_j|^{-1} \leqq \sum_{j=1}^{n} (x - |a_j|)^{-1}.$$

Now if $0<\alpha\leqq\alpha_j\leqq A$ and $0<\beta\leqq\beta_j\leqq B$, $(j=1, \cdots, n)$, then by [7],

$$(\sum \alpha_j^2)(\sum \beta_j^2)/(\sum \alpha_j\beta_j)^2 \leqq (\alpha\beta + AB)^2/(4\alpha\beta AB).$$

Define now $\alpha=\sqrt{x-c}$, $\beta=1/\sqrt{x}$, $A=\sqrt{x}$, $B=1/\sqrt{x-c}$, $\alpha_j=\sqrt{x-|a_j|}$, $\beta_j = 1/\sqrt{x-|a_j|}$. There results $(1/n^2)[\sum(x-|a_j|)][\sum(x-|a_j|)^{-1}] \leqq [(x-c)+x]^2/[4(x-c)x]$. Hence $\sum(x-|a_j|)^{-1}\leqq n(x-(1/2)c)^2/[x(x-c)^2]$. The theorem follows at once.

Increasing coefficients. Theorem I tells us that when the coefficients are positive, then $xP'/P\leqq n$, $x>0$. Theorem II tells us that if moreover the ratios of the coefficients are bounded, then $xP'/P\leqq n/2$, $0<x\leqq c$.

We shall now find conditions such that

$$xP'/P \leqq (n+1)/k, \qquad (k \geqq e).*$$

THEOREM IV. *Define* $P(x)=\sum_{j=0}^{n} c_jx^{n-j}$, $c_j>0$. *If* $c_j\geqq c_{j-1}$, $(j=1, \cdots, n)$, *then*

$$P'/P \leqq \frac{n+1}{k}x^{-1}, \qquad 0 < x \leqq \exp\left(-\frac{k}{e}\right), \qquad k \geqq e.$$

Proof. Define $F(y)\equiv(1/e)-(1/y)\log y$. Then $F'\equiv y^{-2}(\log y-1)\geqq 0$ for $\log y \geqq 1$, that is, for $y\geqq e$. Hence $F(y)$ is increasing for $y\geqq e$.

Now $F(e)=0$. Therefore, $F(y)\geqq 0$, $y\geqq e$. Thus $1/e\geqq(1/y)\log y$ for $y\geqq e$. Consequently $y/e\geqq\log y$, and so $\exp y/e\geqq y$, $y\geqq e$. When $j=0, \cdots, n-1$, then $k(n-j)\geqq e$. As a result, we may take $y=k(n-j)$, $(j=0, \cdots, n-1)$. Therefore,

$$\left(\exp\frac{k}{e}\right)^{n-j} = \exp\frac{k(n-j)}{e} \geqq k(n-j), \qquad (j=0, \cdots, n-1).$$

Hence

$$P' \equiv x^{-1}\sum_{j=0}^{n-1}(n-j)c_jx^{n-j} \leqq \frac{1}{k}x^{-1}\sum_{j=0}^{n-1}c_j\left(x\exp\frac{k}{e}\right)^{n-j} \leqq \frac{1}{k}x^{-1}\sum_{j=0}^{n}c_j.$$

Now by Chebyshev's inequality [3], if $a_j\geqq 0$ and $b_j\geqq 0$ are monotonic sequences, both increasing, and if $p_j\geqq 0$, then

$$\sum p_ja_j\cdot\sum p_jb_j \leqq \sum p_ja_jb_j\cdot\sum p_j.$$

By hypothesis c_j is increasing, and x^{n-j} increases with j since $x<1$. Set $a_j=c_j$, $b_j=x^{n-j}$, and $p_j=1$.

As a result $\sum_{j=0}^{n} c_j\cdot\sum_{j=0}^{n} x^{n-j}\leqq(n+1)\sum_{j=0}^{n} c_jx^{n-j}$. Therefore,

* We originally proved the theorem for $k=3$, and are indebted to the referee for pointing out that the same proof can be used for $k\geqq e$.

$$P \equiv \sum_{j=0}^{n} c_j x^{n-j} \geqq \frac{1}{n+1} \sum_{j=0}^{n} c_j \cdot \sum_{j=0}^{n} x^{n-j} \geqq \frac{1}{n+1} \sum_{j=0}^{n} c_j.$$

Hence

$$P'/P \leqq \frac{n+1}{k} x^{-1}.$$

References

1. S. Bernstein, Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Reelle, Paris, 1926, pp. 45–47.
2. P. Erdös, Extremal properties of derivatives of polynomials, Ann. of Math., vol. 41, 1940, pp. 310–313.
3. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934, p. 43.
4. D. Jackson, Theory of Approximation, New York, 1930, p. 92.
5. D. Jackson, On the application of Markoff's theorem to problems of approximation in the complex domain, Bull. Amer. Math. Soc., vol. 37, 1931, pp. 883–890, especially p. 883.
6. P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 509–513.
7. G. Pólya and G. Szegö, Aufgaben and Lehrsätze aus der Analysis, New York, 1945, vol. 1, p. 57.
8. A. C. Shaeffer, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, Bull. Amer. Math. Soc., vol. 47, 1941, pp. 565–579.
9. I. Schur, Über das Maximum des absoluten Betrags eines Polynoms in einem gegebenen Intervall, Math. Z., vol. 4, 1919, pp. 271–287, especially p. 282.

THE PARTIAL SUMMATION OF SERIES BY MATRIX METHODS

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1. Introduction. In a recently published article [1], the author discussed certain properties of a sequence of interesting matrices. Further study of these matrices led to the deduction of a number of interesting relationships involving binomial coefficients, such as the relationships (1), (2), (3), (4), *etc.* exhibited below.

While most of these relationships are not essentially new, and may be found in more general form elsewhere [2], it is the matrix technique which is used in deriving them which represents a novel approach. If this technique could be applied to establish general theorems of a similar nature, it would prove to be a useful analytical tool.

2. Some relationships for binomial coefficients. To illustrate the method let us consider the following identities:

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n+i-1}{n}\binom{n}{i}=1. \tag{1}$$

$$\sum_{i=1}^{n}(-1)^{i}(i+j-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}=0, \qquad 2\leqq j\leqq n. \tag{2}$$

$$\begin{aligned}(-1)^{j-1}\binom{n+j-1}{n}\binom{n}{j}\sum_{i=1}^{n}\Big[&(-1)^{i}(i+j-1)^{-1}\\ &\times(i+k-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\Big]=\delta_{kj}, \qquad 2\leqq j,k\leqq n.\end{aligned} \tag{3}$$

$$\begin{aligned}\binom{n+j-1}{n}\binom{n}{j}\Big[(-1)^{n-j}+\sum_{i=1}^{n}&(-1)^{i+j-1}i(i-1)(i+j-1)^{-1}\\ &\times(i+k-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\Big]=\delta_{jk}.\end{aligned} \tag{4}$$

These identities are not new. The first two, for example, follow directly from Theorems I and II applicable to polynomials

$$\phi(x)=\sum_{k=0}^{n}a_kx^k$$

of nth degree:

THEOREM I.

$$\sum_{i=0}^{r}(-1)^{i}\phi(i)\binom{r}{i}=\begin{cases}0 & \text{for } n<r\\ (-1)^{n}n!a_n & \text{for } n=r.\end{cases}$$

THEOREM II.

$$\sum_{i=0}^{n}(-1)^{i}(i+x)^{-1}\phi(y-i)\binom{n}{i}=x^{-1}\phi(x+y)\binom{n+x}{n}^{-1}.$$

Theorem I is well known* and is discussed in most books dealing with finite differences, while Theorem II is exhibited in [3]. Putting $\phi(x)=\binom{n+x-1}{n}$ in Theorem I leads to (1). Setting $y=0$, and $\phi(x)=\binom{n-x-1}{n}$ in Theorem II, results in $\sum_{i=0}^{n}(-1)^{i}(i+x)^{-1}\binom{n+i-1}{n}\binom{n}{i}=x^{-1}\binom{n-x-1}{n}\binom{n+x}{n}^{-1}$. Now letting $x=j-1$, and restricting j to the interval $2\leqq j\leqq n$ establishes relation (2). The relationships (3) and (4) follow in a similar manner from this generalization of Theorem II:

* The form in which Theorems I, II, III are stated was indicated by the referee.

THEOREM III. *If* $\phi(x)=\sum_{k=0}^{n+1} a_k x^k$, *then*

$$\sum_{i=0}^{n}(-1)^i(i+z)^{-1}(i+x)^{-1}\phi(y-i)\binom{n}{i}$$
$$=(n+1)^{-1}(x-z)^{-1}\left[\phi(z+y)\binom{n+z}{n+1}^{-1}-\phi(x+y)\binom{n+x}{n+1}^{-1}\right].$$

3. Application of the matrix method. Let us now illustrate the use of the matrix technique to establish our relationships (1) through (4).

Inspection of the identities in question seems to indicate the feasibility of considering them to be of the form

$$\sum_{i=1}^{n} a_{ji}^{(n)}\cdot\alpha_{ik}^{(n)}=\delta_{jk}, \tag{5}$$

where the $a_{ji}^{(n)}$ denote elements of an n-dimensional matrix A_n, and $\alpha_{ik}^{(n)}$ the elements of its inverse A_n^{-1}. The occurrence of the term $(i+j-1)^{-1}$ in the relationships (2), (3), (4), but not in (1), further suggests the choice of $a_{1i}^{(n)}=1$, $a_{ij}^{(n)}=(i+j-1)^{-1}$, so that

$$A_n=\begin{bmatrix}1 & 1 & \cdots & 1\\ 2^{-1} & 3^{-1} & \cdots & (n+1)^{-1}\\ 3^{-1} & 4^{-1} & \cdots & (n+1)^{-1}\\ \vdots & \vdots & & \vdots\\ n^{-1} & (n+1)^{-1} & \cdots & (2n-1)^{-1}\end{bmatrix}. \tag{6}$$

The problem of proving the relationships for the binomial coefficients thus resolves itself in the problem of inverting the A_n, and the subsequent testing of the satisfaction of the relations (1), *etc.*, interpreted in the form (5).

In turns out that* [1]

$$\alpha_{i1}=(-1)^{n-i}\binom{n+i-1}{n}\binom{n}{i}, \tag{7}$$

$$\alpha_{ij}=(-1)^{i+j-1}j(j-1)(i+j-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\binom{n+j-1}{n}\binom{n}{j} \tag{8}$$

for $j\geqq 2$.

The following alternative proof of these inversion formulas, due to D. J. Newman, is based on the relation $AA^{-1}=I$, which the α_{ij} must necessarily satisfy. Consequently,

* For convenience in writing, the indication of the fixed index n has been deleted in the following.

$$\sum_{i=1}^{n} \alpha_{ij} = \delta_{1j} \qquad j = 1, \cdots, n, \tag{9}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{ij} = \delta_{kj} \qquad j = 1, \cdots, n;\ k = 2, \cdots, n. \tag{10}$$

(i) The case $j=1$. We have, by (9) and (10),

$$\sum_{i=1}^{n} \alpha_{i1} = 1, \tag{11}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{i1} = 0 \quad \text{for} \quad k = 2, \cdots, n. \tag{12}$$

The α_{i1} may thus be considered to be the coefficients of the partial fraction expansion of a certain rational function:

$$f_1(x) = \sum_{i=1}^{n} \alpha_{i1}(x + i - 1)^{-1}. \tag{13}$$

By (12) this function $f_1(x)$ must vanish at $x=2, \cdots, n$; because of (11) $f_1(x)$ must asymptotically approach x^{-1} as x becomes unbounded. It may be concluded that

$$f_1(x) = (x - 2) \cdots (x - n)[x(x + 1) \cdots (x + n - 1)]^{-1}. \tag{14}$$

Consequently, it is seen that

$$\alpha_{i1} = \frac{(-i + 1 - 2)(-i + 1 - 3) \cdots (-i + 1 - n)}{(-i + 1)(-i + 2) \cdots (-i + i - 1)\cdot(-i + i + 1) \cdots (-i + n)}$$
$$= (-1)^{n-i}\binom{n + i - 1}{n}\binom{n}{i},$$

as claimed in (7).

(ii) The case $j>1$. Here, from (9) and (10),

$$\sum_{i=1}^{n} \alpha_{ij} = 0, \qquad j = 2, \cdots, n, \tag{15}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{ij} = \delta_{kj}, \qquad k, j = 2, \cdots, n. \tag{16}$$

The appropriate rational functions are now

$$f_j(x) \equiv \sum_{i=1}^{n} \alpha_{ij}(x + i - 1)^{-1}.$$

Conditions (15) and (16) indicate that

$$f_j(x) = c_j(x-2)\cdots(x-j+1)(x-j-1)\cdots(x-n) \cdot [x(x+1)\cdots(x+n-1)]^{-1},$$

with c_j denoting a constant.

Putting $x=j$, and utilizing condition (16) leads to

$$c_j = j(j+1)\cdots(j+n-1)[(j-2)\cdots(j-n)]^{-1},$$

where in the brackets $[\cdots]$ the term $j-j$ is omitted.

Since the α_{ij} are the coefficients of the partial fraction expansion of $f_j(x)$, it follows that

$$\alpha_{ij} = \frac{c_j(-i+1-2)\cdots(-i+1-n)}{(-i+1)(-i+2)\cdots(-i+i-1)(-i+i+1)\cdots(-i+n)},$$

or, equivalently,

$$\alpha_{ij} = (-1)^{i+j-1}\frac{j(j-1)}{i+j-1}\binom{n+i-1}{n}\binom{n}{i}\binom{n+j-1}{n}\binom{n}{j},$$

as claimed in (8).

4. The binomial identities. Having established the expressions (7) and (8) for the α_{ij}, the identities (1) through (4) follow immediately. Inserting the formulas (7) in (11) leads to (1), and putting (7) into (12) results in (2).

Similarly, inserting the expressions (8) in (16) leads to (3).

Making use of $A^{-1}A=I$, one obtains,

$$\alpha_{i1} + \sum_{j=2}^{n} \alpha_{ij}(j+k-1)^{-1} = \delta_{ik} \tag{17}$$

for i, $k=1, \cdots, n$. The substitution of (7) and (8) into (17) results directly in (4).

The essential part of this approach thus lies clearly in the proper choice of a "generating" matrix A_n. Careful inspection of the relationships to be established may frequently provide a sufficient number of clues for the proper composition of such a generating matrix.

References

1. M. Lotkin, A set of test matrices, Math. Tables Aids Comput., vol. 9, 1955, pp. 153–161.
2. I. J. Schwatt, Introduction to the Operations with Series, University of Pennsylvania, 1924.
3. Z. A. Melzak, Problem 4458, this MONTHLY, vol. 58, 1951, p. 636.

ON BOUNDS FOR CHARACTERISTIC VALUES OF A PRODUCT OF MATRICES

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Let $A=(a_{ij})$ be a matrix of complex numbers; set $S(A)=\max_i \sum_j |a_{ij}|$; $T(A)=S(A^*)$. From theorems associated with the work of Hadamard, Geršgorin, Perron, it is known that for every characteristic root λ of A the relations $|\lambda| \leqq S(A)$, $|\lambda| \leqq T(A)$ must hold. Now consider the product $P=(p_{ij})$ of the matrices A, $B=(b_{ij})$. From the rule for multiplying matrices, we have $p_{ij}=\sum_k a_{ik}b_{kj}$, so that

$$\begin{aligned}\sum_j |p_{ij}| &= \sum_j \Big| \sum_k a_{ik}b_{kj}\Big| \leqq \sum_j \sum_k |a_{ik}|\,|b_{kj}| \\ &= \sum_k |a_{ik}| \Big(\sum_j |b_{kj}|\Big) \leqq S(B) \sum_k |a_{ik}| \leqq S(B)S(A),\end{aligned}$$

i.e., $S(P) \leqq S(B)S(A)$. Thus we have proved the

THEOREM 1. *Let A, B be two square matrices of complex numbers; $P=AB$. Then every characteristic root λ of P satisfies $|\lambda| \leqq S(A)S(B)$; $|\lambda| \leqq T(A)T(B)$, and indeed for any fixed α, $0 \leqq \alpha \leqq 1$, we have $|\lambda| \leqq \{S(A)S(B)\}^{\alpha}\{T(A)T(B)\}^{1-\alpha}$.*

We put these bounds forward since they are easily computed from the entries a_{ij}, b_{ij} of the factors A, B.

By using the quantities $S_i(A)=\sum_k |a_{ik}|$, $T_i(A)=S_i(A^*)$, the above results can be further refined. Indeed Barankin showed that every characteristic root λ of AB satisfies $|\lambda| \leqq \max_i \{S_i(AB)\}^{\alpha}\{T_i(AB)\}^{1-\alpha}$, for every α, $0 \leqq \alpha \leqq 1$, and when this is combined with $S_i(P) \leqq S(B)S_i(A)$, the conclusion of Theorem 1 can be improved to read

THEOREM 2. $|\lambda| \leqq S(B)^{\alpha}T(A)^{1-\alpha} \max_i \{S_i(A)^{\alpha}T_i(B)^{1-\alpha}\}$.

References

E. W. Barankin, Bounds for characteristic roots of a matrix, Bull. Amer. Math. Soc. vol. 51, 1945, pp. 767–770.

S. Geršgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk SSSR, vol. 8, 1931, pp. 749–754.

J. Hadamard, Leçons sur la Propagation des Ondes, Paris, 1903, pp. 13–14.

O. Perron, Algebra, 1933.

* I am indebted to the referee for his suggestions in rewording my original note and also to Professor S. M. Shah.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following problems were given the contestants in the seventeenth annual William Lowell Putnam mathematical competition:*

Part I

1. The normals to a surface all intersect a fixed straight line. Show that the surface is a portion of a surface of revolution.
2. A uniform wire is bent into a form coinciding with the portion of the curve $y=e^x$, $0 \leqq x \leqq a$, $a>1$, and the line segment $a-1 \leqq x \leqq a$, $y=e^a$. The wire is then suspended from the point $(a-1,\ e^a)$ and a horizontal force F is applied at the point $(0,\ 1)$ to hold the wire in coincidence with the curve and segment. Assuming the x-axis is horizontal, show that the force F is directed to the right.
3. A and B are real numbers and k is a positive integer. Show that

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| < k^2 - 1$$

whenever the left side is defined.†

4. $P(z)$ is a complex polynomial whose roots (as points in the Argand plane) can be covered by a closed circular disc of radius R. Show that the roots of $nP(z)-kP'(z)$ can be covered by a closed circular disc of radius $R+|k|$, where n is the degree of $P(z)$, k is any complex number, and $P'(z)$ is the derivative of $P(z)$.
5. Given n points in the plane, show that the largest distance determined by these points cannot occur more than n times.
6. $S_1 = \ln a$, and $S_n = \sum_{i=1}^{n-1} \ln (a - S_i)$, $n > 1$.

Show that

$$\lim_{n\to\infty} S_n = a - 1.$$

7. Each member of a set of circles in the xy-plane is tangent to the x-axis and no two of the circles intersect. Show that

(a) the points of tangency can include all the rational points on the axis but

(b) the points of tangency cannot include all the irrational points.

* The results were annouced in this MONTHLY, vol. 64, 1957, pp. 486–488.

† The questions committee originally had a "$\leqq$" in this question, but, in some manner, the "$=$" was lost before the question reached the director. The omission was discovered in reading the proofs, but the chairman of the committee decided that it was just as well to give the contestants the opportunity to discover the error and correct it.

Part II

1. Consider the determinant $|a_{ij}|$ of order 100 with $a_{ij}=i\times j$. Prove that if the absolute value of each of the 100! terms in the expansion of this determinant is divided by 101, then the remainder in each case is 1.
2. If facilities for division are not available, it is sometimes convenient in determining the decimal expansion of $1/A$, $A>0$, to use the iteration $X_{k+1}=X_k(2-AX_k)$, $k=0, 1, 2, \cdots$, where X_0 is a selected "starting" value. Find the limitations, if any, on the starting value X_0 in order that the above iteration converge to the desired value $1/A$.
3. For $f(x)$ a positive, monotone decreasing function defined in $0\leqq x\leqq 1$, prove that

$$\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx} \leqq \frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}.$$

4. Let $a(n)$ be the number of representations of the positive integer n as the sums of 1's and 2's taking order into account. For example, since

$$4 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2 = 1+1+1+1,$$

then $a(4)=5$. Let $b(n)$ be the number of representations of n as the sum of integers greater than 1, again taking order into account and counting the summand n. For example, since $6=4+2=2+4=3+3=2+2+2$, we have $b(6)=5$. Show that for each n, $a(n)=b(n+2)$.
5. With each subset X of a set is associated a second subset $f(X)$. The association is such that whenever X contains Y then $f(X)$ contains $f(Y)$. Show that for some set A, $f(A)=A$.
6. The curve $y=f(x)$ passes through the origin with a slope of 1. It satisfies the differential equation $(x^2+9)y''+(x^2+4)y=0$. Show that it crosses the x-axis between

$$x = \frac{3}{2}\pi \quad \text{and} \quad x = \sqrt{\frac{63}{53}}\,\pi.$$

7. Let C be a closed convex planar disc bounded by a regular polygon. Show that for each positive integer n there exists a set of points $S(n)$ in the plane such that each n points of $S(n)$ can be covered by C, but $S(n)$ itself cannot be covered by C.

*Solutions of the Problems.** The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. In the case of Part I, No. 3 and Part II, No. 7, no contestant had a completely correct solution. The solutions of these two problems were furnished by members of the questions committee. The presentation here is intended as a brief sketch of the method of proof rather than a model of a detailed proof such as is expected from the contestants.

Part I

1. Let L be the line intersected by all the normals to the surface, and consider a plane section of the surface by a plane perpendicular to L at O. If P is any point on the intersection of the plane and the surface and if P is not on L, then the normal to the surface projects into the line OP. It follows that OP is normal to the curve in which the plane cuts the surface. An easy discussion shows that if the normals to a plane curve all pass through a fixed point, then the curve is a circle with the fixed point as center. The conclusion follows.

2. Let F_1 be the x-component of the horizontal force which just suffices to hold the wire in the required position. The force acting at the point $(a-1,\ e^a)$ will be just sufficient to make the sum of all the forces acting on the wire equal to zero. The sum of the moments about this point must also vanish. This requires $F_1(e^a-1)=g\left[\int_0^a(x-a+1)\sqrt{1+e^{2x}}dx+1/2\right]$, where g is the gravitational constant. Thus it is sufficient to show that the right member of this equation is positive for $a>1$. Let $f(a)$ denote the bracketed quantity above. It follows that $f(0)>0$. Then $f'(a)=\sqrt{1+e^{2a}}-\int_0^a\sqrt{1+e^{2x}}dx$ and $f''(a)=-1/\sqrt{1+e^{2a}}>-e^{-a}$. Thus $f'(a)=f'(0)+\int_0^a f''(x)dx\geqq\sqrt{2}+\int_0^a(-e^{-x})dx>0$. Thus f is increasing for $a>0$ and the result follows.

3. We first prove that $|\sin rx|\leqq r|\sin x|$, where r is a positive integer. To prove this, note that $|\sin rx| = |\sin \{(r-1)x+x\}| = |\sin (r-1)x\cos x+\cos (r-1)x\sin x|\leqq|\sin (r-1)x|+|\sin x|$, whence the required relation follows by induction. Then

$$\left|\frac{\sin rx \sin sy}{\sin x \sin y}\right| \leqq rs,$$

or more symmetrically,

$$\frac{1}{2}\left|\frac{\sin rx \sin sy + \sin sx \sin ry}{\sin x \sin y}\right| \leqq rs.$$

Let $r=k+1$, $s=k-1$, $x=(A+B)/2$, $y=(A-B)/2$, and the above inequality becomes (after a few elementary trigonometric identities) the required inequality with "$<$" replaced by "$\leqq$".

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

4. If $P(z)=A\prod_{i=1}^{n}(z-r_i)$ is the factored form of the polynomial, then $n-kP'(z)/P(z)=n-k\sum_{i=1}^{n}1/(z-r_i)$. If $|z|>R+|k|$, it follows at once that the foregoing expression is not zero and the conclusion is immediate.

5. The result is clearly true for $n=2$ and $n=3$. Assume $n>3$ and that n points determine $n+1$ segments of maximum length L. Some point P of the given points must serve as end point of three or more of these segments, for if not there could be at most n such segments. The circle with center P and radius L has on it at least three of the given points and all the given points on the circle must lie on an arc which subtends a central angle not exceeding 60°. Let Q and R be those points of the given set which are on this arc and are farthest apart. All other points of the given set must belong to the intersection of the circular discs with centers at P, Q, and R and common radius L. An easy discussion shows that S, which is on the arc QR, can not be at distance L from any of the given points except P. Thus if S is deleted we have $n-1$ points determining n largest distances, and the induction follows.

6. On the range $x\leqq a-1$ the function $x+\ln(a-x)$ is increasing and its values lie between x and $a-1$. Since for any $a>0$, $\ln a\leqq a-1$, as is easily shown, it follows that the sequence $\{S_n\}$ is monotone increasing and bounded above by $a-1$, and hence convergent. The relation $S_{n+1}=S_n+\ln(a-S_n)$ implies, by continuity, that the limit is $a-1$.

7. Let the rationals be enumerated in a sequence r_n, $n=1, 2, \cdots$. Take a circle tangent to the x-axis at the point $(r_1, 0)$ with radius 1. If nonintersecting circles have been constructed with the ith circle tangent to the x-axis at $(r_i, 0)$, $i=1, \cdots, n$, then since the point $(r_{n+1}, 0)$ is outside all these circles, it is at positive distance from their union. A circle tangent to the x-axis at the $(n+1)$th point and not intersecting any of the previous circles is then readily constructed, and the induction is complete.

If C is a set of nonintersecting circles and each circle is tangent to the x-axis, then we may shorten the radius of each circle until it is less than 1, preserving the point of tangency and keeping the new circle inside the original circle. We thus obtain a class C' of nonintersecting circles in one-to-one correspondence with the original set, with no radius exceeding 1. Let C_n' be those circles of C' with radius r satisfying $1/n\leqq r<1/(n-1)$. All the circles of C_n' which are tangent to the x-axis at points x, where $0\leqq x\leqq 1$, lie in a square of side 3 and hence are finite in number. Thus only a denumerable set of circles of C' can be tangent to the x-axis at points of the above interval. Since the irrational points in this interval are known to be nondenumerable, the class C' (therefore the class C also) must fail to fulfill the required condition.

Part II

1. The absolute value of each term is immediately seen to be $(100!)^2$ and an application of Wilson's theorem completes the proof.

2. By induction, $|X_{k+1}-1/A|=A^{2^k-1}|X_1-1/A|^{2^k}$. It follows that the sequence $\{X_k\}$, $k=1, 2, \cdots$, converges to $1/A$ if and only if $0<X_1<2/A$.

3. Since f is nonnegative and nonincreasing, $\int_0^1\int_0^1 f(x)f(y)(x-y)[f(x)-f(y)]\,dxdy \leqq 0$. Expanding, we have

$$\left[\int_0^1 xf^2(x)dx\right]\int_0^1 f(y)dy - \left[\int_0^1 xf(x)dx\right]\int_0^1 f^2(y)dy$$
$$-\left[\int_0^1 f^2(x)dx\right]\int_0^1 yf(y)dy + \left[\int_0^1 yf^2(y)dy\right]\int_0^1 f(x)dx$$
$$= 2\left[\int_0^1 xf^2(x)dx\right]\int_0^1 f(x)dx - 2\left[\int_0^1 xf(x)dx\right]\int_0^1 f^2(x)dx \leqq 0.$$

The conclusion is immediate.

4. Let A_n be the class of all finite sequences of ones and twos with n as the sum of the terms and let B_n be the class of all finite sequences with terms greater than 1 and sum n. Each sequence of A_n determines a sequence in B_{n+2} according to the following rule. Annex an additional term with value 2 as a final term and replace each maximal run of ones and the two which immediately follows the run by a single term with value equal to their sum. Thus 1, 1, 2, 2, 2, 1, 1, 1, 2, 1, 2, 1 becomes 4, 2, 2, 5, 3, 3. The correspondence is readily shown to be bi-unique and the proof follows.

5. Let C be the class of all subsets X of the given set such that $f(X)\subset X$. Put $T=\bigcap_{x\in C} X$. If $X\in C$, then $T\subset X$ and thus $f(T)\subset f(X)\subset X$. Thus $f(T)\subset T$ and therefore $T\in C$. Since $f(T)\subset T$, it follows that $f(f(T))\subset f(T)$ and therefore $f(T)\in C$. This implies $T\subset f(T)$, and the proof follows.

6. Let y and z be functions of x satisfying the respective equations $y''+h(x)y=0$ and $z''+g(x)z=0$ for $x\geqq 0$, where $h(x)\geqq g(x)$ for $x\geqq 0$ and where y and z satisfy the initial conditions $y'(0)=z'(0)=1$ and $y(0)=z(0)=0$. Let x_0 be the first zero of z to the right of the origin. Suppose y does not vanish in the interval $0<x\leqq x_0$. Then

$$0 = [z(yz' - y'z)/y]_0^{x_0} = \int_0^{x_0}\left[\frac{z(yz' - y'z)}{y}\right]' dx$$
$$= \int_0^{x_0}[(yz' - y'z)^2/y^2]dx + \int_0^{x_0} z^2[h(x) - g(x)]dx > 0,$$

since z does not vanish in the interior of the interval and h is assumed not to be identically equal to g on the interval. This contradiction shows that the first zero of y to the right of the origin is in the interval $0<x\leqq x_0$.

Take $g(x)=4/9$ and $h(x)=(x^2+4)/(x^2+9)$ and apply the above result. We find that the function $f(x)$ of the problem must vanish at some point x_0 such that $0<x_0\leqq 3\pi/2$. Next take $g(x)=(x^2+4)/(x^2+9)$ and $h(x)=53/63$. Then $h(x)>g(x)$ for $0\leqq x\leqq 3\pi/2$ and the above result shows that $\sin(x\sqrt{53/63})$ must vanish sooner than $f(x)$. The proof follows.

7. Let P be the regular polygonal disc of m sides, O its incenter, R the radius of the inscribed circle. Let n be a fixed positive integer and $C(n)$ the circle of center O and radius $R \sec(\pi/2mn)$. It will be shown that if $p_1, \cdots, p_n$ are any n points of $C(n)$, then there exists a rotation of $C(n)$ about the point O which will take the points p_i $(i=1, \cdots, n)$ into P. The numbers in the interval 0 to 2π correspond to the possible rotations of $C(n)$ about the point O. The numbers corresponding to the rotations which will take p_i out of P constitute m intervals in the interval 0 to 2π of total length not exceeding $2m(\pi/2mn)=\pi/n$. Thus the total length of the intervals corresponding to those rotations which take at least one of the points $p_1, p_2, \cdots, p_n$ out of the disc P is at most $n(\pi/n)=\pi$. It follows that there is at least one number on the interval 0 to 2π which corresponds to a rotation which takes none of the points p_i out of P, *i.e.*, there exists a rotation taking all the points p_i into P.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

PARTIAL FRACTIONS

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A genuine rational function $f(x)/g(x)$ can be decomposed into partial fractions by rationalizing and comparing coefficients. Whereas this method is rarely used in computational practice it leads to interesting systems of linear equations which it is worth while to discuss.

In the first principal case g is split into r coprime factors $\phi_1, \cdots, \phi_r$ with nonvanishing initial coefficients. We abbreviate $\phi_2\phi_3 \cdots \phi_r = \psi_1$ and decompose

$$f/g = \overset{*}{\phi_1}/\phi_1 + \overset{*}{\psi_1}/\psi_1\,. \tag{1}$$

Given are $f(x) = \sum k_\iota x^\iota$, $\phi_1(x) = \sum a_\kappa x^\kappa$, and $\psi_1(x) = \sum b_\lambda x^\lambda$; assuming that $\phi_1^*(x) = \sum \alpha_\mu x^\mu$ and $\psi_1^*(x) = \sum \beta_\nu x^\nu$ the equations

$$\sum_{\mu+\lambda=\iota} \alpha_\mu b_\lambda + \sum_{\nu+\kappa=\iota} \beta_\nu a_\kappa = k_\iota, \qquad \iota = n-1, \cdots, 0, \tag{2}$$

(n the degree of g) for the unknown coefficients α_μ and β_ν follow immediately. The determinant of these equations is readily seen to be identical, except for transposition, with the resultant $R(\psi_1, \phi_1)$. Denoting the matrix of the (transposed) resultant R by $\mathbf{R}$, and introducing one-column matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{K}$ which contain the coefficients of ϕ_1^*, ψ_1^* and f, respectively, (2) can be written as

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{B}\\ \mathbf{A}\end{pmatrix} = \mathbf{K}. \tag{3}$$

Setting $\psi_1 = \phi_2\psi_2$ we continue the decomposition:

$$\overset{*}{\psi_1}/\psi_1 = \overset{*}{\phi_2}/\phi_2 + \overset{*}{\psi_2}/\psi_2. \tag{4}$$

The procedure remains the same; with $\phi_2^* = \sum\gamma_\rho x^\rho$ and $\psi_2^* = \sum\delta_\sigma x^\sigma$ the system

$$\mathbf{R}(\psi_2, \phi_2)\begin{pmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\end{pmatrix} = \mathbf{B} \tag{5}$$

for the new unknowns γ_ρ and δ_σ is obtained. (1) and (4) together lead to the decomposition

$$f/g = \overset{*}{\phi_1}/\phi_1 + \overset{*}{\phi_2}/\phi_2 + \overset{*}{\psi_2}/\psi_2 \tag{6}$$

into three partial fractions. In order to obtain it in a single step, without the detour through ψ_1^*, one may interpret (5) as a linear transformation which produces the β_ν from the γ_ρ and δ_σ. Extending this transformation so that it leaves the α_μ untransformed:

$$\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\begin{bmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\\ \mathbf{A}\end{bmatrix} = \begin{pmatrix}\mathbf{B}\\ \mathbf{A}\end{pmatrix} \tag{7}$$

(E the appropriate unit matrix), it serves to eliminate $\mathbf{B}$ from (3) to give the system

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\begin{bmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\\ \mathbf{A}\end{bmatrix} = \mathbf{K}$$

for the coefficients required in (6).

Successive application of transformations analogous to (7) finally leads to a system with the matrix

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\cdots\begin{pmatrix}\mathbf{R}(\phi_r, \phi_{r-1}) & 0\\ 0 & E\end{pmatrix}$$

for the unknown coefficients in the decomposition

$$f/g = \phi_1^*/\phi_1 + \cdots + \phi_r^*/\phi_r. \tag{8}$$

The determinant of this system is

$$R(\psi_1, \phi_1)R(\psi_2, \phi_2) \cdots R(\phi_r, \phi_{r-1}) = \prod_{i>k} R(\phi_i, \phi_k) \tag{9}$$

which cannot vanish; thus the solutions are unique, as was to be expected. However, one can also invert the argument. Assuming the possibility of a decomposition in the form (8), the comparison of coefficients leads to (9) which shows that this decomposition is unique if and only if $\phi_1, \cdots, \phi_r$ are coprime in pairs and have nonvanishing initial coefficients. The existence and uniqueness of the familiar decomposition is thus established in another way.

The other principal case arises when $g(x)$ is the mth power of a polynomial $\phi(x)$ of degree n. Here

$$f/g = \overset{*}{\phi}_m/\phi^m + \cdots + \overset{*}{\phi}_1/\phi \tag{10}$$

with numerators ϕ_μ^* of degrees $<n$. Given are

$$\phi(x) = \sum_{\lambda=0}^{n} a_\lambda x^\lambda \quad \text{and} \quad f(x) = \sum_{\iota=0}^{mn-1} k_\iota x^\iota,$$

where $a_n \neq 0$; we then set with unknown coefficients

$$\phi_\mu^*(x) = \sum_{\nu=0}^{n-1} \alpha_{\mu\nu} x^\nu,$$

abbreviate the polynomial development of $[\phi(x)]^\rho$ by $\phi^\rho = \sum C(\rho, \tau)x^\tau$, substitute in (10), make rational and find

$$\sum_{\iota=0}^{mn-1} k_\iota x^\iota = \sum_{\rho=0}^{m-1} \overset{*}{\phi}_{m-\rho}\phi^\rho = \sum_{\rho=0}^{m-1}\sum_{\nu=0}^{n-1}\sum_{\tau=0}^{\rho n} \alpha_{m-\rho,\nu} C(\rho, \tau)x^{\nu+\tau}.$$

In the triple sum we first interchange to $\sum_{\nu=0}^{n-1}\sum_{\tau=0}^{(m-1)n}\sum_{\rho=[\tau/n]}^{m-1}$ where the summation over ρ begins at $[\tau/n]$, the "next integer after τ/n":

$$[\tau/n] \text{ integer and } \tau/n \leqq [\tau/n] < \tau/n + 1; \tag{11}$$

furthermore we introduce the independent index of summation $\iota = \nu + \tau$ and obtain $\sum_{\iota=0}^{mn-1}\sum_{(A)}\sum_{\rho=[\tau/n]}^{m-1}$ where the middle sum has to be formed according as

$$\text{(A)} \qquad \begin{cases} \nu + \tau = \iota \\ 0 \leqq \nu \leqq n - 1 \\ 0 \leqq \tau \leqq (m-1)n. \end{cases}$$

Now the comparison of coefficients is possible and yields

$$\sum_{(A)} \sum_{\rho=[\tau/n]}^{m-1} \alpha_{m-\rho,\nu} C(\rho, \tau) = k_\iota, \qquad \iota = 0, \cdots, mn - 1. \tag{12}$$

These mn equations are best ordered according to increasing ι; thus every equation gets the number $\iota+1$. Moreover, in each equation the unknowns $\alpha_{\mu\nu}$ shall be put into the order

$$\alpha_{m,0} \cdots \alpha_{m,n-1}\alpha_{m-1,0} \cdots \alpha_{m-1,n-1} \cdots \alpha_{1,0} \cdots \alpha_{1,n-1};$$

thus every unknown gets the number $\rho n+\nu+1$. If the system is written down in this way its matrix contains only zeros below, but no zeros in, the principal diagonal.

Proof in two steps. Firstly, equation number $\iota+1$ contains only unknowns with numbers $\geqq \iota+1$. In order to write down this equation one must begin with the diophantine equation $\nu+\tau=\iota$, according to (A). The unknowns pertaining to the solutions of the latter have the numbers $\rho n+\nu+1$, with $\rho \geqq [\tau/n]$ according to (11). Hence $\rho n-\tau \geqq 0$, or $\rho n+\nu+1=\rho n+(\iota-\tau)+1 \geqq \iota+1$.

Secondly, for the unknown number $\iota+1$ to occur in equation number $\iota+1$ it is sufficient that among the mentioned solutions of $\nu+\tau=\iota$ there be one pair ν_0, τ_0 with $\tau_0 \equiv 0 \bmod n$, because in this case $\tau_0=\rho_0 n$ with integral ρ_0, or $\nu_0+\rho_0 n=\iota$, or $\rho_0 n+\nu_0+1=\iota+1$. This condition of sufficiency is, however, fulfilled for every ι. The congruence $\tau=\iota-\nu\equiv 0 \bmod n$ has a unique solution ν_0 for arbitrary ι if and only if ν_0 can be taken from a complete system of residues mod n, and just this is guaranteed by (A). More explicitly, it is seen that

$$\begin{array}{ll} \text{to } \iota = 0, \cdots, n-1 & \text{belongs } \tau_0 = 0, \\ \text{to } \iota = n, \cdots, 2n-1 & \text{belongs } \tau_0 = n, \\ \cdots\cdots\cdots\cdots & \\ \text{to } \iota = (m-1)n, \cdots, mn-1 & \text{belongs } \tau_0 = (m-1)n. \end{array}$$

Thus there are m groups of n equations; in each group τ_0 and hence $\rho_0=\tau_0/n$ are the same, so that all equations of a group also begin with the same coefficient $C(\rho_0, \tau_0)=C(\rho_0, \rho_0 n)=a_n^{\rho_0}$.

Under these circumstances the determinant of the system (12) becomes

$$D = (a_n^0)^n (a_n^1)^n \cdots (a_n^{m-1})^n = a_n^{(m-1)mn/2}. \tag{13}$$

Since this cannot vanish, the system (12) has always a unique solution. As above, one can invert the argument. Assuming the validity of (10) the result (13) shows that the decomposition exists and is unique if and only if $a_n \neq 0$.

References

1. Encyclopédie des Sciences Mathématiques, tome 1, vol. 2, pp. 59–66.

2. W. Wirtinger, Bemerkung zur Partialbruchzerlegung, Monats. Math. Phys., vol. 31, 1921, p. 58.

3. L. v. Schrutka, Zu Wirtingers "Bemerkung zur Partialbruchzerlegung," Monats. Math. Phys., vol. 43, 1936, p. 415.

A RELATIONSHIP BETWEEN SEMI-MAGIC SQUARES AND PERMUTATION MATRICES

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A semi-magic square is an n by n matrix A over a field F for which the sum of the n elements in any row or column is a constant $a=S(A)$. It was shown in [1] that the semi-magic squares of order n under the usual matrix addition and multiplication form an algebra $\mathfrak{M}$ over F.

THEOREM. *If* Π *is the set of permutation matrices and* $\{\Pi\}$ *is the algebra generated by* Π *over* F, *then* $\mathfrak{M}=\{\Pi\}$.

Proof. Let $\mathfrak{R}$ be the set of matrices of the form aI where a is in F and let $\mathfrak{N}$ be the set of matrices for which $S(A)=0$. It was shown in [1] that the matrices

$$N_{ij} = \begin{array}{c c} & \begin{array}{ccccc} & & & j & \end{array} \\ i & \begin{bmatrix} 1 & 0\cdots 0 & -1 & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & \\ -1 & 0\cdots 0 & 1 & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots 0 \end{bmatrix} \end{array}$$

form a basis of $\mathfrak{N}$ over F. Considering $\mathfrak{M}$ as a vector space over F, $\mathfrak{M}=\mathfrak{R}+\mathfrak{N}$: if A is in $\mathfrak{M}$ where $S(A)=a$, then $A=aI+(A-aI)$ and $A-aI$ is obviously in $\mathfrak{N}$.

Now suppose P is in Π, then $S(P)=1$. Thus P is in $\mathfrak{M}$ and therefore $\Pi\subset\mathfrak{M}$. Since $\mathfrak{M}$ is an algebra the sums and products of any matrices in $\mathfrak{M}$ are again in $\mathfrak{M}$, and since $\Pi\subset\mathfrak{M}$, $\{\Pi\}\subset\mathfrak{M}$. Conversely to show that $\mathfrak{M}\subset\{\Pi\}$ it will be sufficient to show that the basis matrices N_{ij} and I can be written as sums of products of permutation matrices. Since I is in $\{\Pi\}$, we need only show the N_{ij} are in $\{\Pi\}$. Now $N_{ij}=PN_{nn}Q$ for suitable permutation matrices P and Q *i.e.* P interchanges the nth row of N_{nn} and the ith row of N_{nn}, and Q interchanges the nth column of N_{nn} and the jth column of N_{nn}. However

$$N_{nn} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 1 & \\ 1 & 0 & \cdots 0 & 0 \end{bmatrix} = I - P_1$$

is in $\{\Pi\}$ since I and P_1 are in Π. Therefore $N_{ij}=P(I-P_1)Q=PQ-PP_1Q$ is in $\{\Pi\}$ and hence $\mathfrak{M}\subset\{\Pi\}$.

Reference

1. L. M. Weiner, The algebra of semi-magic squares, this MONTHLY, vol. 62, 1955, pp. 237–239.

PARTIAL ORDER AND THE SUCCESSOR FUNCTION

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The successor function; *i.e.*, the function which assigns to each element of a well-ordered set the "next element", is usually defined for just well-ordered sets. The purpose here is to define this function for partially ordered sets* and to prove the well-orderedness of these sets assuming a modified mathematical induction principle. For this we need:

DEFINITION 1. *If S is a partially ordered set and $s\in S$, then the (immediate) successor of s is an element of S which properly includes s and is included in all elements which properly include s, if such exists, or is s in case s is a maximal element of S. "s'" will denote the successor of s whenever it exists.*

DEFINITION 2. *A partially ordered set S with a least element satisfies the mathematical induction principle if and only if every subset S_1 of S, containing the least element of S and containing the successor (whenever it exists) of every element of S_1, is equal to S.*

THEOREM. *If S is a partially ordered set with a least element s_0 and satisfies the mathematical induction principle, then S is well-ordered.*

Proof. Assume the hypothesis, and also that there is a nonnull subset R of S without a least element. Let S_1 be the set of all elements of S which "precede" R; *i.e.*, which are properly included in each element of R. ($s_0\in S_1$ since R has no least element and s_0 is the least element in S.)

The proof is by cases:

Case 1. Each element of S_1 has a successor in S. $s_0\in S_1$, and if $s\in S_1$ then $s'\in S_1$ (otherwise s' would be the least element of R). Hence $S_1=S$ and so R is null, contrary to assumption.

Case 2. There exists an element x of S_1 which has no successor in S. By letting $S_1'=S/x\cup\{x\}$ (where S/x is the set of all elements of S properly included in x), then $s_0\in S_1'$, and if $s\in S_1'$ then $s'\in S_1'$ (whenever s' exists). Therefore $S_1'=S$ and R is null, a contradiction.

The two cases, being exhaustive, establish the theorem.

* "Partially ordered set," "least element," "maximal element," and "well-ordered set" are defined in G. Birkhoff, Lattice Theory, American Mathematical Society, Colloquium Publications, vol. 25 Revised ed., 1949. See pp. 1, 7, 7. and 32, respectively.

ON A GEOMETRICAL THEOREM

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Given a triangle $A_1A_2A_3$ and a line through any point P in the plane, meeting the sides of the triangle at X_1, X_2, X_3 respectively. Let A_iP $(i=1, 2, 3)$ meet the circumcircle at R_i. The three lines X_iR_i are concurrent at a point E on the circumcircle [1, p. 238]. We propose to show how to associate a definite point E with a definite line on P and *vice versa*.

We shall use the circumcircle $A_1A_2A_3$ as the base circle and let the coordinates of the vertices A_i $(i=1, 2, 3)$ be turns t_i, *i.e.* complex numbers with unit modulus. Let T be any other point on the circumcircle with coordinate T, and P a fixed point not on the circle with cordinate p. We choose as the line on P, a line perpendicular to the image line of T [2, p. 422].

The equation of this line is $Tx+\sigma_3\bar{x}=pT+\sigma_3\bar{p}$, where $\sigma_1=t_1+t_2+t_3$, $\sigma_2=t_2t_3+t_3t_1+t_1t_2$, $\sigma_3=t_1t_2t_3$. The coordinates of the points X_i and R_i are readily found to be $(T-t_i)x_i=pT+\sigma_3\bar{p}-(t_j+t_k)\sigma_3$, $(1-\bar{p}t_i)r_i=p-t_i$. The three lines cut the circumcircle of $A_1A_2A_3$ at

$$E=\frac{\sigma_3[p-\sigma_1+\sigma_2\bar{p}-\sigma_3\bar{p}^2+(1-p\bar{p})T]}{T(p^2-\sigma_1p+\sigma_2-\sigma_3\bar{p})-(1-p\bar{p})\sigma_3}=\frac{q-T}{1-\bar{q}T},$$

where q is the coordinate of Q, the isogonal conjugate point of P as to the triangle $A_1A_2A_3$ [2, p. 425]. From the form of the coordinate of E, it is evident that the points T, Q, E are collinear.

Hence, given a triangle $A_1A_2A_3$ and a point P, to determine what transversal through P will produce a chosen E on the circumcircle we proceed as follows. Join E to Q, the isogonal conjugate point of P as to $A_1A_2A_3$, produce EQ cutting the circle $A_1A_2A_3$ at T. Draw the image line of T, and the line through P perpendicular to this image line is the required transversal.

On the other hand given the transversal through P to determine the point E we proceed thus. Draw a line through H, the orthocenter of $A_1A_2A_3$, perpendicular to the transversal. This line will be the line of images of some point T on the circumcircle, and T can be located [2, p. 423]. The line joining T to Q, the isogonal conjugate of P as to $A_1A_2A_3$, cuts the circumcircle at the required point E.

References

1. R. A. Johnson, Modern Geometry, Cambridge, Mass., 1929.
2. J. R. Musselman, On the line of images, this MONTHLY, vol. 45, 1938, pp. 421–430.

AN APPLICATION OF THE GAUSS MULTIPLICATION THEOREM

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In a previous note (this MONTHLY, vol. 62, 1955, p. 120), S. K. L. Rao derives Legendre's duplication formula by means of Mellin transforms. On attempting to extend this method to a proof of the Gauss multiplication theorem, one is led to the evaluation of some interesting integrals.

The Gauss multiplication theorem [1] states that

$$\Gamma(s)\Gamma(s+1/n)\cdots\Gamma(s+(n-1)/n) = (2\pi)^{(n-1)/2}n^{(1/2)-ns}\Gamma(ns). \tag{1}$$

For $n=3$, this reduces to the triplication formula

$$\Gamma(s)\Gamma(s+1/3)\Gamma(s+2/3) = (2\pi)3^{1/2-3s}\Gamma(3s). \tag{2}$$

If we now let

$$F_1(x) = e^{-x}, \qquad F_2(x) = x^{1/3}e^{-x}, \quad \text{and} \quad F_3(x) = x^{2/3}e^{-x}, \tag{3}$$

then the corresponding Mellin transforms are

$$MF_1(x) = \Gamma(s), \quad MF_2(x) = \Gamma(s+1/3), \quad \text{and} \quad MF_3(x) = \Gamma(s+2/3), \tag{4}$$

where by definition $MF(x)=\int_0^\infty F(x)x^{s-1}dx$. By the Faltung theorem [2]

$$MF(x)\cdot MG(x) = M\int_0^\infty F(x/u)G(u)u^{-1}du. \tag{5}$$

Consequently,

$$MF_1(x)\cdot MF_2(x) = M\int_0^\infty e^{-x/u-u}u^{-2/3}du = MG(x). \tag{6}$$

We now evaluate the following generalization of $G(x)$:

$$I(x) = \int_0^\infty e^{-[t^n+xt^{-n}]}dt, \qquad (\text{for } n=3,\ G(x)=3I(x)). \tag{7}$$

By integration by parts and differentiation through the integral sign, it follows that

$$\left[D^2+\frac{n-1}{nx}D-\frac{1}{x}\right]I(x) = 0. \tag{8}$$

The solution to this latter equation [3] is given by

$$I(x) = x^{1/2n}[AI_{1/n}(2\sqrt{x}) + BK_{1/n}(2\sqrt{x})]. \tag{9}$$

Since [4]

$$(10)\qquad I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \qquad K_p(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}, \quad \text{and} \quad \lim_{x\to 0} \frac{K_p(x)}{2^{p-1}\Gamma(p)x^{-p}} = 1,$$

$$I(x) = (2/n)x^{1/2n}K_{1/n}(2\sqrt{x}).$$

For $n=2$, we get a well known result [5] $I(x)=\sqrt{\pi}/2e^{-2\sqrt{x}}$.

By application of the Faltung theorem again, we get

$$(11)\qquad MF_3(x)\cdot MG(x) = 2Mx^{2/3}\int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-3/2}du.$$

Since $MF_3(x)\cdot MG(x)=(2\pi)3^{1/2-3s}\Gamma(3s)$ (by (2) and (5)) $=M((2\pi/\sqrt{3})e^{-3x^{1/3}})$, it follows that

$$(12)\qquad \int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-3/2}du = (\pi/\sqrt{3})x^{-2/3}e^{-3x^{1/3}}.$$

Similarly, by different pairings of the functions F_1, F_2, F_3, we obtain

$$(13)\qquad \int_0^\infty e^{-x/u}K_{2/3}(2\sqrt{u})u^{-1}du = (\pi/\sqrt{3})x^{-1/3}e^{-3x^{1/3}},$$

$$(14)\qquad \int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-1/2}du = (\pi/\sqrt{3})e^{-3x^{1/3}}.$$

[(12) follows directly from (14) by differentiating through the integral sign]. These last three integrals give rise to the following Laplace transforms:

$$(15)\qquad p\int_0^\infty e^{-pt}K_{1/3}(2t^{-1/2})t^{-1/2}dt = (\pi/\sqrt{3})p^{1/3}e^{-3p^{1/3}},$$

$$(16)\qquad p\int_0^\infty e^{-pt}K_{2/3}(2t^{-1/2})t^{-1}dt = (\pi/\sqrt{3})p^{2/3}e^{-3p^{1/3}},$$

$$(17)\qquad p\int_0^\infty e^{-pt}K_{1/3}(2t^{-1/2})t^{-3/2}dt = (\pi/\sqrt{3})pe^{-3p^{1/3}}.$$

Proceeding in a similar manner from the Gauss multiplication theorem for $n=4$, we also find that

$$(18)\qquad \int_0^\infty K_{1/4}(2\sqrt{x/u})K_{1/4}(2\sqrt{u})u^{-1/2}du = (1/8)x^{-1/8}(2\pi)^{3/2}e^{-4x^{1/4}},$$

$$(19)\qquad \int_0^\infty K_{3/4}(2\sqrt{x/u})K_{1/4}(2\sqrt{u})u^{-1}du = (1/8)x^{-3/8}(2\pi)^{3/2}e^{-4x^{1/4}},$$

and

$$\int_0^\infty K_{1/2}(2\sqrt{x/u})K_{1/2}(2\sqrt{u})u^{-3/4}du = (1/8)x^{-1/4}(2\pi)^{3/2}e^{-4x^{1/4}}. \tag{20}$$

(20) corresponds to the known integral $\int_0^\infty e^{-x^2-a^2/x^2}dx=(1/2)e^{-2a}\sqrt{\pi}$, and also to (9) for $n=2$.

References

1. E. T. Whittaker, G. N. Watson, Modern Analysis, New York, 1946, p. 240.
2. I. N. Sneddon, Fourier Transforms, New York, 1951, p. 43.
3. E. Jahnke, F. Emde, Tables of Functions, New York, 1945, p. 146.
4. F. B. Hildebrand, Advanced Calculus for Engineers, New York, 1949, pp. 161–162.
5. B. O. Peirce, A Short Table of Integrals, New York, 1929, p. 63.

CLASSROOM NOTES

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FROM FOURIER SERIES TO FOURIER INTEGRAL

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It is natural to think of the Fourier integral as the extension of the Fourier series to the case where the range is infinite. Heuristic arguments (as in [1], [2], [3], [4]) are sometimes given to show the transition from series to integral. The development that follows is offered as an alternative method of extending the series to the integral. It is most useful for the case when one has already been discussing convergence of the Fourier series, and hence has equation (3) below for the sum of the first N terms of the Fourier series.

Let

$$S = a_0/2 + \sum_1^\infty \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{1}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\, dt, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt. \tag{2}$$

It is well known (see, for example [1], p. 386) that the sum of the first N

terms of the above series may be written

$$S_N = \frac{1}{2L}\int_{-L}^{L} f(t)\,\frac{\sin\,[(N+1/2)(t-x)\pi/L]}{\sin\,[(t-x)\pi/2L]}\,dt. \tag{3}$$

Hence

$$S_N = \frac{1}{2L}\int_{-L}^{L}\frac{f(t)(t-x)}{\sin\,[(t-x)\pi/2L]}\int_0^{(N+1/2)\pi/L}\cos\,[(t-x)q]dqdt \tag{4}$$

$$= \frac{1}{\pi}\int_0^{(N+1/2)\pi/L}\int_{-L}^{L} f(t)\,\frac{(t-x)\pi/2L}{\sin\,[(t-x)\pi/2L]}\cos\,[(t-x)q]dtdq. \tag{5}$$

This is now an expression for the sum of N terms of a Fourier series for $f(x)$ in the interval $(-L, L)$. It will in general fit the function at just N points, with the average spacing between the abscissae of these points being $2L/N$. To improve the fit, one must let $N\to\infty$. If L is fixed, and $f(x)$ satisfies the Dirichlet conditions, then S_N will converge to $[f(x+0)+f(x-0)]/2$. However, if L is increased as N is increased, then it is necessary to relate the rates at which L and N are increased so as to have $2L/N\to 0$ as L and N are increased. This will be the case, for example, if $L=N^k$, where $0<k<1$. Upon setting $L=N^k$ in (5), and letting $N\to\infty$, one obtains

$$S = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos q(t-x)dtdq, \tag{6}$$

which is the Fourier integral.

References

1. I. S. Sokolnikoff, Advanced Calculus, New York 1939, p. 410.
2. R. V. Churchill, Fourier Series and Boundary Value Problems, New York, 1941, p. 88.
3. I. N. Sneddon, Fourier Transforms, New York, 1951, p. 8.
4.. A. Bronwell, Advanced Mathematics in Physics and Engineering, New York, 1953, p. 37.

AN INTERESTING FOURTH-ORDER DIFFERENTIAL SYSTEM

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Although the theory of linear differential equations with constant coefficients is well known, most of the examples and problems given in texts have characteristic equations which are either first- or second-order or are easily factorable. In this note an important subset of the set of all fourth-order differential systems will be described. The solution of members of this subset can be obtained by the solution of at most two quadratic equations instead of the usual quartic equation.

The most general linear homogeneous fourth-order system of differential equations with constant coefficients may be written in the form

A DEVELOPMENT OF LOGARITHMS USING THE FUNCTION CONCEPT

C. L. SEEBECK, JR. and JOHN W. JEWETT, University of Alabama and University of Georgia

The traditional approach to the logarithmic function in freshman college or high school courses is that of the inverse of the exponential function. The exponential function itself is seldom defined for the entire real line and is a difficult concept for immature students for all values except the positive integer. Any experienced teacher has encountered the difficulties inherent in obtaining real understanding with this approach. In keeping with the modern trend to devote more time to the modern concept of function in both college freshman and high school courses, the authors present a development of the logarithmic function which they believe to be direct and simple, and one which serves as an excellent example of the function concept.

DEFINITION. *Let R^+ denote the set of positive real numbers. The logarithm function is a function* log: $R^+ \Rightarrow R$ *satisfying the following assumptions for all a and b in R^+.*

$$\log(a\cdot b) = \log(a) + \log(b), \tag{1}$$

$$\log(10) = 1, \tag{2}$$

$$\textit{If } a > 1, \textit{ then } \log(a) > 0. \tag{3}$$

We insist on retaining the parentheses in $\log(a)$ to emphasize the fact that $\log(a)$ is the value of the logarithm function at a.

Next we prove, or ask the students to prove as exercises the following simple theorems:

THEOREM 1. $\log(1)=0$.

Proof. $\log(1)=\log(1\cdot 1)=\log(1)+\log(1)=2\log(1)$. But the only number that is twice itself is the number zero.

THEOREM 2. $\log(1/a)=-\log(a)$.

Proof. $\log(1)=\log(a/a)=\log(a\cdot 1/a)=\log(a)+\log(1/a)$.
But $\log(1)=0$, so $\log(1/a)$ must be the negative of $\log(a)$.

THEOREM 3. $\log(a/b)=\log(a)-\log(b)$.

Proof. $\log(a/b)=\log(a\cdot 1/b)=\log(a)+\log(1/b)=\log(a)-\log(b)$.

THEOREM 4. *If m is an integer, then* $\log(a^m)=m\log(a)$.

Proof. $\log(a^m)=\log(a\cdot a\cdot\cdot\cdot a)$ with m factors. Using assumption (1) $\log(a^m)=\log(a)+\log(a)+\cdot\cdot\cdot+\log(a)$ with m terms. Hence $\log(a^m)=m\log(a)$.

COROLLARY. *If m is an integer,* $\log(10^m)=m$.

THEOREM 5. *If n is an integer,* $\log(a^{1/n})=(1/n)\log(a)$.

Proof. Let $N=a^{1/n}$. Then $N^n=a$ and n log $(N)=$ log (a). Hence log $N=(1/n)$ log (a).

COROLLARY 1. *If m and n are integers,* log $(a^{m/n})=(m/n)$ log (a).

COROLLARY 2. *If m and n are integers,* log $(10^{m/n})=m/n$.

THEOREM 6. *If $a<b$, then* log $(a)<$ log(b).

Proof. If $a<b$ then $b=ac$, where $c>1$. Hence log $(b)=$ log $(a)+$ log (c). But by Assumption (3), log $(c)>0$. Hence log $(a)<$ log (b).

COROLLARY. *If $a<1$, then* log $(a)<0$.

We now show the student that any positive number can be written as $10^n \cdot p$, where n is an integer and p a number between one and ten. Moreover, since log $(10^n \cdot p)=n+$ log p, the problem of finding the values of the logarithm function is reduced to the problem of finding the values between one and ten. We then pass out an ordinary four place logarithm table which we label, "Values of log (a) for a between one and ten."

Interpolation is justified as a reasonable approximation for $a<x<b$, by noting that log $(a)<$ log $(x)<$ log (b), for $b-a$ arbitrarily small. We call n the "whole number part" and log (p) the "fractional part" of the logarithm. We proceed to the usual computational work using logarithms.

The proof of the uniqueness of the logarithmic function given below is beyond the scope of an elementary course.

THEOREM. *The only function satisfying assumptions* (1) *to* (3) *above is the usual logarithm function to the base* 10.

Proof. We need a result quoted in [1, p. 16] which, when specialized to the real line, reads as follows:

Let f and g be real valued monotone functions defined on R^+, let X be a subset of R^+ on which f and g agree and let Y_0 be $f(X)$. A sufficient condition that $f=g$ is that Y_0 intersect every set of the form $(y: u<y<v)$, where u and v are real numbers.

Since we have let *log* denote a function satisfying our assumptions (1) and (3), let the ordinary logarithm function to the base ten be written *Log*. Let X_0 denote the set of all positive real numbers which can be written as rational powers of ten.

By the second corollary to Theorem 5, log $(a)=$ Log (a) on X. But this corollary also implies that log maps X onto the set of all rational numbers. The set of all rational numbers intersects every open interval of real numbers. Hence the result quoted from [1] implies that log $(a)=$ Log (a) for all positive real numbers.

Reference

1. J. L. Kelley, General Topology, New York, 1955.

THREE THEOREMS ON PERMUTATIONS

A. A. MULLIN, Massachusetts Institute of Technology

As a corollary to the binomial theorem it is often proved that

$$\sum_{r=0}^{n} {}_nC_r \equiv 2^n, \quad \text{where} \quad {}_nC_r \equiv \frac{n!}{(n-r)!r!}, \tag{1}$$

and where n is a nonnegative integer $(0! \equiv 1)$.

Also, if $\Psi(n)$ is defined by

$$\Psi(n) \equiv \sum_{r=0}^{n} {}_nC_r, \tag{2}$$

and if $\Psi(0) \equiv 1$, then a canonic generator for Ψ is

$$\Psi(n+1) \equiv 2\cdot\Psi(n). \tag{3}$$

Now, consider the function defined by

$$\Phi(n) \equiv \sum_{r=0}^{n} {}_nP_r, \quad \text{where} \quad {}_nP_r \equiv \frac{n!}{(n-r)!}, \tag{4}$$

and where n is a nonnegative integer $(0! \equiv 1)$, and $\Phi(0) \equiv 1$.

THEOREM 1. *A canonic generator for* Φ *is*

$$\Phi(n+1) \equiv \{(n+1)\Phi(n)\} + 1. \tag{5}$$

Proof. Consider the right-hand side (*rhs*) of (5),

$$\begin{aligned} &\{(n+1)[1 + {}_nP_1 + \cdots + {}_nP_n)\} + 1 \\ &= \{(n+1) + (n+1){}_nP_1 + \cdots + (n+1){}_nP_n\} + 1 \\ &= \{{}_{n+1}P_1 + {}_{n+1}P_2 + \cdots + {}_{n+1}P_{n+1}\} + {}_{n+1}P_0; \end{aligned}$$

thus, $rhs \equiv \sum_{r=0}^{n+1} {}_nP_r$; but, by definition, the left-hand side (*lhs*) of (5) is $\sum_{r=0}^{n+1} {}_nP_r$ or, $lhs = rhs$.

THEOREM 2. *An asymptotic expression for* $\Phi(n)$ *is*

$$\Phi(n) \sim (n!)\cdot e, \quad \textit{where} \quad \begin{array}{c} n \gg 1 \\ e = 2.718 \cdots. \end{array} \tag{6}$$

Proof. $\Phi(n) \equiv {}_nP_0 + {}_nP_1 + \cdots + {}_nP_n$ factor out ${}_nP_n \equiv n!$ from the *rhs*, yielding

$$\Phi(n) \equiv n!\left[\frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + 1 + 1\right],$$

but for $n \gg 1$ the bracketed expression approaches e, or $\Phi(n) \sim (n!)\cdot e$, where $e = 2.718 \cdots$.

Note. It should not be too surprising that (6) is the asymptotic form of $\Phi(n)$ in view of the form of (5), which implies a factorial function when n is large.

LEMMA. *An asymptotic expression for $n!$ is*

$$n! \sim n^n e^{-n}\sqrt{2\pi n}. \tag{7}$$ (*Stirling's Formula*)

THEOREM 3. *An asymptotic expression for $\Phi(n)$ is*

$$\Phi(n) \sim n^n e^{1-n}\sqrt{2\pi n}. \tag{8}$$

Proof. Substitute Lemma into Theorem 2.
Tabulations:

$$\begin{aligned}
\Phi(0) &= 1,\\
\Phi(1) &= 2,\\
\Phi(2) &= 5,\\
\Phi(3) &= 16,\\
\Phi(4) &= 65,\\
\Phi(5) &= 326,\\
\Phi(6) &= 1957,\\
\Phi(10) &= 9.8641 \times 10^6.
\end{aligned}$$

Using Theorem 3 and a slide rule yields $\Phi(10) = 9.79 \times 10^6$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1286. *Proposed by F. S. Stancliff, Springfield, Ohio*

Establish the identity

$$\sum_{r=0}^{n-2} (2^r - 1)^2 + [3(2^{n-1}) - 1]^2 = \sum_{r=2}^{n} (2^r + 1)^2 + (2^n - 4)^2,$$

thus obtaining a general method for finding two equal sums of n distinct squares.

E 1287. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the radical center P of the circles inscribed in the equilateral triangles BCA', CAB', ABC' constructed exteriorly (or interiorly) on the sides of a triangle ABC having centroid G divides the distance from G to the point of concurrence Q of the lines AA', BB', CC' in the ratio $GP/GQ=1/4$.

E 1288. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this MONTHLY [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

E 1289. *Proposed by Marvin Shinbrot, National Advisory Committee for Aeronautics, Moffett Field, California*

Show that the Fermat equation $x^n+y^n=z^n$ has no nontrivial solution in integers for $n>2$ if $z<2^{1/n}/(2^{1/n}-1)$.

E 1290. *Proposed by R. E. Shafer, University of California Radiation Laboratory, Livermore, California*

If $|n \tan^{-1} x| \leqq \pi/2$, show that

$$n \tan^{-1} x = \tan^{-1} \frac{\operatorname{Im} (1 + ix)^n}{\operatorname{Re} (1 + ix)^n},$$

where $\operatorname{Im} f(z)$ denotes the imaginary part of $f(z)$ and $\operatorname{Re} f(z)$ denotes the real part of $f(z)$.

SOLUTIONS

Discussion of an Approximate Trisection

E 1256 [1957, 197]. *Proposed by W. B. Andreasen, Lockheed Aircraft Corporation*

Discuss the error involved in the following approximate trisection of a circular arc AB. On chord AB locate C such that $BC=BA/3$ and D such that $CD=7AB/6$. With D as center and DC as radius describe an arc to cut arc AB in the approximate trisection point E.

Solution by D. C. B. Marsh, Colorado School of Mines. Establish coordinate axes with origin at the circle's center and axes directed so that chord AB is parallel to and above the positive x-axis. Choose the scale so that the radius is 3. Label the points as: $B(3 \cos \theta, 3 \sin \theta)$, $A(-3 \cos \theta, 3 \sin \theta)$, $C(\cos \theta, 3 \sin \theta)$, and $D(8 \cos \theta, 3 \sin \theta)$. If we assign E the coordinates $(3 \cos \phi, 3 \sin \phi)$, then E on circle $D(C)$ implies

$$(3 \cos \phi - 8 \cos \theta)^2 + (3 \sin \phi - 3 \sin \theta)^2 = (7 \cos \theta)^2,$$

which reduces to

$$3 + \cos^2 \theta = 8 \cos \theta \cos \phi + 3 \sin \theta \sin \phi,$$

and may be solved for ϕ to give

$$\phi = \operatorname{Arctan}\left(\frac{3 \tan \theta}{8}\right) \pm \operatorname{Arccos}\left(\frac{3 + \cos^2 \theta}{\sqrt{9 + 55 \cos^2 \theta}}\right),$$

where $90° \geqq \theta \geqq 0°$, $90° \geqq \phi \geqq -90°$; the plus sign is used for the approximate trisection of the minor arc AB and the minus sign for the major arc. The relative error, valid for any scaled axis, is

$$(3\phi - \theta - 180°)/(180° - 2\theta) \quad \text{and} \quad (180° - \theta + 3\phi)/(180° + 2\theta)$$

in the respective cases.

The method is exact (by setting error equal to zero) only in the case where AB is a diameter ($\theta = 0°$), and trivially for the minor arc when AB is of length zero ($\theta = 90°$). For AB decreasing in length, the relative error in the major arc's trisection increases without bounds, being less than 0.01 for $\theta < 10°$ (roughly), less than 0.10 up to 40°, and 0.25 near 60°. The method is quite good for the minor arc, however, there being a maximum relative error of only about 0.007 when θ is 25° (or the given arc is around 130°). Following is a rough table:

θ	0°	5°	10°	15°	20°	25°	30°	35°	40°
arc	180°	170°	160°	150°	140°	130°	120°	110°	100°
% error	0.00	0.29	0.50	0.62	0.69	0.71	0.69	0.65	0.59

45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
0.51	0.43	0.35	0.27	0.19	0.13	0.07	0.03	0.01	0.00

Also solved by Julian Braun and (partially) by C. S. Ogilvy. Late solution by D. A. Breault.

A Property of the Complete Quadrilateral

E 1257 [1957, 197]. *Proposed by N. A. Court, University of Oklahoma*

(1) The medial triangle of each of the four triangles formed by the sides of a complete quadrilateral (q) taken three at a time is homological to the diagonal trilateral of (q).

(2) The four axes of the four homologies coincide.

Solution by the proposer. Let $(q) = abcd$ be the given quadrilateral. Its pairs of opposite vertices are the points $A = bc$, $P = ad$; $B = ca$, $Q = bd$; $C = ab$, $R = cd$, and the points $D = (BQ, CR)$, $E = (CR, AP)$, $F = (AP, BQ)$ are the vertices of the diagonal trilateral DEF of (q).

The sides $B'C'$, $C'A'$, $A'B'$ of the medial triangle $A'B'C'$ of the triangle $abc = ABC$ pass, respectively, through the midpoints U, V, W of the cevians

AP, BQ, CR of ABC. Now the latter three lines are the diagonals of (q), whence the points U, V, W lie on the Newton line n of (q). Thus the Newton line n is the axis of perspectivity of the medial triangle $A'B'C'$ of abc and the diagonal trilateral DEF of (q).

Similar results may be established for triangles bcd, cda, dab. Both parts of the proposition are thus proved.

Also solved by Josef Langr (synthetically as above) and D. C. B. Marsh (analytically).

An Infinite Radical

E 1258 [1957, 197]. *Proposed by Aaron Herschfeld, Canisius College*

Prove that a necessary and sufficient condition for the rationality of

$$R = \sqrt[3]{a + \sqrt[3]{a + \cdots}},$$

where a is a positive integer, is that $a = N(N+1)(N+2)$, the product of three consecutive integers. In that case find R.

Solution by D. A. Freedman, McGill University. Define $R_1 = \sqrt[3]{a}$, $R_n = \sqrt[3]{a+R_{n-1}}$. Now $R_2 > R_1$, and $R_k^3 - R_{k-1}^3 = R_{k-1} - R_{k-2}$, so that by induction $\{R_n\}$ is monotone increasing. Moreover, $R_1 < 1 + \sqrt[3]{a}$, and $R_{k-1} < 1 + \sqrt[3]{a}$ implies that $R_k^3 < a + 1 + \sqrt[3]{a} < (1+\sqrt[3]{a})^3$, so that by induction $\{R_n\}$ is bounded. It follows that $\{R_n\}$ converges to a limit R. But then $R^3 - R - a = 0$. If R is rational and a integral, then R is integral, and $a = (R-1)R(R+1)$, the product of three consecutive integers. Hence the condition is necessary. It is also sufficient, since $R = N+1$ satisfies the equation $R^3 - R - N(N+1)(N+2) = 0$, and, as it is the only real root, it is the value of the radical.

Also solved by Robert Bart, Julian Braun and J. R. Holdsworth (jointly), R. E. Briney and D. A. Trumpler (jointly), C. N. Campopiano, Germain Casal and Ruben Perelis (jointly), W. V. Gamzon, A. M. Glicksman, Michael Goldberg, Bernard Greenspan, Cornelius Groenewoud, Emil Grosswald, H. J. Hauer, J. M. Howell, A. R. Hyde, I. M. Isaacs, Sidney Kravitz, Joseph Lewittes and Marshall Luban (jointly), Shen Lin, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, J. B. Muskat, C. S. Ogilvy, Donald Passman, C. F. Pinzka, Montfort Plebstnoch, L. A. Ringenberg, T. A. Porsching, Michael Rosen, Azriel Rosenfeld, P. T. Schaefer, R. E. Shafer, Lawrence Shepp, D. D. Strebe, Chih-yi Wang, L. K. Williams, and the proposer. Late solution by D. A. Breault.

Several solvers generalized the problem by showing that a necessary and sufficient condition for the rationality of

$$R = \sqrt[n]{a + \sqrt[n]{a + \cdots}},$$

where a is a positive integer, is that $a = N(N^{n-1} - 1)$.

Ogilvy pointed out that this problem was essentially presented by C. W. Trigg in his solution of E 874 [1950, 186].

For the convergence of the sequence $\{R_n\}$, and of allied sequences, see Aaron Herschfeld, *On infinite radicals*, this MONTHLY [1935, 419–429].

Marsh and the proposer each pointed out that Cardan's formula yields the explicit expression

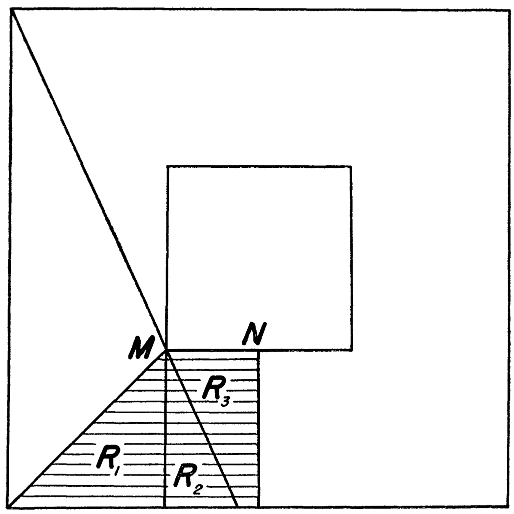
$$R = [a/2 + \sqrt{a^2/4 - 1/27}]^{1/3} + [a/2 - \sqrt{a^2/4 - 1/27}]^{1/3}.$$

The Chance that Two People Can See Each Other

E 1259 [1957, 197]. *Proposed by C. D. Olds, San Jose State College*

A building a feet square has a walk $b<a$ feet wide around it. Two persons are on the walk; find the chance that they can see each other.

Solution by A. R. Hyde, West Hartford, Connecticut, and D. C. B. Marsh, Colorado School of Mines. The figure shows a typical region R of the walk in which a person may be standing. Let convenient coordinate axes be taken.



[Hyde took origin at N and positive x-axis along NM; Marsh took origin at M and positive x-axis along MN.] Then at each point (x, y) of R there is a certain area, say $f(x, y)$, of the surrounding walk which is visible. The probability that a second person on the walk can be seen by the first person is then given by

$$p = \left[\int\int_R f(x, y)dxdy \right] \Big/ TR,$$

where T is the total area of the walk and R represents the area of region R.

Since $f(x, y)$ has discontinuities, met when the first person's position passes from one of the subregions R_1, R_2, R_3 into another, we calculate p by

$$p = \left[\sum_{i=1}^{3} \int\int_{R_i} f(x, y)dxdy \right] \Big/ TR.$$

A straightforward but tedious calculation yields

$$p = [(2 + a/b)^2 + \ln (1 + a/b)]/4(1 + a/b)^2. \tag{1}$$

The above calculation of p holds for $b \leqq a$. It may be shown [as did Hyde] that for $b > a$ two cases arise, according as $b \leqq$ or $> (1+\sqrt{5})a/2$. A figure shows that in the first of these cases R must be divided into four subregions, and in the second case into five subregions. Nevertheless, in both of these cases one again obtains the result (1). Hence (1) gives the desired probability for all relations of a to b. The limits of p are $1/4$ as $b/a \to 0$ and 1 as $b/a \to \infty$, giving two obvious checks on the formula for p. At $b=a$, $p=(9+\ln 2)/16=0.60582$, approximately.

Also solved by Julian Braun.

An Improper Integral

E 1260 [1957, 197]. *Proposed by Viktors Linis, University of Ottawa*

Evaluate $I=\int_0^{\pi/2} \cot\theta \ln \sec\theta d\theta$.

Solution by Emil Grosswald, University of Pennsylvania. The improper integral exists, as the integrand approaches zero at both limits of integration and is continuous over the open interval. Setting $x=\cos\theta$,

$$I = -\int_0^1 [(x\ln x)/(1-x^2)]dx = -\int_0^1 \left(\sum_{n=0}^{\infty} x^{2n+1}\right) \ln x dx.$$

Inversion of summation and integration is easily justified by the uniformity of convergence, so that

$$I = -\sum_{n=0}^{\infty}\left(\int_0^1 x^{2n+1}\ln x dx\right) = \sum_{n=0}^{\infty}(2n+2)^{-2} = (1/4)\sum_{m=1}^{\infty} m^{-2} \doteq \pi^2/24.$$

Also solved by C. A. Church, Jr., F. J. Duarte, C. B. Germain, Bernard Greenspan, Cornelius Groenewoud, G. A. Harris, Jr., A. R. Hyde, I. M. Isaacs, Walter James, Seymour Kass, M. S. Klamkin, Marshall Luban, T. G. McLaughlin, D. C. B. Marsh, G. B. Parrish, Donald Passman, George Richardson and Dale Woods (jointly), L. A. Ringenberg, D. A. Robinson, R. E. Shafer, Lawrence Shepp, Arnold Singer, M. B. Stewart, D. D. Strebe, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by J. L. Alperin.

The integral $I=\int_1^0[(x \ln x)/(1-x^2)]dx$ may be broken up by partial fractions to yield

$$I = (1/2)\int_0^1 [(\ln x)/(1+x)]dx - (1/2)\int_0^1 [(\ln x)/(1-x)]dx.$$

These latter integrals may be evaluated by formulas 509 and 510 of Peirce's *A Short Table of Integrals.*

Other useful substitutions for the transformation of the given integral are $x=\cos^2\theta$ and $x=\ln\sec\theta$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4758. *Proposed by Leonard Carlitz, Duke University*

Let $x_1, \cdots, x_n$ denote the roots of

$$(x - a + \sqrt{1 - 2ax + x^2})^n + (x - a - \sqrt{1 - 2ax + x^2})^n = 2(a^2 - 1)^{n/2}.$$

Evaluate the power sums

$$\sigma_k = x_1^k + \cdots + x_n^k \qquad (0 \leqq k < n).$$

4759. *Proposed by T. K. Pan, University of Oklahoma*

The tangential component of the absolute curvature vector of a unit vector field along a curve on a surface in ordinary space is known as angular spread of the field along the curve and is a generalization of geodesic curvature of a curve. Derive formulas for angular spread generalizing those formulas for geodesic curvature due to Beltrami and Bonnet. (See, *e.g.*, Eisenhart, *Differential Geometry*, 1909, pp. 136, 183.)

4760. *Proposed by G. U. Brauer, University of Minnesota*

Let $f(x)$ be a real function such that $f(0)=0$, $f(1)\neq 0$, $\lim_{n\to\infty} f(n)=0$, where n takes on positive integral values. Construct a sequence of integers $\{a_n\}$, $a_n \to \infty$, and a compact set C such that $f(a_n x)\to 0$ nonuniformly for x in C.

4761. *Proposed by Alfredo Jones, Institute of Mathematics and Statistics, Montevideo, Uruguay*

In a ring with identity and with proper ideals, there always exist maximal ideals. Is the statement true for rings with a nontrivial multiplication and with no identity?

4762. *Proposed by Samuel Beatty, University of Toronto*

Define γ_i by the relations

$$\sum_{r=1}^{n} \frac{\log^i r}{r} = \int_1^n \frac{\log^i x}{x}\, dx + \gamma_i + o(1).$$

Show that

$$\sum_{r=1}^{\infty}(-1)^{r+1}\frac{\log^p r}{r}=\frac{\lambda^{p+1}}{p+1}-\left[\lambda^p\gamma+\binom{p}{1}\lambda^{p-1}\gamma_1+\cdots+\binom{p}{p-1}\lambda\gamma_{p-1}\right],$$

($p=2, 3, \cdots$), where $\lambda=\log 2$ and γ is Euler's constant. This is a generalization of no. 4592 [1954, 350].

SOLUTIONS

Prime Numbers

4711 [1956, 669]. *Proposed by Morgan Ward, California Institute of Technology*

If $\pi(n)$ is the number of prime numbers not greater than n, show that

$$\pi(n)<\frac{5}{4}\frac{n}{\log n}$$

for every composite number n.

Remarks by J. B. Rosser, Cornell University. Lowell Schoenfeld called this inequality to my attention a couple of years ago. I don't know any elegant proof, but one can get a quite direct and straightforward proof by using the results and methods of my paper *Explicit bounds for some functions of prime numbers*, (Amer. J. Math., vol. 63, 1941, pp. 211–232.) By Theorem 25, p. 212, one concludes that the inequality holds for $e^{10}\leqq n\leqq e^{100}$. Finally, by a modification of the method explained on p. 217, one needs only a fairly short computation to verify that it holds for $1<n\leqq e^{10}$, except for $n=113$ (which is not composite.)

Similar results are in a paper by Schoenfeld and Rosser which should appear in the near future.

Related Inequalities

4712 [1956, 669]. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that if $x_i\geqq 0$ ($i=1, \cdots, n$) and $\sum_{i=0}^{n} 1/(1+x_i)\leqq 1$, then $\sum_{i=1}^{n} 2^{-x_i}\leqq 1$.

Solution by B. L. Foster and R. R. Phelps, University of Washington, Seattle. The result is obvious for $n=1$; while for $n=2$, $x_1, x_2\geqq 0$ and $(1+x_1)^{-1}+(1+x_2)^{-1}\leqq 1$ imply $x_1x_2\geqq 1$ and $x_1, x_2>0$. But then $2^{-x_1}+2^{-x_2}\leqq 2^{-x_1}+2^{-1/x_1}$, and the latter is no greater than 1 by problem E1063 [1953, 714].

The proof proceeds by induction. $\sum_{i=1}^{n}(1+x_i)^{-1}\leqq 1$ implies that

$$(1+z)^{-1}+\sum_{i=3}^{n}(1+x_i)^{-1}\leqq 1,$$

where z is defined by $(1+z)^{-1}=(1+x_1)^{-1}+(1+x_2)^{-1}\leqq 1$, and hence $z\geqq 0$. By the induction hypothesis,

$$2^{-z}+\sum_{i=3}^{n} 2^{-x_i}\leqq 1.$$

It remains only to show that $2^{-x_1}+2^{-x_2}\leqq 2^{-z}$ or $2^{-y_1}+2^{-y_2}\leqq 1$, where $y_i=x_i-z$ $(i=1, 2)$. This, however, follows from the case for $n=2$ upon verifying that $y_1, y_2\geqq 0$ and $(1+y_1)^{-1}+(1+y_2)^{-1}\leqq 1$.

Also solved by E. F. Beckenbach and J. W. Green, Peter Henrici, A. R. Hyde, Blagovest Sendov, Lawrence Shepp, Chih-yi Wang, L. E. Ward, Jr., and the proposer. Late solution by Robert Breusch.

Number Systems in Which Squares End in Square Digits

4713 [1956, 729]. *Proposed by Oystein Ore, Yale University*

In the number system with base $n=12$, every square number ends in one of the digits 0, 1, 4, 9, that is, in a square digit. Find all n for which this is the case.

Solution by F. D. Parker, Clarkson College of Technology, Potsdam, N. Y. The numbers 2, 3, 4, 5, 8, 12, 16 have the desired property, and there are no others. The proof runs in four parts, according as n is of one of the forms, $4m+1$, $4m+3$, $4m+2$, $4m$.

Let the base be of the form $n=4m+1$. Then

$$(2m)^2 = (m-1)(4m+1)+3m+1,$$
$$(2m-1)^2 = (m-2)(4m+1)+3m+3.$$

These remainders are the terminal digits if the squares are written in the system having base n unless $3m+3\geqq n$, in which case we have $(2m-1)^2=(m-1)\cdot(4m+1)+2-m$. Since two squares cannot differ by 2 there are no solutions except in this special case, $m\leqq 2$. But $3m+1$ and $2-m$ are both squares only for $m=1$, whence $n=5$.

Suppose $n=4m+3$. Then

$$(2m)^2 = (m-1)(4m+3)+m+3, \quad (2m+1)^2 = m(4m+3)+m+1.$$

Since they differ by 2, both remainders cannot be squares except when $m=0$ and $n=3$.

If $n=4m+2$, then

$$(2m)^2 = (m-1)(4m+2)+2m+2, \quad (2m+1)^2 = m(4m+2)+2m+1.$$

These remainders cannot both be squares unless $m=0$ and $n=2$.

Finally, let the base be $n=4m$, and let $m^2=K\cdot 4m+r$, $0\leqq r<4m$. Then $(m+1)^2=K\cdot 4m+r+2m+1$,

$$(m+2)^2 = (K+1)4m+r+4, \quad (m+3)^2 = (k+1)4m+r+2m+9.$$

Since, from the first equation, r must be divisible by m, it is either 0, m, $2m$ or $3m$. If $m>4$, the requirement that the remainders for m^2 and $(m+2)^2$ be squares demands $r=0$ but then, in order that the remainders for $(m+1)^2$ and $(m+3)^2$ be squares, m must also be 0, an impossibility since $n>1$. Inspection of the cases $m=1, 2, 3, 4$ completes the proof with the result stated at the outset.

Also solved by H. F. Bennett, W. J. Blundon, D. A. Breault, Robert Breusch, N. J. Fine, A. S. Gregory, A. R. Hyde, and D. C. B. Marsh. In several of these solutions the proof that there are no other values of n was incomplete.

Editorial Note. The present result follows as an easy consequence from that of problem no. 4737 [1957, 277] for which a solution will appear in approximately four months.

Extrema of a Polynomial

4714 [1956, 729]. *Proposed by Chandler Davis, Columbia University and the New School for Social Research*

For a real polynomial $P(x)$ of degree n, denote the zeros of $P'(x)$ (multiplicity counted) by $\xi_i, i=1, \cdots, n-1$. Assume all ξ_i real, and let $\xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_{n-1}$. Now to what extent are the numbers $P(\xi_i)$ arbitrary? More precisely, give necessary and sufficient conditions on an $(n-1)$-tuple of real numbers $\eta_1, \cdots, \eta_{n-1}$, in order that there exist a polynomial P such that $P'(\xi_i)=0$, $P(\xi_i)=\eta_i$, $i=1, \cdots, n-1$.

Solution by the proposer. The numbers $(-1)^i(\eta_i-\eta_{i-1})$, $i=2, 3, \cdots, n-1$, must be either all nonnegative or all nonpositive. This condition is obviously necessary. The proof of sufficiency follows, assuming arbitrary $(-1)^i(\eta_i-\eta_{i-1}) \leqq 0$, $\eta_1 \geqq 0$.

It is no restriction to consider only the case $0 \leqq \xi_1 \leqq \xi_{n-1} \leqq 1$; define $\xi_0=0$, $\xi_n=1$. Only polynomials P such that $P(0)=0$ will be considered. It will be shown that the numbers $\phi_i=(-1)^{i-1}\{P(\xi_i)-P(\xi_{i-1})\}$, $i=1, \cdots, n$, can be chosen arbitrarily provided only $\phi_i \geqq 0$. (The trivial case $\phi_i \equiv 0$, $P(x)\equiv 0$ will also be excluded.) Using the notation

$$P(x) = A \int_0^x \prod_{i=1}^{n-1} (\xi_i - x)dx, \qquad A > 0,$$

justified by the assumptions made above, define for convenience $\delta_i = A(\xi_i - \xi_{i-1})$, $i=1, \cdots, n$. Now the main point in the proof is to consider the ϕ_i as n functions of the n variables $\delta_1, \cdots, \delta_n$.

First, $(\delta_1, \cdots, \delta_n)$ ranges over the set $\mathcal{P}^n$ of n-space $\mathcal{E}^n$ defined by the requirement that all coordinates be nonnegative and not all be zero; $(\phi_1, \cdots, \phi_n)$ is in $\mathcal{P}^n$ also, it must be shown that it can have any value in $\mathcal{P}^n$. Now the boundary is mapped into the boundary and the interior into the interior, for $\delta_i=0$ if and only if $\phi_i=0$. Every ray through the origin is mapped onto some ray through the origin (two points on the same ray differ only in the values of A). Evidently $\xi_1, \cdots, \xi_{n-1}, A$ are continuous functions of the δ_i, and the ϕ_i are continuous functions of $\xi_1, \cdots, \xi_{n-1}, A$. The associated Jacobians will be

proved nonzero for $(\delta_1, \cdots, \delta_n)$ in the interior of $\mathcal{P}^n$, that is, for $\xi_i > \xi_{i-1}$, $i=1, \cdots, n$; this will give the result.

It is easy to compute $\partial(\xi_1, \cdots, \xi_{n-1}, A)/\partial(\delta_1, \cdots, \delta_n) = A^{-n+1}$. For $i=1, \cdots, n$, define

$$a_{ij} = \frac{\partial\phi_i}{\phi\xi_j} = (-1)^{i-1}\int_{\xi_{i-1}}^{\xi_i} \frac{P'(x)}{\xi_j - x}\,dx, \quad j = 1, \cdots, n-1,$$

$$a_{in} = \frac{\partial\phi_i}{\partial A} = \frac{(-1)^{i-1}}{A}\int_{\xi_{i-1}}^{\xi_i} P'(x)dx.$$

It must be shown that the determinant of the matrix $\|a_{ij}\|$ is not zero.

If it were zero, there should be some linear relation of the form $\sum_1^n c_j a_{ij} = 0$, holding independent of i. This would mean that the function

$$F(x) = \sum_{j=1}^{n-1} c_j \frac{P'(x)}{\xi_j - x} + \frac{c_n}{A} P'(x),$$

a polynomial of degree $n-1$, would integrate to zero between $x=\xi_{i-1}$ and $x=\xi_i$, $i=1, \cdots, n$; hence that $\int_0^x F(x)dx$, a polynomial of degree n, would be zero at the $n+1$ points $\xi_0, \cdots, \xi_n$, so that $F(x) \equiv 0$. But the n summands of $F(x)$ are easily seen to be linearly independent, so the only possibility is all $c_j = 0$. This completes the proof.

Remark. The argument shows that $\phi_1, \cdots, \phi_n$ determine P uniquely provided no ϕ_i is zero. This restriction can be removed by a refinement of the same argument. It would be interesting to find an "interpolation" formula for P in terms of the ϕ_i.

Editorial Note. A. W. Goodman asks the analogous question for the complex domain: Can the branch points of (the image domain of) a polynomial be prescribed in advance? That is, given $n-1$ complex numbers $B_1, \cdots, B_{n-1}$, are there a polynomial $P(x)$ of degree n, and $n-1$ complex numbers $C_1, \cdots, C_{n-1}$ such that $P'(C_k)=0$, $P(C_k)=B_k$, $k=1, \cdots, n-1$? One may assume the B_k are all distinct for simplicity.

Simultaneous Equations

4715 [1956, 729]. *Proposed by R. S. Underwood, Texas Technological College*

Find real solutions of the equations

$$x^2 + y^2 + z^2 + u + v = D, \qquad x^2 + \frac{y^2}{2} + \frac{z^2}{3} + \frac{u^2}{4} + \frac{v^2}{5} = 1,$$

in case (a) $D = -3$, and (b) $D = 15/4$. What can be said about the solutions for other values of D?

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Subtracting the first from three times the second gives

$$2x^2 + \frac{1}{2}y^2 + \frac{3}{4}\left(u - \frac{2}{3}\right)^2 + \frac{3}{5}\left(v - \frac{5}{6}\right)^2 = \frac{15}{4} - D,$$

so that D is not more than 15/4. If $D = 15/4$, then every term on the left vanishes, giving the solutions $(x, y, z, u, v) = (0, 0, \pm 3/2, 2/3, 5/6)$.

Adding the first to 3/2 times the second gives

$$\frac{5}{2}x^2 + \frac{7}{6}y^2 + \frac{3}{2}z^2 + \frac{3}{8}\left(u + \frac{4}{3}\right)^2 + \frac{3}{10}\left(v + \frac{5}{3}\right)^2 = D + 3.$$

Thus D cannot be less than -3. For $D = -3$, the unique solution is $(0, 0, 0, -4/3, -5/3)$. There are no real solutions for values of D outside the range $-3 \leqq D \leqq 15/4$.

II. *Solution by N. J. Fine, University of Pennsylvania.* Let $\alpha = D - r^2 = D - (x^2 + y^2 + z^2)$, $\beta = 1 - (x^2 + y^2/2 + z^2/3)$. Then the system $u + v = \alpha$, $u^2/4 + v^2/5 = \beta$ has a real solution if and only if $\alpha^2 \leqq 9\beta$; the solution is unique when equality holds. Therefore D must satisfy

$$(1) \qquad r^2 - 3\sqrt{\beta} \leqq D \leqq r^2 + 3\sqrt{\beta}.$$

For all points of the ellipsoid E, defined by $\beta \geqq 0$, we have $r^2 \geqq 0$, $\sqrt{\beta} \leqq 1$, so $r^2 - 3\sqrt{\beta} \geqq -3$, and equality holds only for $(x, y, z) = (0, 0, 0)$. Thus for $D = -3$, there is the unique solution $(0, 0, 0, -4/3, -5/3)$. To determine the largest possible value of D, we maximize $F = r^2 + 3\sqrt{\beta}$ over E. On the boundary of E, $F = r^2 \leqq 3$. In the interior, the partial derivatives of F must vanish, so (i) $x = 0$, (ii) $y = 0$ or $\beta = 9/16$, (iii) $z = 0$ or $\beta = 1/4$. Testing the alternatives, we find the maximum of F to be 15/4, attained for $x = y = 0$, $z = \pm 3/2$. Thus, for $D = 15/4$, we have two solutions $(0, 0, \pm 3/2, 2/3, 5/6)$. For all other D, $-3 < D < 15/4$, there are infinitely many points of E for which $r^2 - 3\sqrt{\beta} = D$ or $r^2 + 3\sqrt{\beta} = D$. For each of these, (1) is obviously satisfied and a solution exists.

Also solved by G. E. Bredon, Emil Grosswald, Edgar Karst, D. C. B. Marsh, R. C. Read, Blagovest Sendov, and the proposer.

Editorial Note. Marsh adds the comment, "For those who are adept at employing the proposer's methods of extended analytic geometry these same results are immediately found by way of the simultaneous solutions of a filled ellipse and parabola." See Underwood, *Extended analytic geometry as applied to simultaneous equations*, this MONTHLY, 1954, pp. 525–542.

An Equation of Motion

4716 [1956, 729]. *Proposed by M. S. Klamkin, A VCO Research and Development, Lawrence, Mass.*

Determine the equation of motion if $\overline{V}_s = \lambda \overline{V}_t$, where $\overline{V}_s$ and $\overline{V}_t$ are the averages of velocity with respect to distance and time, respectively, in any time interval starting at $t = 0$. What is the minimum eigenvalue λ?

Solution by N. J. Fine, University of Pennsylvania. Assuming that $s(0)=0$, we see that $\overline{V}_t=s/t$, and so

$$\frac{1}{s}\int_0^s vds = \lambda\frac{s}{t}\cdot$$

Multiply by s and differentiate, to get

$$v = \lambda\left(2\frac{s}{t} - \frac{1}{v}\frac{s^2}{t^2}\right) \quad\text{and}\quad v = \mu\frac{s}{t},$$

where $\mu=\lambda\pm\sqrt{\lambda^2-\lambda}$. Hence $s=ct^\mu$. Except for the trivial case $\mu=\lambda=0$, we must have $\mu>1/2$ to ensure finiteness of $\overline{V}_s$, and $\lambda=\mu^2/(2\mu-1)\geqq 1$. Therefore every $\lambda\geqq 1$ is an eigenvalue, with the solution $s=ct^\mu$, $\mu=\lambda\pm\sqrt{\lambda^2-\lambda}$.

Also solved by Mary Payne, Chih-yi Wang, and the proposer.

Fixed Points of Entire Functions

4717 [1956, 729]. *Proposed by Patrick Gallagher, Harvard University*

Let $f(z)$ and $g(z)$ be entire functions, each without fixed points, not both linear. Then $f(g(z))$ has infinitely many fixed points.

Solution by Peter Henrici, University of California, Los Angeles. Since the entire function $f(z)$ has no fixed points, the function $f(z)-z$ is an entire function without zeros and hence can be represented in the form $e^{F(z)}$, where $F(z)$ is again entire. Similarly for $g(z)$ in terms of a certain entire function $G(z)$:

$$f(z) = e^{F(z)} + z, \qquad g(z) = e^{G(z)} + z.$$

A point z is a fixed point of $f(g(z))$ if and only if it satisfies the equation $e^{F(e^{G(z)}+z)} = -e^{G(z)} = e^{G(z)+i\pi}$. For this to occur it is necessary and sufficient that, for some integer n, $F(e^{G(z)}+z)=G(z)+(2n+1)i\pi$. Since not both of the functions $f(z)$ and $g(z)$ are linear, at most one of the functions $F(z)$ and $G(z)$ is a constant. It follows that $H(z)=F(e^{G(z)}+z)-G(z)$ is a nonconstant entire function and therefore, by Picard's theorem, assumes every value with at most one exception. Hence it assumes infinitely many of the values $(2n+1)i\pi$, *i.e.*, there are infinitely many fixed points.

Also solved by Joshua Barlaz, Robert Breusch, D. S. Carter and G. M. Wing, D. J. Newman, T. J. Rivlin, Blagovest Sendov, John Wermer, L. K. Williams, and the proposer.

Editorial Note. Rivlin points out that this and related results are contained in P. C. Rosenbloom, *The Fix-Points of Entire Functions* (Comm. Sém. Math. Univ. Lund, Tome Supplémentaire, 1952, dedié à Marcel Riesz.)

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

NOTICE

It has been suggested that this department prepare a "list of mathematical books suitable for small college libraries." If there is sufficient interest, such a list will be prepared. If *you* have any nominations for books especially suitable for small college libraries, please send their titles to Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

The Mathematics of Investment. By Roger Osborn. Harper, New York, 1957. viii+162 pp. +117 pp. of tables. $4.25.

The main difference between this text and the myriad of others on the same subject seems to be that the present work contains not much more material than can be covered in a one-semester, three-hour course. The result is that bonds are given summary treatment and life insurance is not discussed at all. This reviewer looks upon the omission of insurance as a fatal defect in the book, and as one that will make the text unsuitable for most classes in the mathematics of finance. The author also appears to be convinced that students will be required to memorize the plethora of formulae, and thus he devotes attention to developing memorizable ones. In this connection the footnote beginning on page 82 is amusing. For therein the author develops the easily computed

$$S = R(S_{\overline{n+1}|i} - 1)$$

formula for the sum of an annuity due, but rejects it summarily, partly on the grounds that it is difficult to memorize.

On the credit side, the book contains one of the clearest statements known to the reviewer of the distinction between an annuity and an annuity due, as well as carefully explained illustrative exercises and diagrams. The text is readable and the printing job satisfactory.

If one is seeking a text for a course which does not cover life annuities or insurance, and if one desires a book which places a premium on memorization (see the footnote on page 117 for the author's philosophy on this point, and for an unfortunate reflection on the character of students in general), then this text is quite suitable.

R. L. SAN SOUCIE
University of Oregon and Sylvania Electric Co.

Fun with Mathematics. By Jerome S. Meyer. The World Publishing Co., Cleveland and New York, 1952. x+176 pp. $2.75.

Instructors of secondary school or junior college mathematics who are looking for material to interest their promising students will find a variety of useful (although ungraded) material in this book, dedicated "to all young people who love mathematics." Little background is needed, although plane geometry and elementary algebra will be helpful, and the chapter on π, i, e, and logarithms requires the binomial expansion and trigonometry.

The book is readable and inviting, with large type, many charts and illustrations, and chapter headings that pique curiosity: "Babe Ruth Hit 111100 Home Runs" (binary system), "Curves That Control Our Lives" (conic sections). Other topics include number oddities, Fibonacci numbers, magic squares, trigonometry, the limiting process, fallacies and problems, and mathematical "how-to-do's" that will interest the budding engineer. Much of the material is given without explanation, although occasionally hints are offered; on the other hand, the section on the limiting process contains a lengthy and satisfactory exposition on an intuitive basis. Instructions for constructing a sun dial, several specific nomographs, and a slide rule are directive only; the reviewer thought it odd, since a previous chapter dealt with logarithms, that the slide rule directions contained no reference to them, actual or implied.

In general the mathematical presentation is satisfactory, although in a section devoted to fallacies one finds "$\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$ (which is correct)," followed by "$\sqrt{1}$ is either $+1$ or -1," to explain the resulting fallacy, and there are a few statements that could be more precise ("there can be *thousands* of differently shaped ellipses $\cdots$," not long after a section devoted to an explanation of the concept of infinity). It might be added that, while the book can be put into the hands of an individual student, it would be well if there were a well-trained mentor in the background to give direction, and to answer some inevitable questions. Also, it should be remembered that the primary purpose of the book is to stimulate and give constructive amusement, rather than to give formal instruction.

BESS E. ALLEN
Wayne State University

Nonparametric Methods in Statistics. By D. A. S. Fraser, Wiley, New York, 1957. x+299 pp. $8.50.

In *Nonparametric Methods in Statistics* the author, D. A. S. Fraser, has successfully attempted to collect and unify the different developments of nonparametric methods in a book intended as a second course in mathematical statistics. The text presupposes a knowledge of the calculus and familiarity with an introduction to statistics such as found in Hoel's *Introduction to Mathematical Statistics* or Mood's *Introduction to the Theory of Statistics*. The essential ideas of measure theory are introduced and illustrated as needed. Naturally, the stu-

dent who is more familiar with measure theory will find the reading ease improved. The first two chapters consisting of 124 pages treating sample space (measure space, measures and probability measure, expectation and conditional probabilities, sufficient statistics, completeness) and statistical inference (decision problem, estimation of real parameters, hypothesis testing, confidence and tolerance regions) have been used for an undergraduate course surveying recent small sample methods. The remainder of the book is intended for a graduate course on the applications of these methods in the nonparametric branch of statistics. This portion of the text, 172 pages, contains five chapters on nonparametric problems (single sample, randomness, randomized blocks, and more general designs), estimation of real parameters and tolerance regions, theory of hypothesis testing (unbiasedness, most powerful tests, most powerful rank tests, likelihood ratio method), limiting distributions (general theorems concerning limiting distributions, central limit theorems, limiting distribution of **U** statistics, Wald-Wolfowitz limit theorem, limiting distribution or runs, and additive partition functions), and large sample properties of tests (consistency, criterion for the relative efficiency of tests, efficiency of some conditional tests, efficiency of a rank test). Excellent and extensive lists of problems for solution are supplied at the end of each chapter. The author uses these problems to develop portions of the theory as well as to illustrate, apply, and extend it. Fine reference and bibliography lists conclude each chapter. The text and references will impress the student with the recency of the developments of this subject.

JOHN C. BRIXEY
University of Oklahoma
Norman, Oklahoma

Introduction to Statistical Analysis. Second Edition. By Wilfrid J. Dixon and Frank J. Massey, Jr. McGraw-Hill, New York, 1957. xiii+488 pp. $6.00.

This textbook, designed for a one-year basic course in statistics, follows the recommendations of the "committee on teaching of statistics of the National Research Council." Both editions of the book share these characteristics: illustrative examples have been taken from a wide variety of fields such as agriculture, engineering, and medical research; the problem lists are well organized and comprehensive; students are encouraged to carry out simple sampling experiments; discussions are concise; the chapters are unusually self-contained; calculus is not used; in general, formulas are stated without proofs but with some indication of their limitations; and wide use is made of tables contained in the back of the book.

The major changes in the Second Edition are: a chapter on probability; the chapters on statistical inference and analysis of variance have been largely rewritten; almost all problem lists have been augmented; the references have been combined into one extensive list at the back of the book; the old tables have been

expanded and 7 new tables added; and the revolting caricatures have been omitted. In 29 pages the chapter on probability covers the addition and multiplication theorems; the binomial, hypergeometric, and Poisson distributions; and closes with a brief mention of continuous chance variables. Most of the problems deal with sampling situations both with and without replacement.

On the whole the book is carefully written. However, two minor complaints come to mind. On page 30 the authors introduce sampling by stating that "any subset of a population is a sample from that population." On page 36 and again on page 42 the samples enumerated in the illustrative example are *ordered* subsets. It seems to this reviewer that the troublesome question of the relevance of order in sampling might have received more attention. In testing hypotheses set up for possible rejection, this book uses the two alternatives, "reject" and "accept." On pedagogical grounds this reviewer prefers the alternatives, "reject" and either "not reject" or "reserve judgment." The type is large and clear in both the text material and the tables. An interesting misprint "petmutations" occurs on page 356.

Anyone in the market for a statistics text with wide coverage below the calculus level should examine this book.

V. V. LATSHAW
Lehigh University

Theory of Approximations. By N. I. Achieser (translated from the Russian by Charles J. Hyman). Frederick Ungar, New York, 1956. 307 pp. $8.50.

The reader is referred to a review of the original Russian edition of this book by A. Zygmund in *Mathematical Reviews*, vol. 10, 1949, p. 33, for an excellent discussion of its contents. Its translation into English makes available to many readers a very systematic and complete development of the theory of approximation, not previously available in one volume. Not only is the treatment of the subject complete, but those basic concepts germane to the subject are developed in brief but sufficient detail, and the book may serve as a very good reference work in which can be found concise definitions and theorems relating to many topics in the field of analysis. Among those topics particularly well treated are those of a metric space, linear normalized space, Hilbert space, separable and complete spaces, orthonormal systems, linear functionals, Fourier series and transforms, multimonotonic functions, and functionals of exponential type.

The book is extremely well arranged with sufficient groundwork laid in the first chapter of fifty pages for the treatment of special types of approximation given in the remainder of the book. Here for example is given a proof of the fundamental theorem of approximation theory in linear normalized spaces. The many examples of applications of the theory to Hilbert space as well as the geometric interpretations given contribute a great deal to the clarity of the treatment.

In Chapter II, Chebyshev's theorem on the approximation of a function

$f(x)$ in terms of a given function $s(x)$ and a fractional expression involving two polynomials in x is proved. Several examples of applications of the theorem are given, one of considerable interest involving elliptic functions. The treatment is then specialized to consideration of the best trigonometric sum approximation to a periodic function and other particular problems.

Chapter III on harmonic analysis treats Fourier series briefly and contains a proof of the convergence of a Fourier series of an integrable function of bounded variation. Fourier transforms and other integral operators are studied. Although these topics have an interest in themselves, they are introduced because of their application to the theory of approximation and convergence. Chapter IV takes up the special topic of extremal properties of integral transcendental functions of exponential type. The author is thus led to two generalizations of Bernstein's inequality. In Chapter V the relation between rapidity of convergence of the approximating functions to the differentiability properties of the given function is treated and the theorems of Bernstein and Jackson on this subject are proved and generalized. Wiener's treatment of approximation to a given function by functions of very general type is considered in Chapter V.

Pages 243 to 295 are used for statements and solutions of some interesting illustrative extremal and approximation problems. For those who wish to try these problems it should be noted that the author classifies them as elementary. The table of contents reflects the orderly development followed in the text. Notes, references, and the table of contents are also adequate.

It must be said that this treatise on approximation theory is well planned and written in such a way that a graduate student well grounded in the usual graduate courses in analysis or the nonspecialist can study it with profit and pleasure. The translator should be congratulated on the smoothness and readability as well as the accuracy of the English text. The professional mathematician should find the organization of the work as well as the brevity and clarity of the proofs pleasing. The book would be suitable as a reference work for a graduate course or seminar in what is certainly a field of analysis of broad application.

O. H. Hamilton
Oklahoma State University

Proceedings of the International Symposium on Algebraic Number Theory. The Science Council of Japan. Pan-Pacific Press, 1957. xx+267 pp. $5.

This is a collection of research papers on algebraic number theory and allied topics, delivered at an International Symposium in Tokyo and Nikko, Japan, on September 8–13, 1955. There are twenty papers, of which approximately half were contributed by mathematicians from France, Germany, India, and the United States, and the remainder, along with seventeen short notes, by Japanese mathematicians. The subject matter ranges widely over algebraic number theory, such as the idèle-class group of an algebraic number field, Siegel's

modular functions, the generalized principal ideal theorem, and the cohomology of algebraic number fields. There are papers on algebraic geometry, especially on algebraic and abelian varieties, and several papers touching to some degree on number theory. A special emphasis was laid on possible extensions of the class field theory, and several papers deal with a generalization of the theory of complex multiplication. A detailed review paper by paper is not appropriate here, but to give some idea of the outstanding group of mathematicians participating in the Symposium and contributing to this volume, we list the nineteen authors of the major papers in the order of their appearance: A. Weil, G. Shimura, Y. Taniyama, M. Deuring, E. Artin, R. Brauer, K. Iwasawa, T. Tannaka, T. Nakayama, T. Kubota, K. Yamazaki, K. G. Ramanathan, I. Satake, C. Chevalley, A. Néron, Y. Nakai, J.-P. Serre, M. Nagata, and D. Zelinsky.

IVAN NIVEN
University of Oregon

Introduction to Finite Mathematics. By John G. Kemeny, J. Laurie Snell and Gerald L. Thompson, Prentice-Hall, New Jersey, 1957. 372+xi pp., $5.00.

In the growing tide of new elementary texts, this one stands out as the most radical, and to my mind happy departure from tradition. No warming over of high school algebra or gentle calculus background this volume, but a substantial dose of modern topics based upon *finite* sets. The five core chapters are: compound statements, sets and subsets, partitions and counting, probability theory, and vectors and matrices. Careful organization and efficient use of concepts is obvious throughout, giving the exposition a quality reminiscent of more advanced texts. The transition from this book to later courses should be smoother than usual, but the transition to the book itself is another matter.

The authors, although careful to include many examples and problems, may have gone too far in the direction of sophistication: the style, precise and simple, is spare. For example, there are unembellished definitions such as "A *column vector* is an ordered collection of numbers written in a column."—the first sentence of Chapter 5. But, let it be recorded, the Dartmouth freshmen on which the text has been *successfully* tested were not specially selected. However, their instructors were the authors, and the lectures, no doubt, richly supplemented the text. Good lectures—always desirable—may well be necessary when using this book.

The book is also intended for those increasingly common behavioral science graduate students who sense a need to know some mathematics. For them there are two chapters of applications, covering aspects of linear programming and two-person game theory and nontrivial topics from sociometrics, genetics, learning theory, anthropology, and economics.

The main fault of commission is, I believe, the attempt to discuss without sufficiently clear delineation the notions of measure, relative frequency, and credibility of statements in the probability chapter. The book's main omission

(especially unfortunate for the behavioral scientists) is that collection of ideas, beginning with product sets, and including relations, orderings, functions, and axiomatic applications such as utility theory.

In sum, even though I have some specific reservations, I would commend this among beginning texts as an exciting and remarkably successful attempt to tap a different, and clearly important, lode of mathematics. Its influence should be widely felt, even though it may be deemed too mathematically mature for some freshmen courses.

DUNCAN LUCE
Harvard University

Mathematics of Finance. By Hugh E. Stelson. Van Nostrand, Princeton, N. J., 1957. xii+327 pp. $5.50.

This book differs from the usual mathematics of finance text in its treatment of both the nontechnical and the theoretical aspects of this subject. Its emphasis on the practical viewpoint may be indicated by its realistic discussion of "Consumer Loans," "Buying a Home," "The Nature of Life Insurance." These are the headings of three of the book's fifteen chapters. The basic ideas of these topics are presented simply and clearly. The book's attention to technical details and soundness of theory may be indicated by the unusually large number of footnotes. There are twenty footnotes to literature alone, and many others which provide explanatory comments or details for proofs.

The book is well-written in a concise style. The paragraphs are exceptionally short, yet the explanations are clear and sufficiently detailed. All the standard topics in the mathematics of finance are included and treated competently. There is an adequate number of realistic examples and exercises. Each chapter ends with a list of formulas and symbols. The tables are excellent. Eight of the ten tables included are reprinted from Glover's *Tables of Applied Mathematics.*

The preface and the introduction to Chapter 15, which is a review of "Preparatory Topics from Algebra," imply that the students using this text should be well-grounded in algebra. In addition, familiarity with limits is needed for the sections on perpetuities and continuously convertible interest. Also, a knowledge of calculus is essential for the chapter on "Continuous Annuities." However, the author indicates in the preface that all of these portions of the text, along with some others, "can be omitted for a short course with emphasis on applications."

On the assumption that college algebra is a minimum prerequisite for the course in mathematics of finance, the reviewer considers that Professor Stelson has accomplished his purpose of preparing an excellent text for students in mathematics or business administration.

H. S. KALTENBORN
Memphis State University

Nonparametric Statistics. By Sidney Siegel, McGraw-Hill, New York, 1957. xvii+312 pp. $6.50.

Most of the techniques of statistical inference presented in textbooks on statistics for social scientists are valid only under quite stringent assumptions on the nature of the underlying distributions, *e.g.*, that all observations were obtained from normally distributed populations. Since these assumptions are quite often not fulfilled for the available data, these "classical" statistical techniques may, if applied, lead to doubtful conclusions. To remedy this situation, statisticians have been developing a number of procedures which can be correctly applied under very mild restrictions on the class of populations considered. Most of these procedures presuppose only that the variates considered have continuous distributions, but require no further assumptions on the form of the distributions. These very general statistical methods, developed and published mostly in the last two decades, are known to statisticians as "nonparametric" or "distribution-free" methods of statistical inference.

The book by Siegel is written for the use of social scientists or, in the author's terminology, for behavioral scientists, and represents a pioneering effort to introduce the behavioral scientists to the use of distribution-free statistical techniques. The author states specifically that it is his intention to "make this book fully comprehensible to the reader who has had only introductory work in statistics . . . " and "whose mathematical training is limited to elementary algebra." This specific purpose of the book explains some peculiarities of style which, to a professional statistician may be somewhat disconcerting: the presentation is verbose and frequently contains sentences or paragraphs that contribute little to the progress of thought, and the exposition is repetitious to an extent that may at times confuse the reader. More serious than these qualities of style are some material shortcomings. Basic concepts are often presented in a manner which is likely to give the reader an incorrect picture. To mention some examples: from the book it would appear that the main, possibly the only, use of nonparametric statistics in behavioral sciences consists in testing hypotheses, while problems of estimation are mentioned exactly once in the introduction and distribution-free tolerance limits are not mentioned at all. The concept of power of a test is explained without sufficient emphasis on the fact that the power is defined only for a specified alternative, and the frequently occurring phrase "for these data the . . . test exhibits greater power to reject H_0 than the . . . test" adds to the confusion. The discussion of power-efficiency which follows the presentation of each test may suggest to the reader some vague idea of comparative goodness of tests, but it is doubtful that he will clearly understand what this means. The same mathematical models are presented in various sections of the book under different names, for example, the 2×2 table or the binomial test appear repeatedly under different labels; possibly the author wanted to show how the same technique can be applied to rather different types of research

problems, but then the reader would deserve a hint that this is the same procedure over and over again.

In spite of these shortcomings, *Nonparametric Statistics* is a useful and significant book. It is the first presentation in book form of a body of knowledge that has been accumulating in scattered articles in professional periodicals and which is of considerable importance to the practical researcher. The intuitive motivation of the techniques presented is generally plausible and should prove appealing and informative to the behavioral scientist, even if it does not meet the standards of a mathematically trained reader. Every technique presented is illustrated by examples couched in terms of concrete applications and every step is patiently explained. An appendix consisting of fifty-five pages of tables makes accessible for the first time a large amount of tabular material which previously could be found only in professional journals.

In summarizing, this reviewer feels that regardless of the reservations he has with regard to the accuracy of formulation and sometimes to the logic of presentation, Siegel's book is to be considered an important step in the process of introducing hitherto little known but powerful and useful statistical methods into the work of behavioral scientists.

Z. W. Birnbaum
University of Washington

BRIEF MENTION

Mathematics for Psychologists, Examples and Problems. By Robert R. Bush, Robert P. Abelson and Ray Hyman. Social Science Research Council, 1957. vi+86 pp. $2.00.

Mathematicians interested in providing mathematical training for social scientists will welcome this collection of problems. The fields of testing and measurement, psycho-physics, physiological psychology, and learning are each well represented in this book, and there are several examples of current research on small groups, sociometry and related areas of social psychology. In each case there are examples and references to published material. The book is divided into four general parts: applications to calculus 24 pages, mathematical foundations 10 pages, matrix algebra 22 pages, probability theory (including Markov chains) 20 pages. An appendix gives answers to the problems. A bibliography of one hundred and twenty four items concludes this brief but welcome volume.

System Engineering. By Harry H. Goode, and Robert E. Machol. McGraw-Hill, New York, 1957, xii+551 pp. $10.00.

This introduction to the design of large-scale system engineering will certainly find its way into the library of every university in which a large-scale computer is being considered or is in use.

Bibliography of Russian Mathematics Books. By George E. Forsythe. Chelsea, New York, 1957, 106 pp. \$3.95.

The History of Mathematics. By Joseph E. Hofmann. Philosophical Library, New York, 1957. xi+132 pp. \$4.75.

This brief books deals with what today might be called Ancient Mathematics, from about 2000 B.C. to about 1650 A.D.

A Short Dictionary of Mathematics. By C. H. McDowell. Philosophical Library, New York 1957. xiii+63 pp. \$2.75.

This dictionary limits itself deliberately to high school mathematics of arithmetic, algebra, plane geometry and trigonometry. Some of the terms are not defined precisely as this reviewer would define them, even for this limited group. For example, "Absolute means pure, unmixed or exact." Other examples, "Area is any plane surface, the extent of the surface of any figure included within any three or more straight lines or within any closed line or lines," or "Divergent means receding further and further away." This reviewer would much prefer the carefully written James and James mathematical dictionary even for a high school library.

How to Solve It. By G. Pólya. Doubleday, Garden City, N. Y., 1957. 253 pp. \$0.95.

The appearance of a low cost paperback edition of this well-known book will be welcomed by teachers and students everywhere.

Scientific German. By George E. Condoyannis. Wiley, New York, 1957. \$2.50.
Scientific French. By William N. Locke. Wiley, New York, 1957. \$2.25.

These two concise companion volumes fulfil a long-felt need of mature students who desire to acquire a reading (not speaking or writing) knowledge of scientific (not literary) German and French. Grammatical concepts such as the passive voice and impersonal verb, so important in reading scientific work, are not slighted. No childish "John saw the pencil" drill here, but instead a mature, logical approach to reading scientific works. At the end of a dozen pages the student has already read a passage from a standard scientific source. All emphasis is on how to *read* French and German. As Professor Locke says, "This approximation (to French pronunciation) often will not give a pronunciation understandable to a Frenchman, but may be sufficient for the communication of French words between Americans." It is this reviewer's sincere belief that these volumes fill a long-felt need and his hope that a similar volume will be forthcoming for the Russian language.

Computing with Desk Calculators. By Walter W. Varner. Rinehart, New York, 1957. \$2.00.

"This manual is designed to furnish the user of the modern desk calculator

with sufficient understanding of the machine so that he can quickly design his own computing techniques. It is written primarily for a person who uses a machine to perform many different calculations instead of many repetitions of the same prototype."

Large-scale electronic computers are becoming more and more important, and users must become familiar with the desk calculator, which is customarily used for checking purposes. Particularly noteworthy is the inclusion of double precision calculations for numbers larger than the capacity of the desk calculator.

Canon Arithmeticus. By C. G. J. Jacobi. Akademie-Verlag, Berlin, 1956. 432 pp. D.M. 46.

These tables of indices for the prime moduli under 1000 should prove a welcome addition to any mathematical library.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE ACADEMY-RESEARCH COUNCIL

The National Academy of Sciences—National Research Council is a non-governmental, self-perpetuating organization of scientists, chartered by the U. S. Congress to further the natural sciences and to provide scientific and technical advice to the federal government.

One of the eight divisions of the Academy-Research Council is the Division of Mathematics, which has ten major scientific societies represented in its membership.

The Division has unique opportunities to stimulate interchange among sciences, and the activities of its various committees range from a campaign to acquaint young people with mathematics as a career to the preparation of a handbook of mathematical tables. The Division also serves in an advisory capacity to governmental agencies. At the present time, the Division is actively engaged in a campaign to insure a full representation of American mathematicians at the next International Mathematical Congress scheduled for Edinburgh, Scotland, in August 1958.

The Chairman of the Division is Professor P. A. Smith, Columbia University. The Executive Secretary of the Division is Professor H. W. Kuhn, Bryn Mawr College. The Division has appointed a Committee on Educational Policy to consider questions of training especially in secondary schools, and national education policy in mathematics. The Chairman of this Committee is Professor S. S. Cairns, University of Illinois.

PERSONAL ITEMS

Professor A. W. McGaughey, Bradley University, represented the Association at the inauguration of President Robert G. Bone of Illinois State Normal University on October 4, 1957.

The Catholic University of America: A program of late afternoon and evening courses leading to the master and doctoral degrees in mathematics will be offered in the fields of analysis, algebra, theory of probability, mathematical statistics, and numerical analysis. The program of public lectures in the field of mathematical statistics which were given monthly during the past year under the sponsorship of the National Science Foundation was resumed in September.

Michigan State University: Dr. Joseph Lehner, Staff Member, Los Alamos Scientific Laboratory, New Mexico, has been appointed Visiting Professor; Dr. Günther Ewald, University of Mainz, Dr. D. W. Hall, Amherst College, Dr. L. L. Helms, Senior Research Engineer, Convair Corporation, Pomona, California, and Dr. R. H. Wasserman, University of Michigan, have been appointed Assistant Professors; Mr. Yousef Alavi, Mrs. Delia W. Koo and Mr. R. E. Sechler have been appointed Instructors; Dr. Heinrich Larcher has been promoted to Assistant Professor.

University of Denver: The 4th Annual Symposium on Computers and Data Processing, sponsored by the Electronics Division, Denver Research Institute of the University, was held on August 29–30, 1957. This was followed by a special two weeks course on computer applications.

University of Michigan: Associate Professors R. C. F. Bartels, Raoul Bott, and Gail S. Young have been promoted to Professors; Assistant Professors J. W. Carr, III, A. J. Lohwater, and J. G. Wendel have been promoted to Associate Professors; Dr. J. W. Addison, Dr. A. B. Clarke, and Dr. E. L. Griffin, Jr., have been promoted to Assistant Professors; Visiting Assistant Professor D. G. Higman has been promoted to Assistant Professor; Dr. P. E. Conner, Institute for Advanced Study, has been appointed Assistant Professor; Dr. D. S. Greenstein, Professional Engineer, Radio Corporation of America, Camden, New Jersey, Dr. N. J. Hicks, Massachusetts Institute of Technology, and Dr. E. T. Parker, Graduate Student, Ohio State University, have been appointed Instructors; Assistant Professor F. M. Wright, Iowa State College, has been appointed Visiting Assistant Professor; Assistant Professor C. J. Coe has retired with the title Assistant Professor Emeritus; Associate Professor C. L. Dolph and Professor Erich Rothe have been granted Guggenheim Fellowships and are on sabbatical leave during 1957–58; Associate Professor W. J. LeVeque has received a Sloan Foundation Fellowship and is on leave of absence for 1957–58; Professor E. D. Rainville will be on sabbatical leave for the second semester of 1957–58; Dr. E. L. Griffin is on leave as Visiting Professor at Columbia University during 1957–58.

University of Minnesota: Dr. Steven Orey has been promoted to Assistant Professor; Associate Professors B. R. Gelbaum and G. K. Kalisch have been promoted to Professors; Mr. R. C. Bzoch, Illinois Institute of Technology, Dr. Jesus Gil de Lamadrid, Ohio State University, and Assistant Professor D. A. Storvick, Iowa State College, have been appointed Assistant Professors; Associate Professor G. F. Clanton, Baylor University, has been appointed Lecturer; Professor R. W. Brink, Chairman of the Department of Mathematics, has retired; Professor R. H. Cameron has been appointed Chairman of the Department; Professor J. M. H. Olmsted has been appointed Associate Chairman of the Department.

Associate Professor J. C. Abbott, U. S. Naval Academy, has been promoted to Professor.

Mr. D. S. Adorno, Iowa State College, has a position as a senior engineer at Sylvania Electric Products, Waltham, Massachusetts.

Mr. J. T. Ahlin, Applied Science Representative, I.B.M. Corporation, Houston,

Texas, is now General Manager of Applied Programming, I.B.M. Corporation, New York, New York.

Mr. Eugene Albert, Mathematician, General Electric Company, Schenectady, New York, has been appointed Assistant Professor at Union College.

Assistant Professor H. B. Anderson, Michigan College of Mining and Technology, has been promoted to Associate Professor.

Dr. P. H. Anderson, Survey Statistician, Department of the Army, Washington, D. C., is now employed as an analytical statistician, Bureau of Ships, Navy Department, Washington, D. C.

Dr. J. J. Andrews, Teaching Assistant, University of Georgia, is now employed by the Union Carbide and Carbon Corporation, Oak Ridge, Tennessee.

Mr. D. L. Arenson, Manager, Ex-Cel Development Company, Chicago, Illinois, has been appointed Assistant Manager at the American Machine and Foundry Company, Chicago.

Associate Professor J. W. Ault, U. S. Air Force Academy, has been promoted to Professor and Head of the Department of Mathematics.

Assistant Professor J. B. Bartoo, Pennsylvania State University, has been promoted to Associate Professor.

Associate Professor Lulu Bechtolsheim, University of Redlands, has been promoted to Professor.

Mr. J. S. Becker, Engineering Assistant, Studebaker Corporation, South Bend, Indiana, has been appointed Senior Engineer, Glenn L. Martin Company, Denver, Colorado.

Mr. J. G. Bennett, Computer, North American Aviation, Columbus, Ohio, has a position as a mathematician for the I.B.M. Corporation, Poughkeepsie, New York.

Assistant Professor S. K. Berberian, Michigan State University, has been appointed Assistant Professor, State University of Iowa.

Mr. C. J. Biggerstaff, Electronics Engineer, Rheem Manufacturing Company, Downey, California, is a magnetics chief project engineer for California Magnetics Control, North Hollywood, California.

Associate Professor A. H. Black, Southern Illinois University, has been promoted to Professor.

Dr. R. L. Blair, Michigan State University, has been appointed Assistant Professor at the University of Oregon.

Assistant Professor A. A. Blank, University of Tennessee, is on leave for one year as Research Scientist for the Institute of Mathematical Sciences, New York, New York.

Mr. A. F. Bond, Graduate Assistant, West Virginia University, is now employed as a mathematician for the Bendix Aviation Corporation, Ann Arbor, Michigan.

Dr. R. D. Boswell, Jr., Graduate Assistant, University of Georgia, has been appointed Associate Professor at Mississippi State College.

Dr. L. E. Boyer, Chairman of the Department of Mathematics, State Teachers College, Millersville, Pennsylvania, is now Advisor, College Preparatory, Department of Public Instruction, Harrisburg, Pennsylvania.

Rev. E. W. Brande, Graduate Student, St. Louis University, has been appointed Instructor at Canisius High School, Buffalo, New York.

Mr. J. P. Brannen, Sam Houston State College, has been promoted to Assistant Professor.

Mr. J. R. Brashear, Student, Carnegie Institute of Technology, is Director, Computation Department, R. A. Cummings, Jr. & Associates, Pittsburgh, Pennsylvania.

Mr. D. A. Breault, Carnegie Institute of Technology, has accepted a position as an engineer in the Missile Systems Laboratory, Sylvania Electric Products Company, Waltham, Massachusetts.

Dr. D. M. Brown, Research Engineer, Data Reduction and Computation, Willow Run Research Center, University of Michigan, Ypsilanti, Michigan, is now Educational Staff Consultant, Remington Rand Univac, St. Paul, Minnesota.

Mr. G. A. Brown, Graduate Student, Rutgers University, has a position as a mathematician at Okonite Company, Passaic, New Jersey.

Assistant Professor J. W. Brown, Clemson College, has been promoted to Associate Professor.

Mrs. Mary H. Brown, Perkinston Junior College, has been appointed Head of the Department of Mathematics.

Associate Professor H. D. Brunk, University of Missouri, has been promoted to Professor.

Associate Professor C. E. Burgess, University of Utah, has returned after his leave of absence as Visiting Lecturer at the University of Wisconsin.

Dr. J. R. Byrne, San Jose State College, has been appointed Assistant Professor at Portland State College.

Assistant Professor Lonnie Cross, Tuskegee Institute, has been appointed Associate Professor at Atlanta University.

Assistant Professor C. C. Faith, Michigan State University, has been appointed Assistant Professor at Pennsylvania State University.

Professor Ky Fan, University of Notre Dame, is on leave of absence and has accepted a position on the Mathematics Panel, Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Assistant Professor J. R. Foote, University of Oklahoma, has accepted a position as Associate Professor in the Division of Engineering Sciences, Purdue University.

Dr. J. W. Gaddum, Michigan State University, has been appointed Assistant Professor at the University of Florida.

Dr. W. J. Harrington, Pennsylvania State University, has been appointed Professor at North Carolina State College.

Assistant Professor L. H. Kanter, Drexel Institute of Technology, has been promoted to Associate Professor.

Professor Don Kirkham, Iowa State College, has received a Guggenheim Fellowship for research and a Fulbright award for lecturing in soil physics at the State Agricultural University, Ghent, Belgium, for the academic year 1957–58.

Dr. Paul Koosis, Research Assistant, Institute of Mathematical Sciences, New York University, has received a Fulbright grant and will conduct research at the University of Montpellier, France, during the academic year 1957–58.

Mr. Arthur Libenson, Bedford, Massachusetts, has accepted a position as systems engineer at Raytheon Manufacturing Company, Maynard, Massachusetts.

Mr. R. T. J. Mahoney, Teaching Fellow, University of Buffalo, has been appointed Assistant Professor at the U. S. Naval Academy.

Assistant Professor J. H. McKay, Michigan State University, has been appointed Assistant Professor at Seattle University.

Dr. J. A. Schatz, University of Connecticut, has a position as a staff member at the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Edward Silverman, Michigan State University, has been appointed Associate Professor at Purdue University.

Professor F. C. Smith, College of St. Thomas, has a position as Actuary at George V. Stennes & Associates, Minneapolis, Minnesota.

Mr. L. T. Wos, University of Illinois, has accepted a position as Assistant Mathematician at the Argonne National Laboratories, Lemont, Illinois.

Dean Elijah Swift, University of Vermont, died on July 21, 1957. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-EIGHTH SUMMER MEETING OF THE ASSOCIATION

The thirty-eighth summer meeting of the Mathematical Association of America was held at Pennsylvania State University, University Park, Pennsylvania, from Monday, August 26 through Thursday, August 29, 1957, in conjunction with the summer meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, and the Pi Mu Epsilon Fraternity. There were registered 924 persons, including 451 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday and Thursday afternoons. All sessions were held in 121 Sparks Building of Pennsylvania State University. Presiding officers were President G. B. Price, Vice-President B. W. Jones, and Professors Deane Montgomery and W. L. Duren. The sixth series of Earle Raymond Hedrick Lectures was delivered by Professor Leo Zippin of Queens College. The lecture by Professor Albert Edrei was presented on closed circuit television. The sessions on Wednesday and Thursday afternoons were open meetings of the Association's Committee on the Undergraduate Program in Mathematics. The Program Committee for the meeting consisted of N. J. Fine, Chairman; Harley Flanders, and W. R. Transue.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Topological transformation groups;* Lecture I, by Professor Leo Zippin, Queens College.

Elementary problems in the theory of entire functions, by Professor Albert Edrei, Syracuse University.

Intersection properties of convex sets, by Professor V. L. Klee, Jr., University of Washington.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Zippin.

The science teaching improvement program, by Dr. J. R. Mayor, Director, Science Teaching Improvement Program, A.A.A.S.

A special program for gifted undergraduates, by Professor K. O. May, Carleton College.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Zippin.

Business Meeting of the Association.

Survey of research potential and training in the mathematical sciences, by Professor J. W. Green, University of California, Los Angeles.

Mathematicians in industry and government, by Dean Mina S. Rees, Hunter College. (By title).

FOURTH SESSION OF THE ASSOCIATION

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Symposium: *Probability and statistics in undergraduate mathematics.*
Introduction by Professor M. E. Munroe, University of Illinois.
Moderator: Dr. Brockway McMillan, Bell Telephone Laboratories.

Symposium: *The study of logic and foundations in undergraduate courses.*
Introduction by Professor P. R. Halmos, University of Chicago.
Moderator: Professor J. G. Kemeny, Dartmouth College.

FIFTH SESSION OF THE ASSOCIATION

The work of the committee on the undergraduate program in mathematics: C.U.P. books and commercial texts, by Professor R. L. Davis, University of Virginia.

The C.U.P. program, by Professor J. G. Kemeny, Dartmouth College.

Symposium: *Goals of freshman courses.*
Introduction by Professor K. O. May, Carleton College.
Moderator: Professor G. B. Price, University of Kansas.

Symposium: *Calculus for freshmen.*
Introduction by Professor R. V. Andree, University of Oklahoma.
Moderator: Professor W. L. Duren, Jr., University of Virginia.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Beta Theta Pi fraternity house with twenty-six members present. Among the more important items of business transacted were the following:

To fill a vacancy, the Board elected Professor Ernst Snapper of Miami University as Governor from the Ohio Section for a one-year term.

It was voted to authorize future meetings of the Association at the Massachusetts Institute of Technology in August 1958, at the University of Pennsylvania in January 1959, and at the University of Utah in August 1959.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning, with President G. B. Price presiding. The Secretary reported that the membership of the Association was 7,024 on August 23, 1957.

The following committee chairmen reported for their committees: Professor B. W. Jones, Committee on Visiting Lecturers; Professor J. S. Frame, Committee on Employment Opportunities; Professor Edith R. Schneckenburger, Committee on Sections.

At the conclusion of the Tuesday morning session, a protracted discussion was held of some of the problems of mathematicians in industry and government. The following motion presented by Professor Wallace Givens was adopted: "It is the sense of this meeting that the Albert Survey has proved of sufficient value that a continuing or recurrent analysis of a similar sort is fully justified. It is particularly important that any such further study pay special attention to the employment of mathematicians in industry and other non-academic positions."

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Hetzel Union Building. Fifty-five persons were present representing 26 of the 27 Sections.

The national contest for high school students planned for the spring of 1958 was discussed in great detail. The visiting lecturer programs for high schools sponsored by the Kentucky Section and the Northern California Section were described. Several sectional committees reported on their activities pertaining to the strengthening of the teaching of mathematics.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium lectures on *Cohomology operations* were delivered by Professor N. E. Steenrod of Princeton University. Invited addresses were given by Professors A. P. Calderon and M. A. Rosenlicht.

The Society for Industrial and Applied Mathematics and the Pi Mu Epsilon Fraternity met on Monday and Tuesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Evan Johnson, Jr., Chairman; J. B. Bartoo, Orrin Frink, H. M. Gehman, W. O. Gordon, H. L. Krall, R. D. Schafer.

Registration headquarters were located in Waring Hall, with dormitory and cafeteria accommodations in the same area. The Employment Register and the Book Exhibit were on display in Waring Hall.

On Monday evening, Sigma Delta Epsilon Honorary Society held a tea in Hamilton Hall for the women mathematicians. A reception for members of the mathematical organizations was held in the Hetzel Union Building on Wednesday afternoon immediately preceding the dinner. On Thursday afternoon a chicken barbecue was held in Horticultural Woods.

At the banquet on Wednesday evening, Professor H. B. Curry acted as toastmaster. The visitors were welcomed on behalf of the University by Dean Ben Euwema, Professor Orrin Frink, and President Eric A. Walker. Professors E. J. McShane responded for the American Mathematical Society, G. B. Price for the Mathematical Association of America, T. H. Southard for the Society for Industrial and Applied Mathematics, and J. S. Frame for the Pi Mu Epsilon Fraternity.

A resolution prepared by Professor W. L. Hart was unanimously adopted expressing sincere appreciation of the hospitality shown by Pennsylvania State University, and presenting cordial thanks to Professor Evan Johnson and his Committee on Arrangements for their efforts, which commenced long in advance and ensured the success of the meetings.

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 52 persons have been elected to membership by the Board of Governors on applications duly certified.

RICHARD P. AMENT, A.B. (Cornell U.) Instr., College of Wooster.

JOHN P. ANDERSEN, Student, University of Omaha.

ALLENE B. ARCHER, M.Ed. (Virginia) Professor, Maryland State Teachers College, Towson.

DAN B. AYDELOTT, B.S. (Ozarks) Nuclear Engr., Convair, Ft. Worth, Texas.

ROBERT C. BALENTINE, Student, Baylor University.

DONALD L. BARNETT, Student, University of Missouri.

STANLEY D. BATTLE, Student, Baylor University.

HOMER F. BECHTELL, JR., M.S. (Wisconsin) Teaching Asst., University of Wisconsin.

JEROME G. BEERY, Student, Kansas State College.

WILLIAM M. BOLING, Student, Baylor University.

CHARLES M. BRADEN, M.S. (Minnesota) Asst. Professor, Macalester College.

FREDERICK W. BRUNDAGE, B.A. (Alfred) Grad. Student, University of Rochester.

CHARLES W. BURMEISTER, Student, Baylor University.

JAMES T. BYRNE, Student, Kent State University.

KENNETH C. CARTWRIGHT, B.S.E.E. (Purdue) Vandervoort, Arkansas.

LEONARD A. CASCIOTTI, Student, Pennsylvania State University.

MRS. ANGELA L. CHANG, B.A. (Alfred) Jr. Math., Cornell Aeronautical Lab., Buffalo, N. Y.

JAMES D. CHURCH, B.A. (Nebraska) University of Nebraska.

JOSEPHINE A. CURRAN, A.B. (George Washington) Chevy Chase, Maryland.

MAHLON M. DAY, PH.D. (Brown) Professor, University of Illinois.

JOHN W. DEFORD, Student, Carleton College.

SUDHISH G. GHURYE, PH.D. (North Carolina) Asst. Professor, University of Chicago.

ALLEN A. GOLDSTEIN, PH.D. (Georgetown) Design Specialist, Convair Astronautics, San Diego, Calif.

WILLIAM B. GRAGG, JR., Student, University of Denver.

ELI HELLERMAN, B.S. (George Washington) Math.-Programmer, Council for Economic and Industry Research, Arlington, Va.

THOMAS J. HILL, B.S., M.ED. (Oklahoma) Teacher, Classen High School, Oklahoma City, Okla.

EARL D. HILLER, M.S. (Oklahoma) Instr., Oklahoma City University.

JAMES T. HODDE, Student, Baylor University.

DONALD G. HOOK, B.S. (Huron) Grad. Asst., South Dakota State College.

GEORGE W. HORTON, JR., M.S. (Oklahoma A. & M.C.) Asst. Professor, Upper Iowa University.

NORMAN HOSAY, B.S. (Wayne) Grad. Student, University of Wisconsin.

JOHN L. HUBISZ, JR., Student, St. Francis Xavier University.

QUINTIN C. JOHNSON, Student, St. Olaf College.

DWAYNE S. KING, Student, Baylor University.

LOUIS I. LARSON, B.S. (North Dakota) Statistician, North Dakota Geological Survey, Grand Forks.

RALPH L. LONDON, Student, Washington and Jefferson College.

EARL D. LOWERY, M.S. (North Carolina C.) Math., Ballistic Research Lab., Aberdeen Proving Ground, Md.

OSCAR D. LÜSCHER, B.C.L. (Montevideo) Asesor Cientifico, Centro Cooperacion Cientifica de Unesco para America Latina, Uruguay.

GUY K. MAGNUSON, Student, Beloit College.

MARK MAHOWALD, Ph.D. (Minnesota) Math., General Electric Corp., Cincinnati, Ohio.

DAVID MAZKEWITSCH, Ph.D. (Bern) Instr., University of Cincinnati.

M. EVANS MUNROE, Ph.D. (Brown) Asso. Professor, University of Illinois.

EDWARD W. NICHOLS, Roanoke, Virginia.

GENE B. PARRISH, M.A. (North Carolina) Chief, Mechanics Branch, Mathematical Sciences Div., Office of Ordnance Research, Durham, N. C.; Grad. Student, Duke University.

BENOÎT PROVENCHER, Student, University of Montreal.

FRANK D. QUIGLEY, Ph.D. (Chicago) Asst. Professor, Yale University.

ROBERT W. RITCHIE, Student, Reed College.

JOHN V. RYFF, Student, Syracuse University.

THOMAS H. SLOOK, M.A. (Pennsylvania) Instr., Temple University.

IRVING C. STATLER, Ph.D. (Calif. I.T.) Head, Aeronautical Engineering Section, Cornell Aeronautical Lab., Buffalo, N. Y.

HOMER L. TERWILLIGER, B.S. (Montana S. C.) Instr., Montana State College.

EILEEN L. TING, Student, Seattle University.

THE EIGHTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The eighteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, February 8, 1958.* This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed out early in December to the regular mailing list. If an application blank is not received by December 15, you may secure one from the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio by a postcard request. Your application must be filed with the director not later than January 15, 1958. For further details of the examination and the list of prizes (including the $2500 scholarship at Harvard) see the announcement which will be mailed out along with the application blank.

Reports of the seventeen previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952, 1953, 1954, 1955, December 1956, and August–September (announcement of winners) and November (questions and solutions) 1957.

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section, Mathematical Association of America, was held at Muhlenberg College on November 24, 1956 with Professor John Oxtoby, Bryn Mawr College, Chairman of the Section, presiding. There were 64 present including 50 members of the Association.

The following slate of officers was elected: Chairman, Professor Albert Wilansky, Lehigh University; Members Executive Committee, Professor C. W. Saalfrank, Lafayette College (two years to complete the unexpired term of N. J. Fine), Professor E. R. Mullins, Jr., Swarthmore College (three years). Revisions of the By-Laws were approved.

The matter of Section participation in the proposed National Contest for High Schools was discussed and the following resolution adopted:

The Philadelphia Section of the Mathematical Association of America agrees in principle with the establishment of a National Contest for High Schools. The Section believes that considerable autonomy should be left with the Sections to determine the particular form of organization most suitable for the Section within the national regulations.

Professor G. C. Webber discussed briefly the examination for high school students given in Delaware in the spring of 1956 under partial sponsorship of the Mathematical Association of America.

The following papers were presented at this meeting:

1. *Integers*, by Professor Peter Scherk, Universities of Saskatchewan and Pennsylvania, introduced by the Secretary.

The author discussed sets of integers, their sums and counting functions. Some early results by Landau and Besicovitch were reviewed and more recent results and problems connected with Mann's theorem were surveyed (cf. *Ann. Math.*, vol. 43, 1942, pp. 523–527).

2. *How to tell that a simple overhand knot is really knotted*, by Professor E. E. Moise, University of Michigan and Institute for Advanced Study.

This was an expository talk, giving a proof that a trefoil knot does not have the knot-type

* In order to get out the results of the Putnam Competition in time for it to be of maximum use, it is probable that the nineteenth competition will be held about December 1, 1958.

of the circle. The method was to show that the fundamental group of the complement of the trefoil is not commutative.

3. *On the Cauchy criterion for the convergence of an infinite series,* by Professor Albert Wilansky, Lehigh University.

Proofs were given of a theorem of S. Mazur and W. Orlicz, *Studia Mathematica,* vol. 14, 1954, Theorem 4.3 p. 158, to the effect that convergence of a sequence x cannot be concluded from a countable set of restrictions on the sequence of the form "$x_{p_n}-x_{q_n}\to 0$, p_n, q_n being increasing sequences of indices"; or, more generally, of the form "Ax is a null sequence, x being a column vector and A a matrix which transforms each convergent sequence into a null sequence." The general principle of convergence supplies a set of such conditions with the power of the continuum.

4. *Impossibility of computational algorithms for group-theoretic problems,* by Dr. M. O. Rabin, Princeton University, introduced by the Secretary.

G. C. WEBBER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 10, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, November 30, 1957, Harvard College, Haverford, Pennsylvania.

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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DECEMBER 1957

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

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THE CONVERSE OF FERMAT'S THEOREM

RAPHAEL M. ROBINSON, University of California, Berkeley

1. **Basic theorems.** In this paper, we discuss some of the known results concerning the use of the converse of Fermat's theorem as a test for primeness, and prove several results, believed to be new, which clarify certain points and also delimit to some extent the nature of possible improvements. The proofs of the known results are included, so that only the standard material of elementary number theory is presupposed.

Fermat's theorem states that if N is prime and $(a, N)=1$, then

$$a^{N-1} \equiv 1 \pmod{N}.$$

Alternatively, we may say that $a^N \equiv a \pmod{N}$ for any value of a, but we prefer the first form. It is well known that this congruence is not a sufficient condition for N to be prime, even if we exclude trivial cases such as $a=\pm 1$. For example, we have $2^{10}-1=3\cdot 341$, hence $2^{340}\equiv 1 \pmod{341}$, although $341=11\cdot 31$ is not prime.

An interesting class of numbers, not all prime, satisfying Fermat's congruence with $a=2$, was given by Malo [9], namely the Mersenne numbers. If n is prime, then $2^n\equiv 2 \pmod{n}$. Putting $N=2^n-1$, we have $2^n\equiv 1 \pmod{N}$ and $N\equiv 1 \pmod{n}$, hence $2^{N-1}\equiv 1 \pmod{N}$. As pointed out by Sierpiński [16], we do not use the fact that n is prime, but only that $2^n\equiv 2 \pmod{n}$. Thus if n satisfies the congruence $2^{x-1}\equiv 1 \pmod{x}$, so also does $N=2^n-1$. Furthermore, if n is composite, so also is N. This furnishes a simple proof that there are infinitely many composite solutions.

Despite this, Fermat's congruence is important for testing numbers for primeness. It is of course a necessary condition, and it serves to eliminate most composite numbers. One possibility, used, for example, by Lehmer [5], [6], is to make a list of the exceptional cases in a certain range. But, so far as possible, it is desirable to give supplementary conditions which insure that N is prime, or, failing that, at least to give restrictions on the form of the factors.

The following three theorems are of that type. Theorem 1 is essentially contained in Lucas [7], p. 66, and [8], pp. 161–162. (Other references to the early literature may be found in Dickson [1], both in the section on the converse of Fermat's theorem, pp. 91–95, and in the chapter on Fermat numbers, pp. 375–380.) Theorem 2 is due to Lehmer [2] and [3]. I have not seen Theorem 3 in the form given, though of course it is closely related to the known results.

It should perhaps be emphasized that such direct tests for primeness are applicable to much larger numbers than one can be sure of factoring by trial. Numbers as large as $2^{2281}-1$ have been identified as prime by the author [14], whereas much smaller numbers, such as $2^{101}-1$ (which is known to be composite), have resisted factorization. In both cases, use was made of high-speed computers.

THEOREM 1 (Lucas). *If* $N>1$ *and*

$$a^{N-1} \equiv 1 \pmod{N},$$

but

$$a^{(N-1)/q} \not\equiv 1 \pmod{N}$$

for each prime factor q *of* $N-1$, *then* N *is prime.*

Proof. It follows from the hypotheses that a belongs to the exponent $N-1$ mod N, and hence that $N-1 | \phi(N)$. But if N were composite, we would have $\phi(N) < N-1$.

THEOREM 2 (Lehmer). *Let* $N = kq^n+1$, *where* $k>0$, $n>0$, *and* q *is prime. Suppose that*

$$a^{N-1} \equiv 1 \pmod{N}.$$

Then every prime factor p *of* N *which does not divide* $a^{(N-1)/q}-1$ *satisfies the congruence*

$$p \equiv 1 \pmod{q^n}.$$

In particular, if $(a^{(N-1)/q}-1, N)=1$, *then every prime factor* p *of* N *satisfies this congruence.*

Proof. Suppose that a belongs to the exponent d mod p. Then $d | N-1$ but $d \nmid (N-1)/q$, that is, $d | kq^n$ but $d \nmid kq^{n-1}$. It follows that $q^n | d$. Since $d | p-1$, we see that $q^n | p-1$.

THEOREM 3. *Let* $N = kQ+1$, *where* $0<k<Q$. *Suppose that*

$$a^{N-1} \equiv 1 \pmod{N},$$

but

$$(a^{(N-1)/q} - 1, N) = 1$$

for every prime factor q *of* Q. *Then* N *is prime.*

Proof. If p and q are primes such that $p | N$ and $q^n | Q$, then by Theorem 2 we have $p \equiv 1 \pmod{q^n}$. It follows that $p \equiv 1 \pmod{Q}$. Thus

$$p^2 > Q^2 \geqq (k+1)Q > N.$$

That is, for every prime $p | N$ we have $p^2 > N$. It follows that N is prime.

Remark. Comparing Theorems 1 and 3, we see that the former theorem requires, for each prime divisor q of $N-1$, that $a^{(N-1)/q}-1$ should not be a multiple of N, whereas the latter imposes the more stringent condition that this quantity should be prime to N, but for fewer values of q. If we can write N in the form $N = kq^n+1$ with $0<k<q^n$, then only this single value of q need be considered.

The most striking advantage of Theorem 3 over Theorem 1 is that it does not require the complete factorization of $N-1$ to be known. However, we still require a partial factorization in which the factored portion of $N-1$ exceeds the unfactored portion.

It may be remarked that there is no known practical method of testing a number N for primeness, if N is too large to be factored by trial, which is free from such a restriction. It is possible, however, by using the Lucas functions, to make use of a factorization of $N+1$ instead of $N-1$; see, for example, Lehmer [4] and Robinson [14].

2. A more precise result. In this section, we prove a theorem which on the one hand extends Theorem 2, and on the other shows that no improvement of a certain type is possible.

LEMMA. *Suppose that a belongs to the exponent d* mod p^l, *where p is prime and* $l>0$. *If $p\nmid d$, then a also belongs to the exponent d* mod p.

Proof. Suppose that a belongs to the exponent e mod p. Then clearly $e\mid d$. On the other hand, by raising the congruence $a^e\equiv 1 \pmod{p}$ to the pth power $l-1$ times, we find that

$$a^{ep^{l-1}} \equiv 1 \pmod{p^l},$$

since the exponent of p in the modulus can be increased by one unit each time. Hence $d\mid ep^{l-1}$. Since $p\nmid d$, this yields $d\mid e$, and hence $d=e$.

THEOREM 4. *Let* $N=kq^n+1$, *where* $k>0$, $n>0$, *q is prime, and* $q\nmid k$. *Then there exists a number a such that*

$$a^{N-1} \equiv 1 \pmod{N}$$

and

$$(a^{(N-1)/q} - 1, N) = \delta,$$

if and only if $\delta\mid N$, $(\delta, N/\delta)=1$, *and every prime factor p of N/δ satisfies the congruence*

$$p \equiv 1 \pmod{q^n}.$$

Proof. Suppose first that such an a exists. Then clearly $\delta\mid N$. Now choose any prime p such that $p\mid \delta$, and take the largest l such that $p^l\mid N$. Suppose that a belongs to the exponent d mod p^l. Then $d\mid N-1$, hence $p\nmid d$. Thus, by the lemma, a also belongs to the exponent d mod p. Since $a^{(N-1)/q}\equiv 1 \pmod{p}$, we will also have $a^{(N-1)/q}\equiv 1 \pmod{p^l}$, so that $p^l\mid \delta$. Hence $p\nmid N/\delta$, and therefore $(\delta, N/\delta)=1$.

Next choose any prime p which divides N/δ. Then $p\nmid \delta$, that is, $p\nmid a^{(N-1)/q}-1$, and hence, by Theorem 2, we have $p\equiv 1 \pmod{q^n}$.

Now consider the converse. Supposing that the stated conditions are satisfied, we must show how to choose a. The value of a mod p^l may be assigned

independently for each prime power p^l in the canonical factorization of N. The required conditions are

$$a^{(N-1)/q} \equiv 1 \pmod{p^l} \text{ if } p^l \mid \delta,$$

$$a^{N-1} \equiv 1 \pmod{p^l}, \text{ but } a^{(N-1)/q} \not\equiv 1 \pmod{p}, \text{ if } p^l \mid N/\delta.$$

The first condition may be satisfied by taking $a \equiv 1 \pmod{p^l}$ whenever $p^l \mid \delta$, which amounts to supposing that $a \equiv 1 \pmod{\delta}$. If $p^l \mid N/\delta$, it will be sufficient to choose for the value of a mod p^l a number which belongs to the exponent q^n mod p^l. This is possible since $q^n \mid p-1$, hence $p > 2$, so that there is a primitive root mod p^l, and hence a number belonging to any prescribed exponent which divides $\phi(p^l)$, for example q^n. Now since $p \nmid q^n$, we see that p also belongs to the exponent q^n mod p. Since $q^n \mid N-1$, but $q^n \nmid (N-1)/q$, we see that the required congruence and incongruence are both satisfied.

Remark. Theorem 4 gives an improvement of Theorem 2, in that the conclusion $p \equiv 1 \pmod{q^n}$ is drawn for all prime factors p of N/δ rather than for the apparently narrower class of factors of N not dividing δ. This result was once stated by Lehmer [**3**], Theorem 4, but was later weakened to that given in Theorem 2 above (see [**4**], p. 443, footnote), which was all that his proof justified. Of course, the proof shows that the two classes are actually the same.

The most interesting aspect of Theorem 4, however, is the converse. Restricting ourselves to the simplest and most usual case $\delta = 1$, the theorem states that there exists a number a such that $a^{N-1} \equiv 1 \pmod{N}$, but $a^{(N-1)/q} - 1$ is prime to N, if and only if every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{q^n}$. Thus the last sentence of Theorem 2 draws the strongest possible conclusion from the mere existence of such an a. Of course, if the value of a is known, further conclusions might be drawn. One example of such a conclusion is given in Theorem 6 below.

3. Euler's criterion. According to Euler, if N is prime, then

$$a^{(N-1)/2} \equiv (a/N) \pmod{N},$$

where on the right we have a Legendre symbol. This is usually considered as a criterion for a to be a quadratic residue mod N. Interpreting the right side as a Jacobi symbol, we shall see that this congruence is also satisfied for some composite values of N. Notice that, for $(a, N) = 1$, it implies $a^{N-1} \equiv 1 \pmod{N}$, so that it is certainly as strong a condition on N as Fermat's congruence. It is in fact more restrictive. We shall find that it is very useful in testing N for primeness, at least when $(a/N) = -1$. We shall study this case in some detail.

THEOREM 5. *Let $N = k \cdot 2^n + 1$, where $k > 0$, $n > 0$, and k is odd. Then there exists an a such that*

$$a^{(N-1)/2} \equiv -1 \pmod{N},$$

if and only if every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{2^n}$.

Proof. If a satisfies the condition stated, then $a^{N-1}\equiv 1 \pmod N$ and $(a^{(N-1)/2}-1,\ N)=1$. The conclusion follows from Theorem 2. Conversely, by Theorem 4, if the prime factors of N satisfy the given condition, then some a will satisfy the conditions just mentioned. Thus for each prime power $p^l \mid N$ we will have $a^{N-1}\equiv 1 \pmod{p^l}$ but $a^{(N-1)/2}\not\equiv 1 \pmod{p^l}$, hence (since p is odd) $a^{(N-1)/2}\equiv -1 \pmod{p^l}$, and so $a^{(N-1)/2}\equiv -1 \pmod N$.

Remark. Thus from the existence of a such that $a^{(N-1)/2}\equiv -1 \pmod N$, we can conclude only that every prime factor of N satisfies the condition $p\equiv 1 \pmod{2^n}$. However if the value of a is known, it may be possible to draw further conclusions. The next theorem furnishes an example of this.

THEOREM 6. *Let $N=k\cdot 2^n+1$, where $k>0$, $n>0$, and k is odd. Suppose that*

$$a^{(N-1)/2} \equiv -1 \pmod N.$$

Then for every prime factor p of N we have not only $p\equiv 1 \pmod{2^n}$, but also

$$(a/p) = (-1)^{(p-1)/2^n}.$$

Proof. Suppose that a belongs to the exponent d mod p. Then $d\mid N-1$ but $d\nmid (N-1)/2$, hence d is an odd multiple of 2^n, and thus also $2^n\mid p-1$ (as previously shown). We must have

$$a^{d/2} \equiv -1 \pmod p,$$

and hence, raising both sides to the power $(p-1)/d$,

$$(a/p) \equiv a^{(p-1)/2} \equiv (-1)^{(p-1)/d} \pmod p.$$

Thus

$$(a/p) = (-1)^{(p-1)/d} = (-1)^{(p-1)/2^n}.$$

THEOREM 7. *If $N>1$ is odd, and*

$$a^{(N-1)/2} \equiv -1 \pmod N,$$

then $(a/N)=-1$.

Proof. We may put $N=k\cdot 2^n+1$, where $k>0$, $n>0$, and k is odd. Let the canonical factorization of N be

$$N = p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r}.$$

Put $p_i-1=2^n s_i$. Then

$$N = \prod_{i=1}^{r} p_i^{l_i} = \prod_{i=1}^{r} (1+2^n s_i)^{l_i},$$

hence

$$N \equiv 1 + 2^n \sum_{i=1}^{r} l_i s_i \pmod{2^{2n}}.$$

It follows that $\sum l_i s_i$ is odd. Now by Theorem 6, we have $(a/p_i) = (-1)^{s_i}$. Hence

$$(a/N) = \prod_{i=1}^{r} (a/p_i)^{l_i} = \prod_{i=1}^{r} (-1)^{l_i s_i} = -1.$$

Remark. On the other hand, from $a^{(N-1)/2} \equiv 1 \pmod{N}$, it does not follow that $(a/N) = 1$, as the example $a = 2$, $N = 341$ shows.

THEOREM 8. *If* $N = (a^{2^m} + 1)/2$, *where* $a \equiv \pm 3 \pmod{8}$ *and* $m > 0$, *then* $a^{(N-1)/2} \equiv -1 \pmod{N}$.

Proof. Since $a^2 \equiv 1 \pmod{8}$, but $a^2 \not\equiv 1 \pmod{16}$, we find by squaring $m-1$ times that

$$a^{2^m} \equiv 1 \pmod{2^{m+2}}, \qquad a^{2^m} \not\equiv 1 \pmod{2^{m+3}},$$

since the permissible power of 2 in the modulus increases by just one unit each time. Thus $a^{2^m} = k \cdot 2^{m+2} + 1$, where k is odd, and hence $N = k \cdot 2^{m+1} + 1$. Now

$$a^{2^m} = 2N - 1 \equiv -1 \pmod{N},$$

hence

$$a^{(N-1)/2} = a^{k \cdot 2^m} \equiv (-1)^k \equiv -1 \pmod{N}.$$

Remark. This theorem furnishes an interesting example of a class of numbers N which are not all prime, but all of which nevertheless satisfy Euler's criterion. In particular, for $a = 3$ and $m = 3$, we find that the composite number

$$N = 3281 = 205 \cdot 2^4 + 1 = (2^4 + 1)(12 \cdot 2^4 + 1) = 17 \cdot 193$$

satisfies the congruence $3^{(N-1)/2} \equiv -1 \pmod{N}$.

4. Tests for primeness using Euler's criterion. The following theorem was stated by Proth [13], but without any indication of a proof. The special case $k = 1$ had been proved earlier by Pepin [12].

THEOREM 9 (Proth). *Let* $N = k \cdot 2^n + 1$, *where* $0 < k < 2^n$. *Suppose that* $(a/N) = -1$. *Then N is prime, if and only if*

$$a^{(N-1)/2} \equiv -1 \pmod{N}.$$

Proof. If N is prime, then the congruence is satisfied, according to Euler. Now consider the converse. We may suppose that k is odd. By Theorem 5, every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{2^n}$. Hence $p^2 > 2^{2n} \geq (k+1) \cdot 2^n > N$. It follows that N is prime. This conclusion can also be drawn by specializing Theorem 3.

COROLLARY. *Let $N=k\cdot 2^n+1$, where $n>1$, $0<k<2^n$, and $3\nmid k$. Then N is prime, if and only if*

$$3^{(N-1)/2} \equiv -1 \pmod{N}.$$

Proof. This is trivial if $3|N$, since $N>3$ is not prime and the congruence is false. Otherwise, we must have $N\equiv 2 \pmod 3$ as well as $N\equiv 1 \pmod 4$, hence $(3/N)=(N/3)=(2/3)=-1$.

Remark. Thus, in many cases, we can take $a=3$ in Theorem 9. But in any case, a suitable a is easily found. This criterion is a very practical test for primeness, and when applicable appears to be the easiest known test. It was used (in the Pepin special case) by Morehead and Western [10], [11] to show that the Fermat numbers $F_7=2^{128}+1$ and $F_8=2^{256}+1$ are composite, and by the author [14] to show that $F_{10}=2^{1024}+1$ is composite. More recently, the theorem in its general form has been used by the author [15] to identify hundreds of large primes.

We may raise the question, to what extent the restriction $k<2^n$ in Theorem 9 is necessary. The following theorem provides the most complete answer which I have been able to find to this question.

THEOREM 10. *Let $N=k\cdot 2^n+1$, where $n>2$ and $0<k<6\cdot 2^n+7$. Suppose that $(a/N)=-1$. Then N is prime, if and only if*

$$a^{n-1/2} \equiv 1 \pmod{N}.$$

The conclusion is no longer valid, if we extend the range of k to $0<k\leqq 6\cdot 2^n+7$.

Proof. Assuming that the congruence holds, we have to show that N is prime. We may suppose that k is odd. By Theorem 5, every prime factor p of N must satisfy the condition $p\equiv 1 \pmod{2^n}$. Since $N<(2^n+1)^3$, we see that if N is composite, it must be the product of two primes, $N=(x\cdot 2^n+1)(y\cdot 2^n+1)$. It follows that $k=xy\cdot 2^n+x+y$. Hence $x+y$ is odd, and $xy\cdot 2^n+x+y<6\cdot 2^n+7$. We may suppose that x is odd and y is even. Then the only values of x and y satisfying the inequality are

$$x=1,\ y=2; \qquad x=1,\ y=4; \qquad x=3,\ y=2.$$

Now 2^n+1 and $2^{n+1}+1$ cannot both be prime unless $n=1$, and 2^n+1 and $2^{n+2}+1$ cannot both be prime unless $n=2$, since the exponents must be powers of 2. Also, $3\cdot 2^n+1\equiv 0 \pmod 5$, whenever $n+1\equiv 0 \pmod 4$, so that $3\cdot 2^n+1$ and $2^{n+1}+1$ cannot both be prime if $n>1$. Thus in none of the possible cases can $x\cdot 2^n+1$ and $y\cdot 2^n+1$ both be prime if $n>2$. It follows that N must be prime.

On the other hand, if we allow $k=6\cdot 2^n+7$, then for $n=4$ we have

$$N = 103\cdot 2^4+1 = (2^4+1)(6\cdot 2^4+1) = 17\cdot 97,$$

where both factors are prime. By Theorem 5, there must be some value of a such that $a^{(N-1)/2}\equiv -1 \pmod{N}$. (In fact, this congruence is satisfied if $a=12$.)

By Theorem 7, we will also have $(a/N) = -1$.

Remark. We can further relax the bound on k, when n is sufficiently large, if and only if there are only a finite number of values of n for which both 2^n+1 and $6\cdot 2^n+1$ are prime. This appears likely, but I cannot prove it.

References

1. L. E. Dickson, History of the Theory of Numbers, vol. 1, 1919.
2. D. H. Lehmer, Tests for primality by the converse of Fermat's theorem, Bull. Amer. Math. Soc., vol. 33, 1927, pp. 327–340.
3. ———, A further note on the converse of Fermat's theorem, *ibid.*, vol. 34, 1928, pp. 54–56.
4. ———, An extended theory of Lucas' functions, Ann. of Math., ser. 2, vol. 31, 1930, pp. 419–448.
5. ———, On the converse of Fermat's theorem, this MONTHLY, vol. 43, 1936, pp. 347–354.
6. ———, On the converse of Fermat's theorem II, *ibid.*, vol. 56, 1949, pp. 300–309.
7. E. Lucas, Sur la recherche des grands nombres premiers, Association Française pour l'Avancement des Sciences, Compte-Rendu, vol. 5, 1876, pp. 61–68.
8. ———, Considérations nouvelle sur la théorie des nombres premiers et sur la division géométrique de la circonférence en parties égales, *ibid.*, vol. 6, 1877, pp. 159–167.
9. E. Malo, Nombres qui, sans être premiers, vérifient exceptionellement une congruence de Fermat, L'Intermédiaire des Mathématiciens, vol. 10, 1903, p. 88.
10. J. C. Morehead, Note on Fermat's numbers, Bull. Amer. Math. Soc., vol. 11, 1905, pp. 543–545.
11. J. C. Morehead and A. E. Western, Note on Fermat's numbers, *ibid.*, vol. 16, 1909, pp. 1–6.
12. T. Pepin, Sur la formule $2^{2^n}+1$, C. R. Acad. Sci. Paris, vol. 85, 1877, pp. 329–331.
13. F. Proth, Théorèmes sur les nombres premiers, *ibid.*, vol. 87, 1878, p. 926.
14. R. M. Robinson, Mersenne and Fermat numbers, Proc. Amer. Math. Soc., vol. 5, 1954, pp. 842–846.
15. ———, Factors of Fermat numbers, Math. Tables Aids Comput., vol. 11, 1957, pp. 21–22.
16. W. Sierpiński, Remarque sur une hypothèse des Chinois concernant les nombres $(2^n-2)/n$, Colloq. Math., vol. 1, 1947, p. 9.

A MOVING BOUNDARY FILTRATION PROBLEM OR "THE CIGARETTE PROBLEM"*

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In this paper a study is made of filtration through a burning cigarette. Let us consider a uniform cigarette being smoked under the following conditions:

(1) Steady inhalation.
(2) A constant fraction $1-a$ of any component Z in the tobacco is burnt at the tip (the fraction a being transmitted and filtered down the cigarette).

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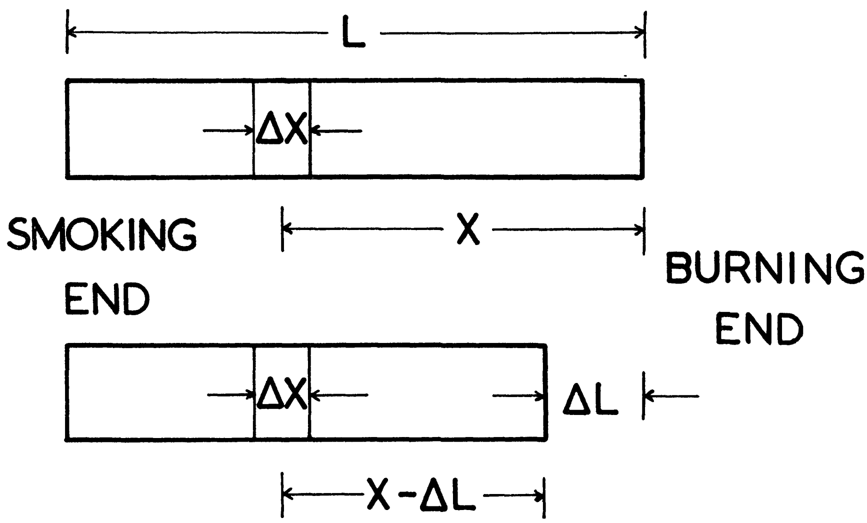
(3) The absorption coefficient of the tobacco is constant and equal to b with respect to the filtration of component Z.

Let $C(x, L)$ denote the concentration of component Z in unit weight per unit length for a cigarette of length L at a distance x from the burning end. The



amount of component Z transmitted down the cigarette after burning ΔL of cigarette is, to first order terms, $\Delta A = aC(0, L)\Delta L$. The fraction of ΔA filtered out by element Δx is

$$\Delta(\Delta A) = \Delta C \cdot \Delta x = abe^{-bx}C(0, L)\Delta L \cdot \Delta x. \tag{4}$$

This follows since the filtration law is $dA/dx = -bA$ or $A = A_0 e^{-bx}$, and thus, to first order terms, $\Delta A = bA\Delta x$.

Equation (4) can be rewritten as

$$\frac{C(x - \Delta L, L - \Delta L) - C(x, L)}{\Delta L} = abC(0, L)e^{-bx}.$$

On letting $\Delta L \to 0$, we obtain the mixed partial differential equation of first order,

$$\frac{\partial C(x, L)}{\partial x} + \frac{\partial C(x, L)}{\partial L} = -abC(0, L)e^{-bx}, \tag{5}$$

subject to the initial condition $C(x, L_0) = C_0$ (constant), where L_0 is the initial length of the cigarette.

To solve (5) we assume a solution of the form $C(x, L) = C_0 + \phi(L)e^{-bx}$, where $\phi(L_0) = 0$. This transforms (5) into $\phi'(L) - b(1-a)\phi(L) = -abC_0$. Thus,

$$C(x, L) = C_0 + \frac{C_0 a e^{-bx}}{1 - a}\left[1 - e^{-b(1-a)(L_0-L)}\right].$$

The amount of component Z transmitted to the smoker after burning ΔL of the cigarette is, to first order terms, $\Delta A_t = aC(0, L)e^{-bL}\Delta L$. Thus the total amount transmitted when the cigarette has been burned down to a length L_f is

$$\begin{aligned} A_t &= \int_{L_f}^{L_0} aC(0, L)e^{-bL}dL \\ &= \frac{aC_0e^{-bL_f}}{(1-a)b}\left[1 - e^{-b(1-a)(L_0-L_f)}\right] = \frac{C(L_f, L_f) - C_0}{b}. \end{aligned} \tag{6}$$

The total amount of Z destroyed by burning is

$$A_b = \int_{L_f}^{L_0} (1-a)C(0, L)dL = C_0(L_0 - L_f) - \frac{aC_0}{(1-a)b}\left[1 - e^{-b(1-a)(L_0-L_f)}\right].$$

The amount of Z left in the cigarette is

$$A_c = \int_0^{L_f} C(x, L_f)dx = C_0L_f + \frac{aC_0}{(1-a)b}\left[1 - e^{-bL_f}\right]\left[1 - e^{-b(1-a)(L_0-L_f)}\right].$$

It follows immediately that

$$A_t + A_b + A_c = C_0L_0. \tag{7}$$

Equation (7) expresses the material balance on component Z.

We can now determine the effect of the initial length L_0 on the amount A_t transmitted to the smoker. Let us consider two cigarettes of lengths L_0 and L_0', and identical otherwise. The amounts A_t and A_t' may be compared on one of the following two bases:

(B_1) Both cigarettes burn down the same amount.
(B_2) Both cigarettes burn down to the same final length.

In the latter case, we would consider the amount A_t per unit length of cigarette smoked.

On either basis, the longer cigarette is more effective in reducing the amount A_t transmitted to the smoker. This follows directly from (6) and the fact that $(1-e^{-x})/x$ is a monotonic decreasing function of x for $x>0$.

We shall now consider the effects of filtration alone on the amount A_t. We let $a \to 1$ (zero burning fraction). Then (6) reduces to $A_t = C_0e^{-bL_f}(L_0 - L_f)$. Using basis ($B_1$) for a comparison we get

$$\frac{A_t}{A_t'} = \frac{e^{-bL_f}}{e^{-bL'_f}}. \tag{8}$$

Thus, the initially longer cigarette is more effective in reducing the amount of A_t transmitted. However, if we now use basis (B_2), we get

$$(9) \qquad \frac{A_t}{L_0 - L_f} = \frac{A_t'}{L_0' - L_f} .$$

In this case, the filtering capacity is independent of the initial length. Since many smokers burn their cigarettes (be they regular or king size) to the same approximate final length before discarding them, basis (B_2) would be indicated. This latter result seems contrary to the claims of certain cigarette advertising.

For those cigarettes having a filter tip of length L_1 and absorption constant b_1, the only term which changes is A_t. In this case we multiply the right-hand side of (6) by $e^{-b_1 L_1}$. If the effective length of the filter tip is L_1' (sometimes part of the filter tip is hollow), then in order that the filter tip be more effective than the tobacco it replaces, we should have $b_1 > bL_1/L_1'$. If our theory is valid, this inequality was apparently not satisfied by many filter-tip cigarettes, since it has been shown experimentally by Consumers Union (*Consumer Reports*, February 1953 and March 1957) that insofar as reducing the nicotine content of the smoke is concerned, there is little to choose between filter-tip or nonfilter-tip cigarettes. However, the average filtered cigarette smoke does contain somewhat less tar than the unfiltered smoke. A direct illustration of this is given by a filter-tip cigarette first put out in 1952. This cigarette did a creditable job of filtering out a high percentage of both nicotine and tar from the tobacco smoke. By 1955 the filter had been more looosely packed for easier smoking, but at the expense of increasing the nicotine and tar levels approximately fourfold and sixfold, respectively, and thus bringing it back to the average filtering level of regular cigarettes.

It is unfortunate that the Consumers Union experiments did not include a determination of $C(x, L)$ after a cigarette had been smoked down. This could then have been used to check the validity of conditions (1), (2), (3), assumed here.

To approximate more closely to actual smoking conditions, we shall consider a cigarette being smoked in the following manner. The cigarette is first smoked a length ΔL_s under steady inhalation and then is allowed to burn a length ΔL_b without inhalation. This process is repeated until the cigarette is discarded.

We now have to solve the differential equation (5) subject to the nonconstant initial condition $C(x, L_0) = C(x)$, where $C(x)$ is a specified function. To solve (5) under this condition, we let

$$C(x, L) = e^{-bx}[R(x, L) + S(L)].$$

Then (5) becomes

$$\frac{\partial R(x, L)}{\partial x} + \frac{\partial R(x, L)}{\partial L} + S'(L) = b[R(x, L) + S(L)] - ab[R(0, L) + S(L)].$$

If we let

$$\frac{\partial R(x, L)}{\partial x} + \frac{\partial R(x, L)}{\partial L} = bR(x, L), \tag{10}$$

subject to the condition

$$R(x, L_0) = e^{bx}C(x), \tag{11}$$

then $S(L)$ must satisfy

$$\frac{dS(L)}{dL} - b(1 - a)S(L) = - abR(0, L), \tag{12}$$

subject to the initial condition $S(L_0) = 0$.

The subsidiary equations for solving (10) are $dx = dL = dR(x, L)/b$. Consequently, $R(x, L) = e^{bx}\psi(L - x)$, where $\psi(x)$ is an arbitrary function of x. To determine $\psi(x)$, we use (11). Thus we find that $R(x, L) = e^{bx}C(L_0 - L + x)$.

We can now solve (12), and the solution is

$$S(L) = abe^{b(1-a)L}\int_L^{L_0} e^{-b(1-a)L}C(L_0 - L)dL,$$

whence $C(x, L) = C(L_0 - L + x) + abe^{b(1-a)L-bx}\int_L^{L_0} e^{-b(1-a)L}C(L_0 - L)dL$.

If we now smoke a length ΔL_s under steady inhalation, the concentration function will be given by

$$C_1(x, L_1) = C_0 + abe^{-bx+b(1-a)L_1}\int_{L_1}^{L_0} e^{-b(1-a)L}C_0 dL,$$

where $L_1 = L_0 - \Delta L_s$. After burning a length ΔL_b, the concentration function will then be $C_1^*(x, L_2) = C_1(x + \Delta L_b, L_1)$. Here we have $L_2 = L_1 - \Delta L_b = L_0 - \Delta L_s - \Delta L_b$. On repeating another cycle we get

$$C_2(x, L_3) = C_1^*(L_2 - L_3 + x, L_2) + abe^{-bx+b(1-a)L_3}\int_{L_3}^{L_2} e^{-b(1-a)L}C_1^*(L_2 - L, L_2)dL,$$

$$C_2^*(x, L_4) = C_2(x + \Delta L_b, L_3).$$

By induction, we find that after N cycles,

$$\begin{aligned} C_N(x, L_{2N-1}) &= C_{N-1}^*(L_{2N-2} - L_{2N-1} + x, L_{2N-2}) \\ &\quad + abe^{-bx+b(1-a)L_{2N-1}}\int_{L_{2N-1}}^{L_{2N-2}} e^{-b(1-a)L}C_{N-1}^*(L_{2N-2} - L, L_{2N-2})dL, \\ C_N^*(x, L_{2N}) &= C_N(x + \Delta L_b, L_{2N-1}), \end{aligned}$$

where

$$L_{2N-1} = L_0 - N\Delta L_s - (N - 1)\Delta L_b, \qquad L_{2N} = L_0 - N\Delta L_s - N\Delta L_b.$$

The amount A_{t_N} transmitted to the smoker during the Nth cycle is given by

$$A_{t_N} = \int_{L_{2N-1}}^{L_{2N-2}} aC_{N-1}(0, L)e^{-bL}dL.$$

Hence the total amount A_T absorbed by the smoker after N cycles is

$$A_T = \sum_{n=1}^{N} A_{t_n} = \sum_{n=1}^{N} \int_{L_{2n-1}}^{L_{2n-2}} aC_{n-1}(0, L)e^{-bL}dL.$$

Again, in order to determine the effect of filtration alone, we set $a=1$. It then can be established by induction that

$$\frac{C_N(x, L)}{C_0} = 1 + b\Delta L_s e^{-bx}\left[\frac{e^{-(N-1)b\Delta L_b} - 1}{1 - e^{b\Delta L_b}} + \frac{L_0 - L}{\Delta L_s} - (N-1)\left(1 + \frac{\Delta L_b}{\Delta L_s}\right)\right],$$

and that also

$$A_T = C_0\Delta L_s e^{b(N\Delta L_s - L_0)}\left[\frac{1 - e^{Nb\Delta L_b}}{1 - e^{b\Delta L_b}}\right].$$

For a comparison of the filtration efficiency (under discontinuous inhalation) of two cigarettes, identical except for length, it is reasonable to keep ΔL_s and ΔL_b fixed. Then, using basis (B_1) we get

$$\frac{A_T}{A'_T} = \frac{e^{-bL_0}}{e^{-bL'_0}},$$

which corresponds to (8), and to which it reduces if $\Delta L_b = 0$. By using basis (B_2) we get

$$\left(\frac{A_T}{N\Delta L_s}\right)\bigg/\left(\frac{A'_T}{N'\Delta L_s}\right) = \left(\frac{1 - e^{-Nb\Delta L_b}}{N}\right)\bigg/\left(\frac{1 - e^{-N'b\Delta L_b}}{N'}\right).$$

Since $(1 - e^{-x})/x$ is a monotonic decreasing function, it follows that under either basis, (B_1) or (B_2), the longer cigarette is a better filter under discontinuous inhalation which more nearly approximates actual smoking conditions than continuous inhalation.

ANOTHER NOTE ON QUASI-IDEMPOTENT MATRICES

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Huff [1] and Marathe [2] have considered quasi-idempotent matrices. Huff calls the matrix A over the complex field K quasi-idempotent, if there exists a polynomial matrix $F(x)$ over $K[x]$ such that $A^r = F(r)$ for all positive integers r. Use of the canonical form of A permits us to decompose A into pieces, study the pieces separately, and then put the pieces back together again. This seems to give a clearer picture of the subject.

The Jordan canonical form of a matrix yields immediately that $A(A-I)^k=0$ for some positive integer k, if and only if A is similar to the direct sum $J_1 \dotplus \cdots \dotplus J_s \dotplus 0$, where J_i is the Jordan form of the companion matrix of $(x-1)^{s_i}=0$, $i=1, \cdots, s$. Thus $J_i - I_{s_i}$ is nilpotent of index s_i, *i.e.* $(J_i - I_{s_i})^{s_i-1} \neq 0$, but $(J_i - I_{s_i})^{s_i}=0$. For notational simplicity let us drop the subscripts and consider the nilpotent matrix $J-I$ of index k. Since $J=I+(J-I)$ and $(J-I)^k=0$, we have by the binomial expansion

$$J^n = I + n(J-I) + \frac{n(n-1)}{2!}(J-I)^2 + \cdots + \frac{n(n-1)\cdots(n-[k-2])}{(k-1)!}(J-I)^{k-1}$$

for all positive integers n. (In fact, the expansion holds for all real numbers n, as a finite expansion certainly converges. We understand that if $n=1/3$, say, the resulting matrix $J^{1/3}$ is a cube root of J. There are, of course, other cube roots of J.) Hence if $F(x)$ is the polynomial matrix,

$$(1)\qquad I + x(J-I) + \cdots + \frac{x(x-1)\cdots(x-[k-2])}{(k-1)!}(J-I)^{k-1},$$

we have that $J^r = F(r)$ for all real numbers r. Thus the matrix J (and hence the matrix A above) is quasi-idempotent.

Conversely if A is quasi-idempotent, let the Jordan canonical form of A be $J_1 \dotplus \cdots \dotplus J_t$. Then J_i is quasi-idempotent for $i=1, \cdots, t$. Say $G_i(x)$ is the polynomial matrix such that $J_i^r = G_i(r)$ for all positive integers r. Now J_i is nilpotent implies that $G_i(x)$ is the 0 matrix. Hence J_i is either 0 or nonsingular.

Let J be one of these nonsingular blocks and $G(x)$ the corresponding polynomial matrix whose elements are polynomials of degree $k-1$, say. If $F(x)$ is as defined in (1) above, then the elements of both $F(x)$ and $G(x)$ are polynomials in x of degree $k-1$ (or less). Since $G(i)=F(i)$ for $i=0, \cdots, k-1$, the polynomial elements of both $G(x)$ and $F(x)$ coincide and so $F(x)=G(x)$. But this implies that $(J-I)^n=0$ for $n>k-1$. Hence J is the Jordan form of the companion matrix of $(x-1)^k=0$.

Thus the necessary and sufficient condition that $A(A-I)^k=0$ for some

positive integer k is also a necessary and sufficient condition for A to be quasi-idempotent. Moreover $A^r=F(r)$ for all positive integers r implies $A^r=F(r)$ for all real numbers r. Note that if A is nonsingular $A^0=I$. But if A is singular (and quasi-idempotent), then $A=P(A_1\dotplus 0)P^{-1}$, where A_1 is nonsingular and quasi-idempotent, and $A^0=P(I\dotplus 0)P^{-1}$, where I is the unique identity for A_1.

An idempotent matrix is similar to the direct sum $I\dotplus 0$. A quasi-idempotent matrix, as noted above, allows the use of connecting 1's in the nonsingular portion of the direct sum. Thus any idempotent matrix of order 3 is similar to one of I, 0, $I_1\dotplus 0$, $I_2\dotplus 0$. Any quasi-idempotent matrix of order 3 which is not idempotent is similar to one of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In particular a singular quasi-idempotent matrix A of order 3 which is not idempotent is similar to $J\dotplus 0$, where

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since $(J-I)^2=0$, we have $J^n=I+n(J-I)$ for all real numbers n. Thus

$$F(x) = I + x(J - I) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

for the matrix J. If $A=P(J\dotplus 0)P^{-1}$, then

$$F(x) = P\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}P^{-1}$$

for the matrix A. If

$$P = \begin{bmatrix} -9 & -12 & -2 \\ -1 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad \text{then} \quad P^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 3 & -2 \\ 1 & 0 & 3 \end{bmatrix},$$

and we get the A and $F(x)$ of the example of Huff [**1**].

References

1. G. B. Huff, On quasi-idempotent matrices, this MONTHLY, vol. 62, 1955, pp. 334–339.

2. C. R. Marathe, A note on quasi-idempotent matrices, this MONTHLY, vol. 63, 1956, pp. 632–635.

SOME RESULTS ON RECURRENT EVENTS*

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The purpose of this paper is to develop and illustrate a fundamental theorem for recurrent events occurring at times t where t is a *continuous* real variable, analogous to the results obtained in [1] for the discrete case.

We shall consider the event E to be recurrent when the following conditions are met:

(a) There exists a rule which determines whether E does or does not occur at any time t.
(b) If E occurs at time t, then it occurs at time $t+\tau$, if and only if it occurs at time τ after the time axis has been shifted so that its origin is at t.
(c) If $u(t)dt$ is the element of probability that E occurs at time t, and $u(t, t+\tau)dtd\tau$ is the element of probability of the joint occurrence of E at times t and $t+\tau$, then $u(t, t+\tau)=u(t)u(\tau)$.

Let us denote by $f(t)dt$ the element of probability that a recurrent event E occurs *for the first time* at time t, the process starting at $t=0$, and let us write the generating functions corresponding to $u(t)$ and $f(t)$ as

$$U(\theta) = \int_0^\infty u(y)e^{\theta y}dy, \tag{1}$$

$$F(\theta) = \int_0^\infty f(x)e^{\theta x}dx. \tag{2}$$

FUNDAMENTAL THEOREM. *If* $U(\theta)$ *and* $F(\theta)$ *are defined as in* (1) *and* (2) *above, then*

$$F(\theta) = \frac{U(\theta)}{1 + U(\theta)} \cdot$$

If E is a recurrent event, it can easily be shown that $u(t)$ and $f(t)$ are related as follows,

$$u(t) = f(t) + \int_0^t u(t - x)f(x)dx. \tag{3}$$

Either E occurs for the first time at t or it occurs previously at time $t-x$, where $0\leqq x\leqq t$. Writing the product of $U(\theta)$ and $F(\theta)$ as a double integral, we get

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$$U(\theta)F(\theta) = \int_0^\infty \int_0^\infty u(y)f(x)e^{\theta(y+x)}dxdy, \tag{4}$$

and letting $t=y+x$ with x fixed, (4) becomes

$$U(\theta)F(\theta) = \int_0^\infty \int_0^t u(t-x)f(x)e^{\theta t}dxdt \tag{5}$$

$$= \int_0^\infty \left[\int_0^t u(t-x)f(x)dx \right] e^{\theta t}dt. \tag{6}$$

Substituting (3) for the quantity in brackets, we get

$$U(\theta)F(\theta) = \int_0^\infty [u(t) - f(t)]e^{\theta t}dt = U(\theta) - F(\theta). \tag{7}$$

Upon solving for $F(\theta)$, we get

$$F(\theta) = \frac{U(\theta)}{1 + U(\theta)}, \tag{8}$$

and this completes the proof of the theorem.

Example 1. Let us consider a Poisson process, for which the probability element of the occurrence of an event A at time t is λdt. (As can easily be checked, the occurrence of A is a recurrent event.) Since $u(t) = \lambda$, we have

$$U(\theta) = \int_0^\infty \lambda e^{\theta t}dt = -\frac{\lambda}{\theta} \ (\theta < 0),$$

and the fundamental theorem yields $F(\theta) = \lambda/(\lambda - \theta)$. It is readily verified that this is the generating function of $f(t) = \lambda e^{-\lambda t}$, and we, thus, have an alternative way of proving the well-known result that in a Poisson process the waiting times between successive occurrences are exponentially distributed.

Example 2. Let us consider two independent Poisson processes for which the element of probability of the occurrence of events A_1 and A_2 at time t are $\lambda_1 dt$ and $\lambda_2 dt$, respectively. The recurrent event which we shall consider is the event that either A_1 or A_2 occurs, and that with this occurrence A_1 and A_2 will have occurred an equal number of times. (This is analogous in the discrete case to two players being even in a sequence of independent two-person games with constant probability and constant pay-off.)

Using the known result that the probability element for the nth occurrence of A_1 at time t is

$$\frac{\lambda_1^n e^{-\lambda_1 t} t^{n-1}}{(n-1)!} dt,$$

and the probability that A_2 occurs exactly n times on the interval from 0 to t is $\lambda_2^n e^{-\lambda_2 t} t^n/n!$, we have by symmetry

$$u(t) = 2\sum_{n=1}^{\infty} \frac{(\lambda_1\lambda_2)^n e^{-(\lambda_1+\lambda_2)t} t^{2n-1}}{(n-1)!n!}. \tag{9}$$

Substitution into (1) yields

$$\begin{aligned} U(\theta) = \int_0^{\infty} u(t)e^{\theta t}dt &= \sum_{n=1}^{\infty}\binom{2n}{n}\left[\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^n \\ &= \left[1 - \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^{-1/2} - 1, \end{aligned} \tag{10}$$

and according to the fundamental theorem we finally get

$$F(\theta) = \frac{U(\theta)}{1+U(\theta)} = 1 - \left[1 - \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^{1/2}. \tag{11}$$

To find $f(t)$, let us first write $F(\theta)$ as

$$F(\theta) = \sum_{n=1}^{\infty} \frac{1}{2n-1}(-1)^n\binom{-\frac{1}{2}}{n}\left[\frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^n, \tag{12}$$

and the individual terms of the summation can readily be identified with the generating functions of gamma distributions. Using this fact we get

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{2n-1}\cdot\frac{(\lambda_1\lambda_2)^n e^{-(\lambda_1+\lambda_2)t} t^{2n-1}}{(n-1)!n!}. \tag{13}$$

It is of interest to note that if we write (9) as

$$u(t) = \sum_{n=1}^{\infty} g(n, \lambda_1, \lambda_2, t), \tag{14}$$

then (13) becomes

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{2n-1} g(n, \lambda_1, \lambda_2, t), \tag{15}$$

which parallels the result given in [1] for the discrete case.

If we refer to a recurrent event as *certain* if $\int_0^\infty f(t)dt = F(0) = 1$ and as *uncertain* if $\int_0^\infty f(t)dt = F(0) < 1$, it can readily be shown in our example that the recurrent event is *certain* if $\lambda_1 = \lambda_2$ and *uncertain* if $\lambda_1 \neq \lambda_2$.

Reference

1. W. Feller, An Introduction to Probability Theory and its Applications, vol. 1, New York, 1950.

NUMERICAL DIFFERENTIATION FORMULAS

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1. **Introduction.** In a recent paper R. T. Gregory [1] has described an algorithm for finding the coefficients in certain formulas, which express the derivatives of an interpolation polynomial at a set of equally spaced points, as linear combinations of the given functional values. This algorithm involves the solution of systems of linear equations, or equivalently, the inversion of certain Vandermonde matrices.

In the following, we obtain explicit representations for the same coefficients, and in such a form that their use is in no way restricted to the tabular points. These representations are in terms of Stirling numbers, which are well known and extensively tabulated. This in turn enables one to determine directly the matrix inverses mentioned above. The authors expect to present further applications of the above to more general matrices, and to some general interpolation problems.

2. **Numerical Differentiation Formulas.** Let values y_i be given at points $x_i = x_0 + ih$, $i = 0, 1, \cdots, n$. (For simplicity of notation we consider n as being fixed throughout.) Further, define the factorial polynomial of degree k by

$$p_k(x) = \sum_{j=1}^{k} S_k^j x^j,$$

where the S_k^j are the Stirling numbers of the first kind (*cf.*, for example, Jordan [2], p. 163). If $y = f(x)$ is the polynomial of degree n passing through the $n+1$ points (x_i, y_i), we have

$$h^k f^{(k)}(x) = h^k \frac{d^k f}{dx^k} = \sum_{j=k}^{n} \frac{1}{j!} \sum_{r=k}^{j} p_k(r) S_j^r m^{r-k} \Delta_{j/2}^{(j)},$$

where m is any real number and $x = x_0 + mh$ (Kopal [3], p. 98). Since $\Delta_{j/2}^{(j)}$ is independent of r, we have

$$h^k f^{(k)}(x) = \sum_{j=k}^{n} \frac{\Delta_{j/2}^{(j)}}{j!} C_{mj}^k,$$

with

$$C_{mj}^k = \sum_{r=k}^{j} p_k(r) S_j^r m^{r-k}. \tag{1}$$

Applying the relation $\Delta_{j/2}^{(j)} = \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} y_i$ ([2], p. 9), we obtain

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$$h^k f^{(k)}(x) = \sum_{j=k}^{n} \frac{C_{mj}^k}{j!} \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} y_i,$$

and, finally,

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i, \tag{2}$$

where

$$A_{mi}^k = \sum_{j=k}^{n} \frac{(-1)^{i+j}\binom{j}{i} C_{mj}^k}{j!}, \qquad \binom{j}{i} = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i > j \end{cases}. \tag{3}$$

If the values of y_i are those of a function $y=y(x)$ at the points $x_i=x_0+ih$, where the function has a continuous derivative of order $n+1$, we have

$$h^k y^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i + O(h^{n+1}).$$

This equation, along with (3) and (1), enables one to write formulas for the derivatives of $y(x)$, with coefficients of the y_i given explicitly. The derivatives of $y(x)$ at the tabular points may be found from the equation by setting $m = 0, 1, \cdots, n$.

3. A matrix inverse. Even though the determination of the coefficients A_{mi}^k is now complete, it is of interest to invert the Vandermonde matrix

$$M(m) = \begin{pmatrix} 1 & 1 & 1 & \cdot & 1 \\ -m & 1-m & 2-m & \cdot & n-m \\ (-m)^2 & (1-m)^2 & (2-m)^2 & \cdot & (n-m)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-m)^n & (1-m)^n & (2-m)^n & \cdot & (n-m)^n \end{pmatrix},$$

which appears in Gregory's algorithm. In fact, the algorithm can be extended and written in the form

$$\begin{pmatrix} A_{m0}^0 & A_{m0}^1 & \cdots & A_{m0}^n \\ A_{m1}^0 & A_{m1}^1 & \cdots & A_{m1}^n \\ \cdot & \cdot & \cdot & \cdot \\ A_{mk}^0 & A_{mk}^1 & \cdots & A_{mk}^n \\ \cdot & \cdot & \cdot & \cdot \\ A_{mn}^0 & A_{mn}^1 & \cdots & A_{mn}^n \end{pmatrix} = M^{-1}(m) \begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & \cdot & 0 \\ 0 & 1! & 0 & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & k! & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & \cdot & n! \end{pmatrix}.$$

If we now write $M^{-1}(m)=\{a^m_{\lambda\mu}\}$; $\lambda,\mu=1,2,\cdots,n+1$, we have immediately

$$a^{m}_{\lambda\mu} = \frac{A^{\mu-1}_{m,\lambda-1}}{(\mu-1)!},$$

where the $A^{\mu-1}_{m,\lambda-1}$ is given by (3) and (1). Computationally, the easily obtained identity

$$a^{m}_{\lambda,\mu} = (-1)^{\mu+1} a^{n-m}_{n+2-\lambda,\mu}$$

may be of interest. (For the special case where m is an integer, *cf.* [3], p. 91.)

References

1. R. T. Gregory, A method for deriving numerical differentiation formulas, this MONTHLY, vol. 64, 1957, pp. 79–82.
2. Charles Jordan, Calculus of Finite Differences, 2nd ed., New York, 1950.
3. Zdenek Kopal, Numerical Analysis, New York, 1955.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE PRODUCT OF CERTAIN POLYNOMIALS ANALOGOUS TO THE HERMITE POLYNOMIALS

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The Hermite polynomial $H_m(x)$ defined by $e^{2xt-t^2}=\sum_{n=0}^{\infty} H_m(x)t^m/m!$ satisfies the identities

$$(1)\qquad H_m(x)H_n(x) = \sum_{r=0}^{\min(m,n)} 2^r r!\binom{m}{r}\binom{n}{r} H_{m+n-2r}(x),$$

$$(2)\qquad H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-1)^r 2^r r!\binom{m}{r}\binom{n}{r} H_{m-r}(x)H_{n-r}(x).$$

These formulas are due to Nielsen [5, pp. 31–33]; (1) was rediscovered by Dhar [1] and Feldheim [4].

If we put

$$e^{3xt-t^3}=\sum_{n=0}^{\infty}\phi_m(x)\,\frac{t^m}{m!} \tag{3}$$

(compare [3], p. 268, formula (4)), then

$$\begin{aligned}\sum_{m=0}^{\infty}\phi_m(x)\frac{t^m}{m!}\sum_{n=0}^{\infty}\phi_n(x)\frac{z^n}{n!} &= e^{3x(t+z)-(t+z)^3}e^{3tz(t+z)} \\ &= \sum_{h=0}^{\infty}\phi_h(x)\,\frac{(t+z)^h}{h!}\sum_{k=0}^{\infty}\frac{3^k(tz)^k(t+z)^k}{k!} \\ &= \sum_{h=0}^{\infty}\phi_h(x)\sum_{k=0}^{\infty}3^k(tz)^k\,\frac{(t+z)^{h+k}}{h!k!} \\ &= \sum_{k,h=0}^{\infty}3^k\binom{h+k}{h}\phi_h(x)\sum_{r+s=h+k}\frac{t^{k+r}z^{k+s}}{r!s!}\,.\end{aligned}$$

Thus $k+r=m$, $k+s=n$, $r+s=h+k$, which yields $r=m-k$, $s=n-k$, $h=m+n-3k$. Hence we get

$$\frac{\phi_m(x)\phi_n(x)}{m!n!}=\sum_{k=0,\,3k\leq m+n}^{\min(m,n)}3^k\,\frac{(m+n-2k)!}{k!(m-k)!(n-k)!(m+n-3k)!}\cdot\phi_{m+n-3k}(x),$$

$$\phi_m(x)\phi_n(x)=\sum_{k=0,\,3k\leq m+n}^{\min(m,n)}3^kk!\binom{m}{k}\binom{n}{k}\frac{(m+n-2k)!}{(m+n-3k)!}\,\phi_{m+n-3k}(x). \tag{4}$$

For the proof compare Watson [6].

Similarly it follows from

$$\sum_{h=0}^{\infty}\phi_h(x)\,\frac{(t+z)^h}{h!}=\sum_{m=0}^{\infty}\phi_m(x)\frac{t^m}{m!}\sum_{n=0}^{\infty}\phi_n(x)\,\frac{z^n}{n!}\cdot\sum_{k=0}^{\infty}(-3)^k\,\frac{k(tz)^k(t+z)^k}{k!}$$

that

$$\phi_{m+n}(x)=\sum_{r,s}(-3)^{r+s}\binom{m}{r}\binom{n}{s}\frac{(m-r)!(n-s)!}{(m-2r-s)!(n-r-2s)!}\phi_{m-2r-s}(x)\cdot\phi_{n-r-2s}(x), \tag{5}$$

the summation extending over all nonnegative r, s such that $2r+s\leqq m$, $r+2s\leqq n$.

The formulas (4) and (5) may be compared with (1) and (2), respectively.

It is immediate from (3) that

$$\phi_m(x)=\sum_{3r\leq m}(-1)^r\,\frac{m!}{r!(m-3r)!}\,(3x)^{m-3r}. \tag{6}$$

Substituting from (6) in (4), we find that the latter is equivalent to

$$\sum_s \binom{v}{s}\binom{m+n-3v}{m-3s} = \sum_r (-3)^r \binom{v}{r}\binom{m+n-2r}{m-r}. \tag{7}$$

But as was pointed out by Drazin [2], (7) is an immediate consequence of the relation

$$(1+x^3)^v(1+x)^{m+n-3v} = \left\{1 - \frac{3x}{(1+x)^2}\right\}^v (1+x)^{m+n}.$$

Thus we have another proof of (4).

References

1. S. C. Dhar, On the product of parabolic cylinder functions, Bull. Calcutta Math. Soc., vol. 26, 1934, pp. 57–64.

2. M. P. Drazin, Another note on Hermite polynomials, this MONTHLY, vol. 64, 1957, pp. 89–91.

3. A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Higher Transcendental Functions, vol. 3, New York, 1955.

4. E. Feldheim, Quelques nouvelles relations pour les polynomes d'Hermite, J. London Math. Soc., vol. 13, 1938, pp. 22–29.

5. N. Nielsen, Recherches sur les polynomes d'Hermite, Danske Vid. Sels., Math.-Fys. Medd., I, 6, pp. 1–78.

6. G. N. Watson, A note on the polynomials of Hermite and Laguerre, J. London Math. Soc., vol. 13, 1938, pp. 29–32.

A PROOF OF A THEOREM ON JACOBIANS*

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1. Let x, y denote points in a d-dimensional Euclidean space E^d, and let F be a C^1-mapping from E^d into another d-dimensional space.

We denote by $x^1, \cdots, x^d$, the coordinates of x and by $f^1(x), \cdots, f^d(x)$ those of $F(x)$. It is well known that if the Jacobian det. $J(z) = \text{det.}\ (\partial f^i(x)/\partial x^j)_{x=z} \neq 0$, *then $f(x)$ is a one-to-one and open mapping around z. The purpose of this note is to give a simple proof of this theorem.*

2. Proof of the theorem. We may assume without loss of generality that $z=0$. We shall use the vector and matrix notations, and shall consider matrices as linear operators on this vector space. Since $J(x)$ is continuous in x, there exists a small positive ϵ such that if $|x| = \text{Max}_i |x^i| < \epsilon$ then for any vector u

$$|J(x)u - J(0)u| \leqq (1-\epsilon)|J(0)u|. \tag{1}$$

This is possible because $J(0)$ is nonsingular. Now consider a curve $F(x+ty)$ where t ranges over the unit interval $[0, 1]$. We assume that $|x+ty| < \epsilon$ for all t in $[0, 1]$.

* This work was done with the support of the National Science Foundation and the Sloan Foundation.

Define

$$(\partial/\partial t)F(x + ty) = \lim_{h\to 0} (1/h)(F(x + (t + h)y) - F(x + ty)). \tag{2}$$

Then by easy computations,

$$(\partial/\partial t)F(x + ty) = J(x + ty)y. \tag{3}$$

Integrate the left hand side from 0 to 1, and we have

$$F(x + y) - F(x) = \int_0^1 J(x + ty)ydt. \tag{4}$$

Now suppose that $F(x+y)=F(x)$. Then

$$\begin{aligned} 0 &= \left| \int_0^1 J(x + ty)ydt \right| \\ &= \left| \int_0^1 J(0)ydt + \int_0^1 (J(x + ty) - J(0))ydt \right| \\ &\geqq \left| J(0)y \right| - (1 - \epsilon)\left| J(0)y \right| = \epsilon \left| J(0)y \right| . \end{aligned}$$

Hence $J(0)y=0$, which implies $y=0$ because $J(0)$ is nonsingular.

Next, we assume $F(0)=0$ and prove that the image of $|x| < \epsilon/2$ contains the sphere $|z| < \delta = \epsilon/2 \inf_{|x|=\epsilon/2} |J(0)x|$. Let z_0 be given, $|z_0| < \delta$. Find x_0 such that

$$\left| F(x_0) - z_0 \right| = \inf_{|x| \leqq \epsilon/2} \left| F(x) - z_0 \right| .$$

For any x, $|x| = \epsilon/2$, by (4) and (1)

$$\begin{gathered} \left| F(x) \right| = \left| J(0)x + \int_0^1 (J(tx) - J(0))xdt \right| \geqq \epsilon \left| J(0)x \right| \geqq 2\delta, \\ \left| F(x) - z_0 \right| \geqq \delta > \left| F(0) - z_0 \right| \geqq \left| F(x_0) - z_0 \right| . \end{gathered}$$

Hence $|x_0| < \epsilon/2$, and y can be chosen so that $|x_0+ty| \leqq \epsilon/2$ for $0 \leqq t \leqq 1$, and $J(0)y = -\eta(F(x_0)-z_0)$ for some small positive η. Then, by (4) and (1) $|F(x_0+y) - F(x_0) - J(0)y| \leqq (1-\epsilon)|J(0)y|$, $|F(x_0+y)-z_0| \leqq |F(x_0)-z_0+J(0)y| + (1-\epsilon)|J(0)y| \leqq (1-\eta\epsilon)|F(x_0)-z_0|$. From the definition of x_0 it follows that $F(x_0)-z_0 = 0$. The theorem is hence proved.

Remark. We can extend this result to a certain more general infinite dimensional space if we can give a proper formulation to the continuity and the nonsingularity of $J(x)$.

CUBIC INVERSION

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The purpose of this paper is to develop inversion, using a cubic as the base curve instead of a conic section. To simplify algebraic manipulations, any cubic of the pencil of syzygetic cubics has been chosen as the base curve rather than the general cubic. This is permissible; for any nonsingular cubic by proper transformations can be put in the form of one of these cubics.

The equations of the inverse transformation. Take a fixed point $A_3(0, 0, 1)$ as origin, and take as the fixed fundamental base curve a cubic of the syzygetic pencil $F \equiv x_1^3+x_2^3+x_3^3-3\lambda x_1x_2x_3=0$. Points collinear with the origin and conjugate with respect to the base curve are said to be inverse. A line through the origin and a general point $P'(x_1', x_2', x_3')$ intersects the polar line of P' with respect to the base cubic in a point $P(x_1, x_2, x_3)$. P and P' are conjugate points.

When the point $P(x_1, x_2, x_3)$, the intersection of A_3P' and the polar line of P', is found, the following cubic inverse transformation is established:

$$x_1 : x_2 : x_3 = f_1 : f_2 : f_3, \tag{1}$$

where

$$f_1 = \lambda x_1'^2 x_2' - x_1' x_3'^2, \qquad f_2 = \lambda x_1' x_2'^2 - x_2' x_3'^2,$$
$$f_3 = x_1'^3 + x_2'^3 - 2\lambda x_1' x_2' x_3'.$$

The cubics $f_1=0, f_2=0, f_3=0$ have seven points in common.* Thus the cubics $\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=0$, where the α_i are general constants, form a net of cubics through these seven base points.

Inverses of points and curves. Any point on the base cubic, F, is its own inverse; thus all points of the base cubic are invariant under this transformation.

Each of the points $A_1(1, 0, 0)$ and $A_2(0, 1, 0)$ inverts into $A_3(0, 0, 1)$. The inverse of A_3 is indeterminate. Using the method of infinitesimals, and taking a point $(\epsilon_1, \epsilon_2, 1+\epsilon_3)$ close to A_3 on the line $x_1+\alpha x_2=0$, where $\alpha=-(\epsilon_1/\epsilon_2)$ we find that points close to A_3 invert into points on $x_3=0$. Any point of $x_1=0$ inverts into some other point of $x_1=0$, and likewise the points of $x_2=0$ invert into other points of $x_2=0$, but no point of $x_3=0$ inverts into a point of $x_3=0$.

The inverse of a line through the fixed origin A_3 is a degenerate cubic through all the vertices of the fundamental triangle $A_1A_2A_3$, and it consists of the original line and the conic $\lambda x_1x_2-x_3^2=0$.

The inverse of a line through A_1 or A_2 is a nonreducible cubic through A_3, but not through A_1 or A_2.

The inverse of a general line is a cubic which passes through A_3 but not through A_1 or A_2.

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* A certain cubic transformation, this MONTHLY, vol. 59, 1952, p. 314.

The inverse of a general conic is a sextic through the seven base points of the net of cubics $\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=0$, and it has at least a double point at each of these base points.

The inverse of a general n-ic is a $3n$-ic through the seven base points. If the n-ic is of order higher than the first, the base points are at least double points of the $3n$-ic.

Every n-ic through A_3 is transformed into a degenerate $3n$-ic. If A_3 is a k-ple point of the n-ic, then the inverse curve contains the conic $\lambda x_1x_2-x_3^2=0$ to the k-th power.

The reader will recall that under quadric inversion, the inverse of each curve through A_3 contains a line as part of it. Thus in cubic inversion we have a similar situation.

The effects of the transformation on the multiple points of an n-ic. An n-ic which has a k-ple point at A_1 but does not go through A_2 or A_3 is transformed into a $3n$-ic which does not go through A_1 or A_2, but has an n-ple point at A_3.

Since $k\leqq n$, the order of the singularity is never decreased. If the n-ic is reducible and $k=n$, the order of the singularity is the same. While if $k<n$, the order is increased.

An n-ic which has a k-ple point at A_2, but does not go through A_1 or A_3, is transformed into a $3n$-ic which does not go through A_1 or A_2, but has an n-ple point at A_3.

An n-ic which does not pass through A_1 or A_2, but has an s-ple point at A_3, is transformed into a $3n$-ic with s-ple points at both A_1 and A_2 and an n-ple point at A_3, $(s\leqq n)$.

An n-ic with a k-ple point at A_1, an r-ple point at A_2, and an s-ple point at A_3 is transformed into a $3n$-ic with an n-ple point A_3 and s-ple points at both A_1 and A_2.

Thus an s-ple point at A_3 can reduce a singularity at A_1 and A_2, but a singularity at A_3 is never reduced.

Some examples of the effects of the cubic inverse transformation are:

(a) A cubic with a double point at A_1 or A_2 is transformed into a nonic with a triple point at A_3.
(b) A cubic with a double point at A_3 is transformed into a nonic with double points at A_1 and A_2, and a triple point at A_3.
(c) A conic through A_1 or A_2 is transformed into a sextic with a double point at A_3.
(d) A conic through A_3 is transformed into sextic through A_1 and A_2 with a double point at A_3.

A general n-ic of order higher than the first which has no singularities is transformed into a $3n$-ic with at least a double point at each of the seven base points.

Invariants. There are $3n$ invariant points on any n-ic. These are the points

in which it intersects the base cubic since all points of the base cubic are invariant.

Any cubic G_3 intersects the base cubic in nine invariant points. Three of these points lie on the straight line, of which G_3 is the inverse. By Noether's Law the other six points of intersection lie on a conic.

In general the cross ratio of four collinear points under this transformation is not invariant, but the cross ratio on the lines $x_1=0$ and $x_2=0$ is a covariant.

In general, distance is not invariant but is a covariant on the lines $x_1=0$ and $x_2=0$.

The reader will recall that in quadric inversion the inverse points P and P', and the fixed origin A_3 are collinear, and are so related that the product $A_3P \cdot A_3P'=k^2$, where the base curve is a circle with center A_3 and radius k. Thus the product $A_3P\cdot A_3P'$ is always constant.

In cubic inversion, the product

$$A_3P\cdot A_3P' = \frac{(x^2+y^2)(\lambda xy-1)}{x^3+y^3-2\lambda xy}$$

is a variable quantity. This is easily understood since each point P' has a different polar conic.

If $A_3P\cdot A_3P'=\lambda$, the cubic inverse transformation reduces to the quadratic transformation $x_1:x_2:x_3=\lambda x_1':\lambda x_2':x_1'^2+x_2'^2$. This is equivalent to quadric inversion with respect to a circle with center at A_3 and $r^2=\lambda$.

References

1. Surendramohan Ganguli, The Theory of Plane Curves, University of Calcutta, 1925.
2. William C. Graustein, Introduction to Higher Geometry, New York, 1930.
3. Harold Hilton, Plane Algebraic Curves, Oxford University, 1920.
4. Frank Morley and F. V. Morley, Inversive Geometry, New York, 1933.
5. A. B. Basset, Cubic and Quartic curves, Cambridge, 1901.
6. Heinrich Schroeter, Theorie der Ebenen Kurven, Leipzig, 1888.
7. Julian Lowell Coolidge, A Treatise on Algebraic Plane Curves, Oxford, 1931.
8. Arnold Emch, Lecture Notes on Algebraic Geometry, unpublished.

THE PROBABILITY OF A SADDLEPOINT

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A *saddlepoint* of a matrix (A_{ij}) is a particular pair (r, s) such that A_{rs} is the minimum element in the rth row and the maximum element in the sth column. (See Chapter I of [1] for the importance of this concept in the Theory of Games.) The object of this note is to make precise the statement (intuitively plausible and confirmed by experience) that "large matrices rarely have saddlepoints."

Theorem. *Let the elements of the $m\times n$ matrix (A_{ij}) be independent random variables with the same continuous distribution function. Then the probability that (A_{ij}) has a saddle-point is $P(m, n)=m!n!/(m+n-1)!$*

Proof. Consider the events

$$E\colon (A_{ij}) \text{ has a saddlepoint.}$$

$$E(r, s)\colon (A_{ij}) \text{ has } (r, s) \text{ as saddlepoint,}$$

$$F\colon (A_{ij}) \text{ has all its elements distinct.}$$

Our continuity assumption ensures $\Pr(F)=1$, so that

$$P(m, n) = \Pr(E) = \Pr(E \cap F).$$

It is easy to show that if (r, s) and (t, u) are both saddle-points, then $A_{rs}=A_{tu}$. Thus $E \cap F$ is the *disjoint* union of the events $E(r, s) \cap F$, and so

$$P(m, n) = \sum_{r,s} \Pr(E(r, s) \cap F).$$

By symmetry (we omit the formal argument), $\Pr(E(r, s) \cap F)$ has the same value for all pairs (r, s), and so

$$P(m, n) = mn \Pr(E(1, 1) \cap F).$$

The evaluation of $\Pr(E(1, 1) \cap F)$ is a matter of elementary combinatorial probability; the $(m+n-1)!$ possible orderings of the elements of the union of the first row and first column of (A_{ij}) are clearly equiprobable (again we omit the formalization) and precisely $(m-1)!(n-1)!$ of them make $(1, 1)$ a saddlepoint, so that

$$\Pr(E(1, 1) \cap F) = (m-1)!(n-1)!/(m+n-1)!$$

and the proof is complete.

Note that $P(m, n)$ does not depend on the distribution function. This is *not* the case for discontinuous distributions, however. For a nontrivial example, consider a 2×2 matrix (A_{ij}) of independent random variables with the same Bernoulli distribution

$$\Pr(A_{ij} = 0) = p, \qquad \Pr(A_{ij} = 1) = q = 1 - p;$$

we find by enumerating the 16 possible numerical matrices (each with the appropriate probability) that the probability of a saddlepoint is $1-2p^2q^2$, which of course depends on p.

Reference

1. J. C. C. McKinsey, Introduction to the Theory of Games, New York, 1952.

NOTE ON VERTICES IN EUCLIDEAN 3-SPACE

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The object of this note is to indicate by theorem and example that, strangely enough, a point of a curve where the radius of the osculating sphere has an extreme value need not be the same as a point where the osculating sphere has at least five consecutive points in common with a given curve. Throughout, the notation and results of *Lectures on Classical Differential Geometry* by D. J. Struik are used.

The following relationships are well known:

$$K^2 = \vec{x}'' \cdot \vec{x}'', \qquad \tau = (\vec{x}'\vec{x}''\vec{x}''')/K^2; \tag{1}$$

$$\text{Frenet formulae } \vec{t}' = K\vec{n}, \qquad \vec{n}' = -K\vec{t} + \tau\vec{b}, \qquad \vec{b}' = -\tau\vec{n}; \tag{2}$$

$$\begin{gathered} r = \text{radius of osculating sphere} = [R^2 + (TR')^2]^{1/2}, \\ T = 1/\tau, \qquad R = 1/K, \qquad R' = -K'/K^2 \end{gathered} \tag{3}$$

DEFINITION 1. *A space curve* $C: \vec{x} = [x^1(s),\ x^2(s),\ x^3(s)]$ *is called a curve of type* I *if and only if*

(a) $x^i(s)$ *is analytic for each* $i = 1, 2, 3$;
(b) *the parameter s represents arc length*;
(c) *neither* $K = 0$ *nor* $\tau = 0$ *at any point of* C;
(d) *the curve is not spherical.*

From (b) and (c), neither $\vec{x}'$ nor $\vec{x}' \times \vec{x}''$ is zero at any point of a curve of type I, since $\vec{x}' = \vec{t}$ and $\vec{x}' \times \vec{x}'' = K(\vec{t} \times \vec{n})$. Also, since $\vec{x}$ is analytic, it follows from equation (1) that K is finite. Hence, since $K \neq 0$, by (c), it follows readily from equation (1) that K' is also finite. Since $\tau \neq 0$, it follows from equation (3) that for every curve of type I

$$0 < r = [R^2 + (TR')^2]^{1/2} < \infty. \tag{4}$$

Throughout, the letter C represents a curve of type I, the letter Σ represents a given sphere, and P represents the point corresponding to parameter value s_0.

DEFINITION 2. *Let* C *and* Σ *meet in point* P. *Let* A *be a point on* C, *near* P, *and let* AD *be its distance to* Σ. *Then* Σ *has a contact of order n with* C *if and only if*

$$\lim_{A \to P \text{ on } C} \frac{AD}{(AP)^k}$$

is finite and not zero for $k = n+1$, *but is zero for* $k = n$.

Let C be given by $\vec{x} = x(s)$, and let Σ be given by $f(x^1, x^2, x^3) = (\vec{x} - \vec{a}) \cdot (\vec{x} - \vec{a})$

* The author is now at Purdue University.

$-d^2=0$. Let $f(s)=(\vec{x}(s)-\vec{a})\cdot(\vec{x}(s)-\vec{a})-d^2$. Then the following is known (Struik, p 24):

THEOREM 1. *Necessary and sufficient conditions that Σ have contact of order n at P with C are that $f(s)=f'(s)=\cdots=f^n(s)=0$; $f^{(n+1)}(s)\neq 0$, all at $s=s_0$.*

DEFINITION 3. *A set of $n+1$ points is called spherical if and only if they lie on a sphere.*

DEFINITION 4. *If in any neighborhood of P on C there are $n+1$ spherical points, then the osculating sphere to C at P is said to have at least $n+1$ consecutive points in common with C at P.*

THEOREM 2. *A necessary condition that C and its osculating sphere Σ have at least $n+1$ consecutive points in common at P is that Σ have contact of order at least n with C at P. (This follows analogously as in Struik, p. 10 and p. 24.)*

DEFINITION 5. *A point P of C is defined to be a vertex if and only if the osculating sphere at P has at least five consecutive points in common with C.*

THEOREM 3. *A necessary condition that P be a vertex of C is that, at P*

$$-K\tau K''+2K'^2\tau+KK'\tau'+K^2\tau^3=0.$$

Proof. Let Σ represent the osculating sphere to C at P. By Theorem 1, Theorem 2, and the Frenet formulae, for P to be a vertex, it is necessary that, at $s=s_0$

$$\text{(a)}\qquad f'(s)=2(\vec{x}-\vec{a})\cdot\vec{t}=0$$

$$\text{(b)}\qquad f''(s)=2[(\vec{x}-\vec{a})\cdot K\vec{n}+1]=0$$

$$\text{(c)}\qquad f'''(s)=2(\vec{x}-\vec{a})\cdot(K'\vec{n}+K\tau\vec{b}-K^2\vec{t})=0$$

$$\text{(d)}\qquad \begin{aligned} f''''(s)&=-2K^2+2(\vec{x}-\vec{a})\cdot[K''\vec{n}+K'(-K\vec{t}+\tau\vec{b})+K'\tau\vec{b}+K\tau'\vec{b}\\ &\qquad+K\tau(-\tau\vec{n})-2KK'\vec{t}-K^2K\vec{n}]=0\end{aligned}$$

Since Σ is the osculating sphere, (a), (b), and (c) are satisfied, since

$$(\text{a}_1)\qquad (\vec{x}-\vec{a})\cdot\vec{t}=0$$

$$(\text{b}_1)\qquad (\vec{x}-\vec{a})\cdot\vec{n}=-R=-1/K$$

$$(\text{c}_1)\qquad (\vec{x}-\vec{a})\cdot\vec{b}=RK'/(K\tau)=-R'T=K'/(K^2\tau)\ \text{(See Struik, p. 25)}$$

Hence, it is necessary that $f''''(s)=0$. Using then (a_1), (b_1), (c_1), equation (d) becomes

$$(\text{d}_1)\qquad f''''(s)/2=-K''/K+2K'^2/K^2+\tau'K'/(K\tau)+\tau^2=0$$

But since C is a curve of type I, $K\neq 0$, $\tau\neq 0$, so that (d_1) is equivalent to

$$(5)\qquad -K''K\tau+2K'^2\tau+KK'\tau'+K^2\tau^3=0.$$

DEFINITION 6. *A stationary point Q of C is a point where $r'=0$, where r is given by* (3).

THEOREM 4. *Every vertex point P of C is a stationary point.*

Proof.

$$r' = [RR' + TR'(T'R' + TR'')]/[R^2 + (TR')^2]^{1/2}$$

By equation (4), it follows then that a necessary and sufficient condition that $r'=0$ is that

$$RR' + TR'(T'R' + TR'') = 0. \tag{6}$$

Since $R=1/K$, $R'=K'/K^2$, $R''=-K''/K^2+2K'^2/K^3$, $T=1/\tau$, $T'=-\tau'/\tau^2$, substitution into (6) and use of Definition 1, part (c), yields the equivalent condition

$$K'[-K\tau K'' + 2K'^2\tau + KK'\tau' + K^2\tau^3] = 0 \tag{7}$$

However, if P is a vertex, by Theorem 3, equation (7) is satisfied. Hence $r'=0$ at P, so that P is a stationary point.

It is now of interest to note that every stationary point need not be a vertex. This is motivated from (7). Since the expression in brackets is necessary for a point to be a vertex, one can still make $r'=0$ by making $K'=0$. The following example results from such reasoning.

Consider a helix which is given by the natural equations $K=\tau=1+s^2$. First it is shown that the conditions of Definition 1 are satisfied.

(a) Consideration of the system of linear differential equations, $\alpha'=K\beta$, $\beta'=-K\alpha+\tau\lambda$, $\lambda'=-\tau\beta$, where in this case K and τ are analytic, yields readily, by use of the argument of Struik, pp. 29–31, the fact that the helix is analytic.

(b) By assumption, s represents arc length.

(c) This condition is obviously satisfied.

(d) A necessary and sufficient condition that a curve be spherical is that $R^2+(TR')^2$ be identically constant (see Struik, p. 32). For the given helix

$$r^2 = R^2 + (TR')^2 \equiv (1 + 8s^2 + 6s^4 + 4s^6 + s^8)/(1 + s^2)^6. \tag{8}$$

Since this last expression is not identically constant, all conditions are satisfied and the helix is a curve of type I.

Moreover, simple computation yields

$$r = (1 + 8s^2 + 6s^4 + 4s^6 + s^8)^{1/2}/(1 + s^2)^3 \tag{9}$$

$$r = 1, \quad r' = 0, \quad r'' = 2, \text{ at } s = 0. \tag{10}$$

Hence, at $s=0$, r not only attains a stationary value, but actually attains an extreme value. However, at $s=0$, the point is not a vertex, since $K=\tau=1+s^2$, $K'=\tau'=2s$, $K''=\tau''=2$, so that equation (5) is not satisfied since $-2+0+0+1=-1\neq 0$.

CLASSROOM NOTES

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TRIGONOMETRY ABSTRACTLY TREATED

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Trigonometry when taught to thirteen year olds is (rightly) treated as a branch of physics. If a pupil asks, "What is an angle of 40°?" the teacher points to this angle on a protractor, that is, explains the idea by means of a physical object. Is it possible to make the ideas of angle, sine, cosine precise without any appeal to figures?

With calculus known, one can of course define arc sin k as $\int_0^k(1-t^2)^{-1/2}dt$, and derive its properties. But at school we form an idea of "angle" before we have met calculus, limits, or any infinite process, even before we have met radian measure. It should be possible to explain angle by *algebraic* reasoning.

The natural way to explain angles would be by stepping off equal intervals around the arc of a circle. If $A, B, C, D, E \cdots$ are evenly spaced points on the circumference of a circle with center O, we should recognize statements such as $\angle AOB=\angle BOC$, $\angle AOC=2\angle AOB$. In marking out the points A, B, C, D, E, we use compasses to make equal not the *arcs* AB, BC, CD, DE but the direct distances AB, BC, CD, DE. In drawing the circle $ABCDE$ with center O we also use distance.

The starting point is therefore the idea of *distance between points*. This we can handle quite well by means of coordinate geometry, so we may begin with the axioms, or definitions—

(1) *A point is a pair of real numbers* (x, y).

(2) *The distance s between two points (a, b) and (c, d) is a positive (or zero) number such that* $s^2=(a-c)^2+(b-d)^2$.

(3) *If we have two sets of points, $(x_1, y_1) \cdots (x_n, y_n)$ and $(X_1, Y_1) \cdots (X_n, Y_n)$, such that the distance between a pair of points in the first set equals the distance between the corresponding points in the second set, the sets form congruent figures.*

(4) *We call the operation of replacing points by the points of a congruent set a rigid displacement.*

We want to show that there are rigid displacements in which every point of space goes to a corresponding point. We first investigate what conditions such displacements must satisfy; we are particularly interested in rotations, which we must define.

(5) *A movement about a point (h, k) means a rigid displacement in which (h, k) goes to itself.*

We do not straight away call this a rotation, since it might be for example the reflection; (x, y) goes to $(x, -y)$. This is a movement about (0, 0) ,since (0, 0) goes to (0, 0) and distances are preserved. We must find some way of distinguishing rotations from reflections.

Consider any movement about the origin (0, 0). Let (x_1, y_1) go to (X_1, Y_1) and (x_2, y_2) to (X_2, Y_2). Then

$$\text{(I)} \qquad x_1^2 + y_1^2 = X_1^2 + Y_1^2$$

$$\text{(II)} \qquad x_2^2 + y_2^2 = X_2^2 + Y_2^2$$

$$\text{(III)} \qquad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2$$

since distances are preserved. Adding (II) to (I) and subtracting (III), we find after removing a factor 2, $x_1x_2+y_1y_2=X_1X_2+Y_1Y_2$.

Hence

THEOREM 1. *Any movement about the origin leaves $x_1x_2+y_1y_2$ unaltered in value.*

(One may comment on the significance of this as scalar product, the work done by a force in a displacement, and so on.)

There is a well known algebraic identity $(x_1^2+y_1^2)(x_2^2+y_2^2)-(x_1x_2+y_1y_2)^2 =(x_1y_2-x_2y_1)^2$. As the left-hand side is unaltered in any movement about the origin, the right-hand side also must be. Accordingly we have

THEOREM 2. *$x_1y_2-x_2y_1$ is either unchanged, or changes only in sign in any movement about the origin.*

There are now three possibilities.

(i) In the movement, for all values of x_1, x_2, y_1, y_2, the expression $x_1y_2-x_2y_1$ is unchanged.

(ii) In the movement, this expression changes its sign for all values of the variables.

(iii) There are some values for which the expression is unchanged, and some for which it changes sign.

(6) A movement of type (i) is called a *rotation*; a movement of type (ii) is called a *reflection*; a movement of type (iii) does not exist, as will be shown later.

Formula for rotation. Let (1, 0) go to (p, q), in a rotation about the origin. Then, since distance is preserved, $p^2+q^2=1$. Let (x, y) go to (X, Y). Since $x_1x_2+y_1y_2$ is unchanged in any motion, and $x_1y_2-x_2y_1$ is unchanged for every point in a rotation, we have

$$\text{(A)} \qquad \begin{aligned} Xp + Yq &= x\cdot 1 + y\cdot 0 = x, \\ -Xq + Yp &= -x\cdot 0 + y\cdot 1 = y. \end{aligned}$$

Solving for X, Y and using $p^2+q^2=1$ we have

(B) $$X = px - qy, \quad Y = qx + py.$$

It is easily verified that this formula makes all distances invariant. Hence, if $p^2+q^2=1$, there is one and only one rotation that sends $(1, 0)$ to (p, q).

Formula for reflection. Let $(1, 0)$ go to (p, q) where $p^2+q^2=1$, and (x, y) to (X, Y). For a reflection, $x_1x_2+y_1y_2$ remains the same, while $x_1y_2-x_2y_1$ changes sign. We have

(C) $$Xp + Yq = x, \quad -Xq + Yp = -y;$$

whence

(D) $$X = px + qy, \quad Y = qx - py.$$

Thus there is one and only one reflection that sends $(1, 0)$ to (p, q), while leaving the origin fixed.

Impossibility of mixing rotations and reflections. Possibility (iii) envisages $x_1y_2-x_2y_1$ being unchanged for some values of the variables, while changing sign for others. It is easily verified that, for equations (B), the value is unchanged for all x_1, y_1, x_2, y_2 while for equations (D), the expression always changes sign. A third possibility could only arise if some points obeyed equations (B) while others obeyed equations (D). Let (x_1, y_1) obey (B) and (x_2, y_2) obey (D). A simple calculation shows $X_1X_2+Y_1Y_2=x_1x_2-y_1y_2$. This result can only be reconciled with Theorem (1) if $y_1y_2=0$, that is, if one of the points lies on $y=0$. But when $y=0$, equations (B) and (D) give the same values of (X, Y). Thus there is no possibility of a genuine mixing of the equations. Thus

THEOREM 3. *The only possible types of motion about the origin are the rotations, given by equations* (B), *and the reflections, given by equations* (D).

Rotations and reflections are linear operations; they preserve relations such as $x_1=kx_2$, $y_1=ky_2$ or $x_1+x_2=x_3$, $y_1+y_2=y_3$. This is evident from formulae (B) and (D).

The set of points (ka, kb), where a, b are fixed and k takes all positive values, form a *ray* through the origin. Rotations and reflections transform rays into rays.

Combination of rotations and reflections. Since a rotation leaves all distances, the origin, and all expressions $x_1y_2-x_2y_1$ unaltered, it is evident that a rotation followed by a rotation gives a rotation.

Since reflections change the sign of $x_1y_2-x_2y_1$ it is evident that two successive reflections produce a rotation.

Angles. The measurement of angle is not really investigated in elementary trigonometry except for rational ratios of angles, in effect when $P=n\theta$, $Q=m\theta$,

where m, n are integers. Hence, if the operation given by equations (B) is called rotation through θ, the rotation through $n\theta$ will be the result of carrying out this operation n times.

Rotation through $-\theta$ is the operation that undoes the effect of rotation through θ. The equations for this operation can be written down by inspecting equation (A).

DEFINITION. *The value of p that occurs in equations* (B) *for rotation through θ is called the cosine of θ. The value of q that occurs in equations* (B) *is called the sine of θ.*

By comparing equations (B) and (A), one can see that $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$.

Since $p^2+q^2=1$, $\cos^2\theta+\sin^2\theta=1$.

Rotation through θ thus has the equations

$$X = x\cos\theta - y\sin\theta, \qquad Y = x\sin\theta + y\cos\theta.$$

Sine and cosine of sum. Rotation through $\theta+\phi$ is understood to mean the effect of rotation through θ followed by rotation through ϕ. It can be verified that $\phi+\theta$ gives the same result.

Let rotation through θ send (x_1, y_1) to (x_2, y_2), and rotation through ϕ send (x_2, y_2) to (x_3, y_3).

The equations for this can be written down, and x_3, y_3 found from these equations in terms of x_1, y_1. The resulting equations are, of course, of type (B). The values of p and q can be read off, and give the usual formulas for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

THE MECHANICAL BRAIN

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Misconceptions concerning high speed computers often arise from newspaper articles suggesting that a computer is some sort of a sensational miracle machine or "mechanical brain." A student may, for example, get such a distorted picture from such an article that he concludes that the computer can do the problems for him, so that there is now no need for him to learn any mathematics. Simply telling him that the machine "can only add, subtract, multiply, and divide" is not sufficient. He has heard, for example, that it can integrate and has no idea how the labor of integration is divided between him and the machine. We can give him a clearer picture by leading him through the coding of one problem than by a plethora of comments about the machine. The following oversimplified example can easily be presented in half of one class period.

Problem. Given the equation $x^3+2x^2+1=0$, take x_0 to be -2 for the first approximation to a root and compute, by Newton's Method, the next n approximations.

ANOTHER LOOK AT THE PROBABILITY INTEGRAL

C. P. NICHOLAS, U. S. Military Academy

A well-known method of showing that $\int_0^\infty e^{-x^2}dx=\sqrt{\pi}/2$ is first to demonstrate that the integral is convergent, and then by a transformation of coordinates to set up the equation

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^{\pi/2} \int_0^\infty e^{-\rho^2}\rho d\rho d\theta.$$

From this $[\int_0^\infty e^{-x^2}dx]^2=\pi/4$, and we complete the solution by taking square roots.

An adaptation to instruction at the sophomore level appeared in this MONTHLY, vol. 57, 1950, pp. 412–413. Simplicity was achieved by certain geometric devices, and by the *assumption* that the improper integral is convergent.

The feeling of uneasiness that goes with this sweeping assumption can be avoided by the evaluation offered below. It is believed easy enough for a first course, and the final step establishes both the existence and the value of the integral.

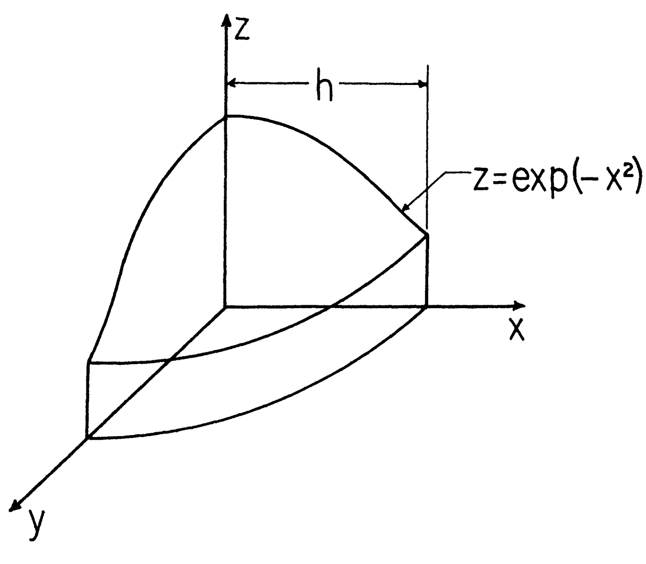

FIG. 1

Consider the surface generated by revolving about the Z-axis the curve $z=e^{-x^2}$, and let V_1 be the volume (Fig. 1) in the first octant bounded by this surface, the coordinate planes, and the cylinder $x^2+y^2=h^2$.

By the method of cylindrical shells

$$V_1 = \int_0^h \frac{\pi}{2} xe^{-x^2}dx = \frac{\pi}{4}[1 - e^{-h^2}].$$

Consider next the volume V_2, bounded as before except that we use the planes $x=h$ and $y=h$ instead of the cylinder $x^2+y^2=h^2$ (Fig. 2).

By double integration

$$V_2 = \int_0^h \int_0^h e^{-(x^2+y^2)}dxdy = \left[\int_0^h e^{-x^2}dx\right]^2.$$

Consider finally the volume V_3, bounded as for V_1 except that we use the cylinder $x^2+y^2=2h^2$ in lieu of $x^2+y^2=h^2$ (Fig. 3).

Again by the method of cylindrical shells $V_3=(\pi/4)[1-e^{-2h^2}]$.

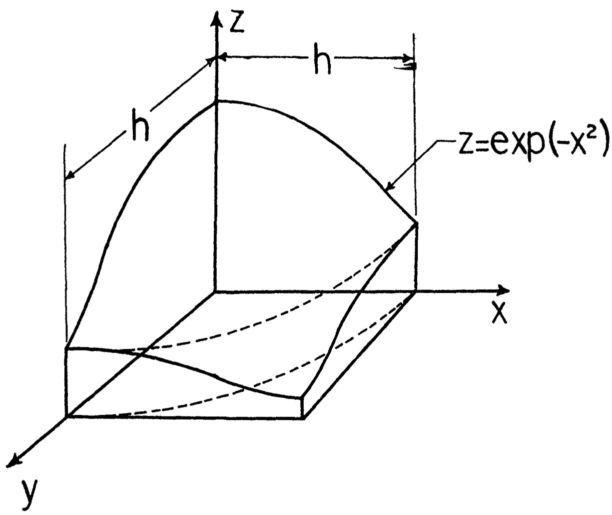

FIG. 2

Since by construction $V_1 < V_2 < V_3$, then for all positive values of h

$$\frac{\pi}{4}[1 - e^{-h^2}] < \left[\int_0^h e^{-x^2}dx\right]^2 < \frac{\pi}{4}[1 - e^{-2h^2}].$$

Now let h increase without bound, and seek limits of all three members. Evaluating the limits of the first and third members, we have

$$\frac{\pi}{4} \leqq \left[\lim_{h\to\infty} \int_0^h e^{-x^2}dx\right]^2 \leqq \frac{\pi}{4}.$$

Hence the middle limit must exist, and its value must be $\pi/4$. Therefore $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$.

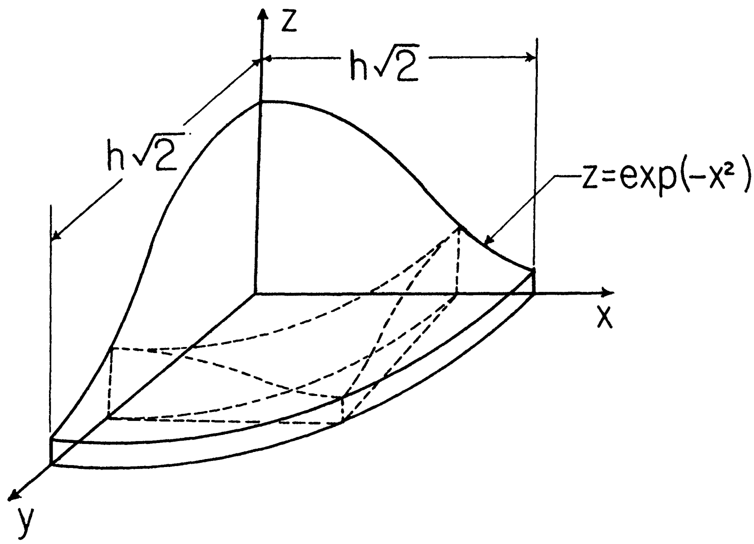

FIG. 3

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1291. *Proposed by G. B. Thomas, Jr., Massachusetts Institute of Technology*

Let a, b, c, d be nonnegative numbers such that $c+d \leqq \min(a, b)$. Prove that $ad+bc \leqq ab$.

E 1292. *Proposed by Jose Gallego-Diaz, Stanford University*

A pirate decided to bury a treasure on an island near the shore of which were two similar boulders A and B and, farther inland, three coconut trees C_1, C_2, C_3. Stationing himself at C_1, the pirate laid off C_1A_1 perpendicular and equal to C_1A and directed outwardly from the perimeter of triangle AC_1B. He similarly laid off C_1B_1 perpendicular and equal to C_1B and also directed outwardly from the perimeter of triangle AC_1B. He then located P_1, the intersection of AB_1 and BA_1. Stationing himself at C_2 and C_3, he similarly located points P_2 and P_3, and finally buried his treasure at the circumcenter of triangle $P_1P_2P_3$.

Returning to the island some years later, the pirate found that a big storm had obliterated all the coconut trees on the island. How might he find his buried treasure? (Dedicated to Howard Dachslager.)

E 1293. *Proposed by J. B. Roberts, Reed College*

Prove that for every nonnegative integer t

$$3(1 + 6^t + 8^t) \equiv 1^t + 2^t + 3^t + \cdots + 9^t \pmod{18}.$$

E 1294. *Proposed by G. A. Harris, Jr., Yale University*

Having chosen two numbers a_1 and b_1 from the open interval $(0, 1)$, define the sequences $\{a_n\}$ and $\{b_n\}$ recursively as follows:

$$a_{n+1} = a_1(1 - a_n - b_n) + a_n, \qquad b_{n+1} = b_1(1 - a_n - b_n) + b_n.$$

Prove that both sequences approach limits as $n \to \infty$, and find these limits.

E 1295. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Show that all the roots of $\tan z = z/(1+m^2z^2)$, where m is real, are real.

SOLUTIONS

A Proposition Equivalent to Dirichlet's Theorem

E 1218 [1956, 342; 1957, 46]. *Proposed by Robert Spira, Berkeley, Calif.*

Consider the two propositions:

I. If $(a, b) = 1$, then $ax+b$ assumes infinitely many prime values.

II. If $(a, b) = 1$, then $ax+b$ assumes at least one prime value.

I is Dirichlet's theorem. Clearly I implies II. Show that II implies I.

II. *Solution by Virginia S. Hanly, Ohio State University.* Assume $|a| \neq 1$ and $(a, b) = 1$. The class $\{ax+b\}$ contains the two disjoint subclasses $\{a^2x+b\}$ and $\{a^2x+a+b\}$. Let p_1 be a prime of the form $ax+b$. Then p_1 is not in both of the subclasses (perhaps in neither). Hence, since $(a^2, b) = (a^2, a+b) = 1$, we let p_2 be a prime in one of the subclasses above not containing p_1. We may repeat the argument upon the subclass, obtaining an infinite sequence of distinct primes $p_1, p_2, \cdots$, all of the form $ax+b$. The case $|a| = 1$ follows trivially.

Editorial Note. Solutions previously received for this problem were patterned like that of [1957, 46], and become invalid if we are not restricted to positive integers. Thus, granted there is by II some x_1 such that ax_1+b is prime, if b is itself prime we cannot conclude that ax_1+b is distinct from b. In fact (see the solution [1957, 46]), letting $x_1 = x_2 = x_3 = \cdots = 0$, we have the sequence $b, b, b, \cdots$ containing just one prime. The new solution above was designed to avoid the restriction.

A Curve Defined by an Area Relation

E 1261 [1957, 272]. *Proposed by C. S. Ogilvy, Hamilton College*

If the sectorial area bounded by a curve and the radii vectores to any two

points on the curve equals the area bounded by the curve, the x-axis, and the ordinates of the two points, find the curve.

Solution by L. A. Ringenberg, Eastern Illinois State College. Equating the differentials of the two areas we have $|(xdy-ydx)/2| = |ydx|$, whence the required locus is any one of the curves $y=cx^3$ or $xy=c$.

Also solved by R. J. Adler, W. A. Al-Salam, J. W. Armstrong, Leon Bankoff, Robert Bart, D. A. Breault, Lawrence Conlon and Victor Manjarrez (jointly), L. R. Ford, David Freedman, Michael Goldberg, G. A. Harris, Jr., A. S. Hendler, Vern Hoggatt, J. R. Holdsworth, Bill Holter, R. H. Hou, A. R. Hyde, C. J. Kirchen, M. S. Klamkin, Frank Kocher, L. I. Lokomowitz, D. C. B. Marsh, E. J. Miller, John Osborn, G. B. Parrish, D. S. Passman, Montfort Plebstnoch, T. A. Porsching, John Rainwater, C. E. Rieck, D. A. Robinson, Anina Schub, Lawrence Shepp, J. A. Tierney, Roscoe White, T. L. Williams, R. H. Wilson, Jr., David Zeitlin, and the proposer.

Most of these solutions gave only one of the two classes of curves.

Cubics with Roots on the Unit Disc

E 1262 [1957, 272]. *Proposed by A. J. Goldman, National Bureau of Standards*

For which real values of A do all roots of $z^3-z^2+A=0$ obey $|z|\leqq 1$?

Solution by Arthur Steger, University of New Mexico. The graph of the real function $A=x^2-x^3$ has a relative maximum at $(2/3, 4/27)$, a relative minimum at $(0, 0)$, and passes through the points $(-1, 2)$, $(-1/3, 4/27)$, $(1, 0)$. From this graph we conclude the following concerning the roots of $z^3-z^2+A=0$:

(1) If $A<0$ or $A>2$, then there is a real root with absolute value greater than one.

(2) If $0\leqq A\leqq 4/27$, then all the roots are real and lie between $-1/3$ and 1.

(3) If $4/27<A\leqq 2$, then there is a real root between -1 and $-1/3$ and two imaginary roots. To investigate the imaginary roots, let r be the real root. Then $A=r^2-r^3$, and the depressed equation is $z^2+(r-1)z+r^2-r=0$ with roots

$$z = (1 - r \pm \sqrt{1 + 2r - 3r^2})/2.$$

For $-1\leqq r<-1/3$, the radicand is negative and $|z|^2=r^2-r$. The condition $|z|\leqq 1$ yields $r^2-r\leqq 1$. Therefore $(1-\sqrt{5})/2\leqq r<-1/3$. Finally, if $r=(1-\sqrt{5})/2$, then $A=(\sqrt{5}-1)/2$.

From (1), (2), and (3) we conclude that all the roots of $z^3-z^2+A=0$ have absolute value less than or equal to one if and only if

$$0\leqq A\leqq(\sqrt{5}-1)/2.$$

Also solved by Robert Bart, Julian Braun, C. N. Campopiano, Michael Goldberg, A. R. Hyde, D. C. B. Marsh, G. B. Parrish, D. S. Passman, Marlow Sholander, E. P. Starke, and the proposer.

Sholander established the general theorem: *When a, b, c are real, the roots of $z^3+az^2+bz+c=0$ satisfy $|z|\leqq 1$, if and only if $|a+c|-1\leqq b\leqq$ min $(3, 1+ac-c^2)$.* For $a=-1$, $b=0$, $c=A$, this reduces to the answer above.

Another Sequence of Triangles

E 1263 [1957, 272]. *Proposed by W. B. Carver, Cornell University*

Let T_0 be any triangle none of whose angles is a multiple of 45°. The tangents to the circumcircle of T_0 at its vertices form a new triangle T_1; and repetition of this process gives an infinite sequence of triangles $\{T_n\}$. If the angles of T_0 in degrees are integers, show that for $n \geqq 2$, T_{n+12} is similar to T_n. Is there a similar theorem for the case when the angles are rational in degrees?

Solution by D. C. B. Marsh, Colorado School of Mines. It is easy to show that: (1) if T_{n-1} is acute-angled, then $A_n = 180° - 2A_{n-1}$, $B_n = 180° - 2B_{n-1}$, $C_n = 180° - 2C_{n-1}$, (2) if T_{n-1} is obtuse-angled, say at A_{n-1}, then $A_n = 2A_{n-1} - 180°$, $B_n = 2B_{n-1}$, $C_n = 2C_{n-1}$. The fact that T_0 contains no angle which is a multiple of 45° guarantees that the construction process will not fail.

For any n, m we have $A_{n+m} \equiv 2^m A_n$ (mod 180), for some choice of sign, no matter what sequence of the two recursive relations is involved. In the special case where $m = 12$ and $n \geqq 2$, we have

$$A_{n+12} \equiv \pm\, 2^{12} A_n \equiv \pm\, 2^{14} 2^{-2} A_n \equiv \pm\, 2^2 2^{-2} A_n \equiv \pm A_n \pmod{180}.$$

Similarly, for $n \geqq 2$,

$$B_{n+12} \equiv \pm\, B_n \pmod{180}, \qquad C_{n+12} \equiv \pm\, C_n \pmod{180},$$

where the same choice of sign occurs as for angle A_{n+12}. Since the sum of the angles of T_{n+12} and T_n is each 180°, it follows that, for $n \geqq 2$, $A_{n+12} = \pm A_n$, $B_{n+12} = \pm B_n$, $C_{n+12} = \pm C_n$, and T_{n+12} is (directly or inversely) similar to T_n.

If T_0 has angles rational in degrees, there is a corresponding theorem (which may be established analogously). Let the angles of T_0 have l.c.d. of $d = 2^e p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, with the p_i distinct odd primes and $e_i \geqq 1$; let c_i be the exponent to which 2 belongs, mod p_i, and define E as the l.c.m. of 12 and the set of $c_i p_i^{e_i - 1}$; then, if no angle of T_0 is of the form $45a/2^b$ (a, b integers), the same method of construction yields T_{n+E} similar to T_n for $n \geqq 2 + e$.

Also solved (partially) by D. S. Passman.

Editorial Note. This sequence of triangles is something like an inverse of the sequence of triangles of problem E 1233 [vol. 64, 274]. If triangle T_{n-1} is acute-angled, its circumcircle will be the incircle of T_n, and the operation from T_{n-1} to T_n is exactly the inverse of the operation of problem E 1233. But in the sequence $\{T_n\}$ we sooner or later obtain an oblique triangle T_{k-1}, whose circumcircle will be, not the incircle, but one of the excircles of triangle T_k.

The Steensholt Inequality for a Tetrahedron

E 1264 [1957, 272]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If an interior point P of a tetrahedron $ABCD$ is projected orthogonally into A', B', C', D' on the planes of the faces BCD, CDA, DAB, ABC, and if the areas of these faces are denoted by A, B, C, D, show that

$$A(PA) + B(PB) + C(PC) + D(PD) \geqq 3[A(PA') + B(PB') + C(PC') + D(PD')].$$

I. *Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.* Represent the volume of the tetrahedron by V. Then $V=(1/3)\sum A(PA')$. Also, $V=(1/3)h_A A$ and $h_A \leqq PA+PA'$. Hence $(1/3)\sum A(PA')=(1/12)\sum h_A A \leqq (1/12)\sum (PA+PA')A$, or $\sum A(PA) \geqq 3\sum A(PA')$.

II. *Solution by N. D. Kazarinoff, University of Michigan.* The proposed inequality follows from a theorem of Pappus generalized to E_3. This generalized theorem states: Construct three triangular prisms which have for their bases three faces of a tetrahedron T, which have a lateral edge in common, and of which all or none lie entirely outside of T; construct a fourth prism on the remaining face whose lateral edges are translates of the common lateral edge of the first three prisms; then, the sum of the volumes of the first three prisms is equal to the volume of the fourth prism. Apply this theorem to $ABCD$, using PA as the common lateral edge of the prisms constructed upon the faces having common vertex A. Then

$$A(PA) \cos (PA, A'P) = B(PB') + C(PC') + D(PD').$$

Replacing $\cos (PA, A'P)$ by 1 and combining the resultant inequality with the three others obtained from it by cyclic permutation, we obtain the desired inequality. Equality clearly holds if and only if $ABCD$ is orthocentric and P coincides with the orthocenter of $ABCD$.

Also solved by J. W. Armstrong, D. C. B. Marsh, C. S. Ogilvy, D. S. Passman, Chih-yi Wang, and the proposer.

Editorial Note. This problem extends to the tetrahedron a property of the triangle given by Gunnar Steensholt, this MONTHLY [1956, 571]. If the tetrahedron is isosceles (that is, equifacial), the inequality reduces to

$$PA + PB + PC + PD \geqq 3(PA' + PB' + PC' + PD'),$$

which establishes the Erdös-Mordell inequality for the tetrahedron ($2\sqrt{2}$ in place of 3) for this special type of tetrahedron.

The Trivial Function

E 1265 [1957, 272]. *Proposed by F. L. Wolf, Carleton College*

Is there a nonnegative, nontrivial, continuous function $f(x)$ such that

$$\int_0^x f(t)dt \geqq f(x)$$

for all x on $0 \leqq x \leqq 1$?

Solution by L. R. Ford, Illinois Institute of Technology. No. Excluding the trivial $f(x) \equiv 0$, suppose a nonnegative continuous $f(x)$ is not identically zero in $0 \leqq x \leqq b < 1$. Its maximum value $M > 0$ in this subinterval is the value of $f(x)$

at some point a therein. Then

$$\int_0^a f(t)dt \leqq Ma < f(a),$$

contrary to the inequality of the question.

Also solved by D. S. Adorno, W. A. Al-Salam, Robert Bart, F. G. Brauer, D. A. Breault, D. G. Brennan, J. L. Brown, Jr., R. G. Buschman, E. M. Cabaña, C. N. Campopiano, T. S. Chihara, R. M. Conkling, Frederic Cunningham, Jr., David Ellis, M. A. Feldstein, David Freedman, D. S. Greenstein, G. A. Harris, Jr., A. S. Hendler, Aaron Herschfeld, Vern Hoggatt, W. H. Holter, J. Horváth, R. H. Hou, A. R. Hyde, Ronald Jacobowitz, Seymour Kass, N. D. Kazarinoff, M. S. Klamkin, Paul Knopp and Victor Manjarrez (jointly), P. Kolar, L. I. Lokomowitz, Marshall Luban, L. A. MacColl, D. C. B. Marsh, C. T. Molloy, Jr., J. B. Muskat, Dale Nelson, C. S. Ogilvy, W. R. Orton and Harold Shniad (jointly), Hiram Paley and D. A. Robinson (jointly), G. B. Parrish, D. S. Passman, Edward Pincus, John Rainwater, C. E. Rieck, G. S. Rinehart, L. A. Ringenberg, Michael Rosen, Lawrence Shepp, W. G. Spohn, Jr., R. W. Wagner, David Zeitlin, and the proposer.

The problem was located as a special case of Prob. 1, p. 37, in Coddington and Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, 1955). Orton and Shniad established the answer *no* by weakening the hypothesis of continuity to Lebesgue integrability. Ogilvy pointed out that if the lower end of the interval is open, then the answer to the problem is *yes*.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4763. *Proposed by Ky Fan, Oak Ridge National Laboratory*

For n positive numbers x_i, let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ denote respectively the least and the greatest of the $n+1$ quantities

$$x_1, \frac{1}{x_1} + x_2, \cdots, \frac{1}{x_{n-1}} + x_n, \frac{1}{x_n} \cdot \tag{1}$$

Prove that

$$\operatorname*{Max}_{x_i>0} f(x_1, \cdots, x_n) = \operatorname*{Min}_{x_i>0} g(x_1, \cdots, x_n) = 2\cos\frac{\pi}{n+2} \cdot \tag{2}$$

4764. *Proposed by A. E. Currier, U. S. Naval Academy*

Prove

$$\lim_{n\to\infty} \sum_{j=0}^{n} \binom{2n-j}{n} (-4)^j \Big/ \binom{2n}{n} = \frac{1}{3}\cdot$$

4765. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $y=f(x)$ be a real function defined for $x \geqq 0$. If (1) f has a finite upper bound in any finite interval, and (2) there are two positive numbers h, k such that $x'-x'' \geqq h$ implies $f(x')-f(x'') \geqq k$, then there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-h) < f(x) < g(x)$ for all sufficiently large x.

4766. *Proposed by Louis Weisner, University of New Brunswick, Canada*

Prove that the polynomials $g_n(x)$ defined by the generating function

$$(1+y^2)^{-1/2} e^{x \arctan y} = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!} y^n, \qquad |y| < 1, \qquad g_0(x) = 1,$$

are orthogonal over the interval $[-\infty, \infty]$ with weight function $e^{\pi x/2}/(1+e^{\pi x})$. Prove that they satisfy the recurrence relations

$$g_{n+1}(x) = x g_n(x) - n^2 g_{n-1}(x); \qquad n = 0, 1, \cdots; g_{-1}(x) = 0.$$

4767. *Proposed by J. L. Brenner, Stanford Research Institute*

Let a, b, d be integers, w real, $w \geqq 1+\sqrt{2}$, $d > wa > w^2 b > 0$, $a^2 - bd = 1$. Show that

$$A = \begin{pmatrix} a & b \\ d & a \end{pmatrix}, \qquad B = \begin{pmatrix} a & d \\ b & a \end{pmatrix}$$

generate a free group.

SOLUTIONS

Irreducible Polynomials

4709 [1956, 669]. *Proposed by R. W. Marsh, Arlington, Va., and A. M. Gleason, Harvard University*

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial with coefficients in the finite field of q elements. Suppose that the roots of f are primitive; that is, they generate the multiplicative group of their extension field. Prove that

$$F(x) = x^{q^n - 1} + a_{n-1} x^{q^{n-1}-1} + \cdots + a_0$$

is irreducible.

Solution by Leonard Carlitz, Duke University. This result is due to Ore, *Contributions to the Theory of Finite Fields*, Trans. Amer. Math. Soc., vol. 36, 1934, p. 260, Theorem 1. An outline of the proof follows. In view of the hypothesis on $f(x)$, q^n-1 is the smallest positive exponent such that

$$x^{q^n-1} - 1 \equiv 0 \pmod{f(x)};$$

this implies that the smallest positive exponent s such that

$$x^{q^s} - x \equiv 0 \pmod{F(x)}$$

is given by $s=q^n-1$. Consequently $F(x)$ is irreducible.

Also solved by the proposers.

Pascal Hexagon and Cubic

4718 [1957, 49]. *Proposed by V. F. Ivanoff, San Carlos, California*

The six points of intersection of a conic and a cubic determine a Pascal hexagon. Show that the residual six points of intersection of the sides of the hexagon with the cubic form two collinear sets, and the lines determined by these sets meet on the Pascal line.

Solution by O. J. Ramler, Catholic University of America. Let the conic Γ cut the cubic Σ in the six points A, B, C, D, E, F which we take as a simple (Pascal) hexagon. Let AB meet DE in L, BC meet EF in M, CD meet FA in N. Then LMN is the Pascal line for this hexagon. Let the sides AB, CD, EF BC, DE, FA meet the cubic again in X, Y, Z, X', Y', Z', respectively. For brevity, let $\Sigma=0$, $\Gamma=0$, $(AB)=0$ represent the equations of the cubic, the conic, and the line AB.

We observe that $\Sigma+k(AB)(CD)(EF)=0$ is the equation of a system of cubics through the nine points A, B, C, D, E, F, X, Y, Z. Since six of these points are on the conic Γ, it follows that the remaining three, X, Y, Z, are collinear. A similar procedure shows X', Y', Z' collinear. Now consider the equation

$$(AB)(CD)(EF)(X'Y'Z') + k(BC)(DE)(FA)(XYZ) = 0,$$

which represents a pencil of quartics through the sixteen points A, B, C, D, E, F, X, Y, Z, X', Y', Z', L, M, N, P, where P is the intersection of lines XYZ and $X'Y'Z'$. However, the first twelve of these points are on the cubic Σ, hence the remaining four, L, M, N, P, are collinear, *i.e.* the lines XYZ, $X'Y'Z'$, LMN are concurrent as required.

Also solved by J. Basile, R. Deaux, R. C. Lyness, and E. J. F. Primrose.

A Determinant of Legendre Symbols

4719 [1957, 49]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2. Show that the determinant of order $p-1$

$$\Delta_p = \left| \left(\frac{r-s}{p} \right) \right| \qquad (r, s = 0, 1, \cdots, p-2),$$

where (r/p) is the Legendre symbol, satisfies $\Delta_p = p^{(p-3)/2}$.

Solution by the proposer. The well known formula for a circulant

$$\left| x_{r-s} \right|_{r,s=0,\cdots,p-1} = \prod_{s=0}^{p-1} \sum_{r=0}^{p-1} \epsilon^{rs} x_r \quad (\epsilon = e^{2\pi i/p})$$

yields, when $\sum x_r = 0$,

$$\left| x_{r-s} \right|_{r,s=0,\cdots,p-2} = \frac{1}{p} \prod_{s=1}^{p-1} \sum_{r=0}^{p-1} \epsilon^{rs} x_r. \tag{1}$$

Now take $x_r = (r/p)$; then (1) becomes

$$\Delta_p = \frac{1}{p} \prod_{s=1}^{p-1} \sum_{r=1}^{p-1} \left(\frac{r}{p} \right) \epsilon^{rs}. \tag{2}$$

Next we recall that the Gauss sum $G(s)$ satisfies

$$G(s) = \prod_{r=1}^{p-1} \left(\frac{r}{p} \right) \epsilon^{rs}, \qquad G(s) = \left(\frac{s}{p} \right) G(1), \qquad G^2(s) = \left(\frac{-1}{p} \right) p.$$

Thus (2) gives

$$\begin{aligned} \Delta_p &= \frac{1}{p} \prod_{s=1}^{p-1} G(s) = \frac{1}{p} \prod_{s=1}^{p-1} \left\{ \left(\frac{s}{p} \right) G(1) \right\} \\ &= \frac{1}{p} (-1)^{(p-1)/2} \left\{ \left(\frac{-1}{p} \right) p \right\}^{(p-1)/2} \\ &= \frac{1}{p} \left(\frac{-1}{p} \right) \cdot \left(\frac{-1}{p} \right) p^{(p-1)/2} = p^{(p-3)/2}. \end{aligned}$$

Also solved by N. J. Fine.

Coefficients in a Power Series

4720 [1957, 49]. *Proposed by S. W. Golomb, University of Oslo, Norway*

Show that in the power series expansion

$$\frac{1}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < 1,$$

the coefficients are given by $a_n = \sum_{k=0}^{n} f_k^2$, where $f_0 = 1, f_1 = 1, f_2 = 2, \cdots$ is the Fibonacci sequence.

Solution by Peter Henrici, University of California, Los Angeles. The desired conclusion is equivalent to

$$\frac{1-x}{1-2x-2x^2+x^3}=\sum_{n=0}^{\infty} f_n^2 x^n$$

or, multiplying by the denominator and comparing coefficients, $(f_{-1}=f_{-2}=\cdots=0)$,

$$f_n^2-2f_{n-1}^2-2f_{n-2}^2+f_{n-3}^2=\begin{cases}1, & n=0,\\ -1, & n=1,\\ 0, & n\geqq 2.\end{cases} \tag{1}$$

This relation is evident for $n=0, 1$; to prove it for $n\geqq 2$, we have

$$\begin{aligned} f_n^2 &= (f_{n-1}+f_{n-2})^2 = 2f_{n-1}^2+2f_{n-2}^2-(f_{n-1}-f_{n-2})^2\\ &= 2f_{n-1}^2+2f_{n-2}^2-f_{n-3}^2,\end{aligned}$$

as desired.

Note that the roots of $1-2x-2x^2+x^3=0$ are $1, \frac{1}{2}(3\pm\sqrt{5})$, so that the radius of convergence is $\frac{1}{2}(3-\sqrt{5})<1$.

Also solved by J. L. Alperin, Anders Bager, J. D. Baum and Samuel Goldberg, Louis Brand, Robert Breusch, W. E. Briggs, P. L. Chessin, Charles Conley, M. Delcourt, J. S. Denton, Jr., P. L. Duren, N. J. Fine, M. F. Friedell, H. E. Goheen, Cornelius Groenewoud, Emil Grosswald, Joseph Hershenov, L. N. Howard, Richard Kelisky, M. S. Klamkin, R. C. Lyness, D. C. B. Marsh, Yoshio Matsuoka, Norman Miller, J. B. Muskat, Živadin Pantič, F. D. Parker, B. I. Penkov, R. S. Pinkham, R. C. Read, L. A. Ringenberg, D. A. Robinson, Lawrence Shepp, P. Somanadham, J. R. Trollope, Chih-yi Wang, D. G. Wertheim, L. K. Williams, David Zeitlin, and the proposer.

A Function Summable over the Entire Plane

4721 [1957, 49]. *Proposed by D. J. Newman, AVCO Research and Advanced Development, Lawrence, Mass.*

Let $u(x, y)$ be continuous and summable $(\int\int |u| dxdy<\infty)$ over the entire plane. Suppose the line integral, $\int_L u(x, y)ds$, vanishes for all straight lines L infinite in both directions. Prove that $u(x, y)$ is identically zero.

Solution by the proposer. We prove $u(0, 0)=0$. By the symmetry of the problem this will suffice. First of all, we have

$$S_m=\int u(x, y)ds=0,$$

where the integration is over the line $y\cos\theta=x\sin\theta+m$, so that

$$\int_{-\infty}^{\infty} e^{-Nm^2}S_m dm=0.$$

Now we have, writing this as a double integral,

$$\iint_{\text{plane}} e^{-N(y\cos\theta - x\sin\theta)^2} u(x, y)\,dxdy = 0.$$

Next, integrate with respect to θ from 0 to 2π. Upon interchanging the order we have

$$\iint_{\text{plane}} u(x, y)F_N dxdy = 0, \qquad F_N = N\int_0^{2\pi} e^{-N(y\cos\theta - x\sin\theta)^2} d\theta.$$

But the integral F_N is equal to $2\pi N e^{-Nr^2/2} I_0(Nr^2)$, where I_0 is the Bessel function and $r = (x^2+y^2)^{1/2}$.

Now we employ the δ-function-like behavior of F_N, namely

$$\iint_{\text{plane}} F_N = C \neq 0; \qquad \iint_{r>\delta} F_N \to 0, \qquad N \to \infty.$$

It follows immediately that

$$\iint_{\text{plane}} u(x, y)F_N dxdy \to C\cdot u(0, 0).$$

Since this integral is identically zero, it follows that $u(0, 0) = 0$.

Editorial Note. As remarked by B. I. Penkov, the theorem of the problem is a generalization of a theorem of Radon and is stated and solved in A. Renyi, *On projections of probability distributions*, *Acta Math. Acad. Sci. Hungar.*, vol. 3, 1952, fasc. 3, pp. 131–141. See also, *Mathematical Reviews*, vol. 15, 1954, p. 139.

Hölder's Inequality

4722 [1957, 49]. *Proposed by R. C. Warner, Toronto, Canada*

Establish the following inequality

$$\left\{\sum_{i=1}^{n}\prod_{j=1}^{r} a_{ij}^{m/r}\right\}^{r} \leqq \prod_{j=1}^{r}\sum_{i=1}^{n} a_{ij}^{m}.$$

Editorial Note. As pointed out by Chih-yi Wang and O. Mourmaki, this is the special case of Hölder's inequality in which every exponent has been made equal to $1/r$. There is no loss in generality in taking $m=1$. For proofs and a number of generalizations see Hardy, Littlewood and Pólya, *Inequalities*, Cambridge, 1934, pp. 21–26.

Also solved by Robert Breusch, N. J. Fine, and the proposer.

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Studies in Differential Equations. Ed. by Harold T. Davis, Northwestern University, 1956. 114 pp. $1.75.

Studies in Differential Equations is a collection of papers by Harold T. Davis, George Springer, W. T. Scott, and Daniel Resch. The first half of the study is concerned with certain special ordinary differential equations of the second order. The second half of the study deals with the general topic of Baecklund Transformations.

In the first part, the nonlinear equation

$$A(y)\frac{d^2y}{dx^2} + B(y)\frac{dy}{dx} + C(y)\left(\frac{dy}{dx}\right)^2 + D(y) = 0 \tag{1}$$

is considered. Special cases of this equation are the Van der Pol Equation and the Volterra Equation. After a general survey of the field we are introduced to the six Painlevé Equations which are special cases of equation (1). The simplest of the Painlevé Equations is the equation

$$\frac{d^2y}{dx^2} = \sigma y^2 + \lambda x. \tag{2}$$

Solutions of the Painlevé Equations are called Painlevé Transcendents. Much of this section of the book is devoted to the study of Painlevé Transcendents. Numerical solutions are obtained for a number of different values of the parameters involved. While the topic of Painlevé Transcendents deals with certain special cases of equation (1), the section dealing with linear fractional transformations of the dependent variable in (1) is of general validity. A number of theorems dealing with invariant properties of (1) under linear fractional transformations are interesting.

The second part of the volume deals with the use of Baecklund Transformations in partial differential equations. A Baecklund Transformation is defined by the four equations

$$F_i(x, y, z, z_x, z_y, x', y', z', z'_{x'}, z_{1\cdot}) = 0,$$

where z and z' are unknown functions of (x, y) and (x', y') respectively. If this system can be solved, it defines not only a surface Σ in (x, y, z)-space and a surface Σ' in (x', y', z')-space, but also a correspondence (but not usually a one-one mapping) between the two. This transformation is called a Baecklund Transformation. The notion of a Baecklund Transformation can be extended to higher dimensions. In this volume, George Springer investigates the Baecklund

Transformations of a certain type under which a system of first order partial differential equations remains invariant. W. T. Scott establishes the existence, under suitable conditions, of linear Baecklund Transformations linking systems of second order equations. Daniel Resch investigates Baecklund Transformations which will link second order partial differential equations in three independent variables. It is shown in particular that if the Baecklund Transformation is linear and in z, z', z_x, z_y, $x'_{x'}$, $z'_{y'}$, then second order equations can be linked to the wave equations.

As can be seen from the description, the authors deal with a variety of topics of interest. The "*Study*" should be of use to individuals who work in either the pure or the applied branches of differential equations.

F. HAAS
Wayne State University

Mathematical Logic. By R. L. Goodstein. Leicester University, 1957. viii+104 pp. 21/-.

"The aim of this little book is to introduce teachers of mathematics to some of the remarkable results which have been obtained in mathematical logic during the past twenty-five years." Probably the most interesting of these results are the Gödel completeness and incompleteness theorems and some of their consequences. The book is essentially self-contained, and the first two chapters outline the necessary background in the sentence and predicate calculus ending with a detailed proof (Henkin's) of the Gödel completeness theorem for the predicate calculus. Many other topics, such as Intuitionistic logic, bracket-free notation, are briefly treated. In every case the author illustrates the ideas involved by one or more examples. Chapter III introduces formal number theory (Z) and discusses various facets of recursive function theory. At the end of this chapter, he begins the proof of the Gödel incompleteness theorem by proving that recursive functions are, essentially, expressible in Z. In the next chapter, Z plus its syntax is mapped into intuitive number theory via a Gödel numbering and, in quite reasonable detail, it is shown that appropriate parts of the syntax are taken into recursive functions. Assuming ω-consistency, Z is then proved incomplete. The concept of ω-consistency is further illustrated by constructing a nonstandard model for Z. The chapter closes with a proof that Z and predicate calculus do not have a decision procedure. In the last chapter a class logic—specifically Quine's *Mathematical Logic*—is described, and the sequence of definitions leading to a formalization of number theory in this system is given.

It is clear that the author has covered a great deal of interest in 100 pages. Of particular interest is the treatment of the incompleteness theorem, since "semi-popular" accounts often leave much to be desired in, for example, discussing the Gödel numbering. The mere fact that Z plus its syntax is countable is trivial; the important thing is the connection between recursive definitions and recursive functions made possible by a Gödel numbering. The author includes sufficient detail so that careful reading (and careful reading will probably

be required of its intended class of readers) puts such things in evidence. However, on another level the book leaves some things to be desired. The book does not give much meaning to the particular undecidable sentence used (and it is probably too much to expect of the reader to supply this), or, more important, an overall guide to the Gödel theorem such as given in the introduction to Mostowski's *Sentences Undecidable in Formalized Arithmetic.* Similarly, the nonstandard model for Z may be disquieting in view of the general feeling that the Peano Axioms are categorical,—it would have been easy to point out that this depends on the relatively narrow class of sentences allowed for induction in Z. However, such remarks can not alter the fact that the author has written a commendable and worthwhile book.

Misprints appear to be very scarce and most of them are minor. However, the reviewer is convinced that the definition of $Sb(e, t, x)$ on page 76 is incorrect, —for example, instead of $e=2^{13}\cdot 3^n \rightarrow Sb(e, t, x)=2^{13}\cdot 3^{Sb(n,t,x)}$ it should be $e=2^{13}\wedge B(n)\rightarrow Sb(e, t, x)=2^{13}\wedge B(Sb(n, t, x))$. Analogous remarks apply to the definition of $t(n)$. (Reason: the Gödel numbering used assigns odd integers to signs, even integers whose prime power factorization uses odd exponents to sequences of signs or sentences, and even integers with even powers for the primes to sequences of sentences.)

J. B. GIEVER
The University of Oklahoma

Advanced Calculus. By R. Creighton Buck. McGraw-Hill, New York, 1956. vi+423 pages. $8.50.

The work under review may be described briefly, and perhaps with fair accuracy, by saying that it is rigorous, carefully composed, easy reading for the most part, more unified than the majority of texts on the subject, modernistic, and advanced in viewpoint. These last two characteristics are major marks of distinction and give the book a definite individualistic character.

A fair proportion of the material and methods presented are drawn from fields that are either relatively new or are currently in vogue for study and research. For example, a variety of topological concepts are introduced, and there is heavy emphasis on mappings and transformations of various kinds. In addition, there is included an exposition of the algebra and calculus of Cartan's exterior differential forms together with applications. Thus the book has more twentieth century overtones than most treatments of advanced calculus, and it is in this sense that it may be said to be modernistic. The material on the various meanings for the symbols *dx etc.* in differential and integral calculus, together with the discussions on change of variables in integrals, should be a good antidote for the semantic difficulties inherent in the archaic use of the notation in some scientific writings.

This text is advanced with regard to viewpoints, in the sense that the language and formulations used in certain fundamental definitions, while not abstruse and recondite, emphasize aspects which are merely implicit in the usual

statements. The superiority of some of these departures from the traditional is immediately apparent, as witness:

"**Definition.** *An infinite series of real numbers is a pair of real sequences* $\{a_n\}$ *and* $\{A_n\}$ *whose terms are connected by the relations*:

$$A_n = \sum_1^n a_k = a_1 + a_2 + \cdots + a_n \tag{4–1}$$

$$a_1 = A_1, \qquad a_n = A_n - A_{n-1} \qquad n \geqq 2.\text{" (p. 105).}$$

In other cases class-room discussions may be necessary for clarification or to bring out the advantages of the formulation given, as for example:

"**Definition.** *A curve* γ *in n-space is a mapping or transformation from* E^1 *to* E^n." (p. 251.)

"**Definition.** *Let* f *be any function of class* C' *in a region* D *of n-space. Then, the differential of* f *at a point* $p \in D$ *is the linear function* L *of* n *variables specified by the matrix* $[f_1(p), f_2(p), \cdots, f_n(p)]$.

When we write $w=f(x, y)$, for example, the differential of f is given by $[\partial w/\partial x, \partial w/\partial y]$, and its value at a point $(\Delta x, \Delta y)$ is $(\partial w/\partial x)\Delta x+(\partial w/\partial y)\Delta y$." (p. 184.)

In one particular instance, it seems to the reviewer that a change in the order of the material presented would be desirable. On pages one and two the dot product is introduced as the inner product of two points $p(x_1, \cdots, x_n)$, $q(y_1, \cdots, y_n)$ in the form

$$p \cdot q = x_1y_1 + \cdots + x_ny_n,$$

without mention of the nature of the coordinate system or the question of invariance, and on page 328 it is stated "the inner product of two points, p and q is an algebraic function of their coordinates whose value is unchanged if the underlying 3-space is subjected to any orthogonal linear transformation (rotation) resulting in new coordinates for p and q."

To sum up, this is not just an ordinary textbook presenting the usual material in the usual way, but one that is fresh and modern in many respects. It should prove to be a very useful and welcome addition to the literature on its subject.

HOMER V. CRAIG
The University of Texas

BRIEF MENTION

The Logical Problem of Induction (2nd Edition). By G. H. Von Wright. Macmillan, New York, 1957. xii+249 pp. $4.00.

This revision of Von Wright's earlier (1941) dissertation on Hume's problem will be of interest to students of the traditional problem of the justification of induction.

Elementary Theory of Angular Momentum. By M. E. Rose. Wiley, New York, 1957. x+248 pp. $10.

This clear presentation of the modern theory of angular momentum based on quantum mechanics is the outgrowth of a series of lectures given at Oak Ridge National Laboratory.

Experimental Designs (2nd Edition). By William G. Cochran and Gertrude M. Cox. Wiley, New York, 1957. xiv+617 pp. $10.25.

It is always encouraging to find a second edition of a nonelementary text. This presentation of statistical experimental design should be welcomed by scientists in many fields.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics of the National Academy of Sciences—National Research Council wishes to call attention to the fact that several foundations and offices will offer financial support for research in mathematics during the year 1958–59. A number of fellowships will be made available, as well as opportunities for mathematicians to engage in basic research. A partial list, with comments, is given below.

1. *National Science Foundation. Fellowships.*

Predoctoral fellowships are awarded annually at the First Year, Intermediate, and Terminal Year levels of graduate study. Applications for 1958–1959 are available from the National Academy of Sciences—National Research Council until the closing date in early January 1958; award date—March 15, 1958.

Science Faculty fellowships for college science teachers (including mathematics) who plan to continue teaching and wish to increase their competence as teachers are at the present time offered semi-annually. Eligibility requirements include a baccalaureate degree and three years of full-time experience in teaching natural science subjects at the collegiate level. The program was opened to application in October 1957 and will be closed early in January 1958. Awards will be made on March 20, 1958. The program will also be reopened the following summer. Address all inquiries for information and applications to National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C.

Regular postdoctoral fellowships, primarily for recent recipients of the doctoral degree, are awarded semi-annually. Program for 1958–1959 is concurrent with above predoctoral program except that program closes in December. Information and applications are available from NAS-NRC. The program will also be open from July to early September 1958. Awards are announced in March and October.

Senior postdoctoral fellowships, which are open to persons who have held a doctoral degree in one of the basic fields of science for a minimum of five years at time of application, or who have had equivalent training and experience, are awarded semi-annually. Applications are available from the National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C. The program is open from October 1957 until January 1958. Awards will be announced on March 18, 1958. The program will be reopened in the summer of 1958.

Research Grants. The National Science Foundation also supports basic research in the mathematical sciences by means of grants. While proposals for such support are accepted at any time, individuals desiring support to begin in the summer or at the beginning of a fall semester should preferably submit their proposals in the mathematical sciences by November 1; persons desiring support to begin in the spring semester should preferably submit their proposals by May 1. Instructions for the preparation of proposals, contained in a booklet entitled Grants for Scientific Research, may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foundation, 1520 H Street, N. W., Washington 25, D. C.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are being established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations and industrial laboratories. Such organizations are encouraged to submit proposals for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn: Mathematics Division, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army.* Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress).* Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Australia, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines, Thailand (competition for the preceding countries closes April 15, 1958); Austria, Belgium-Luxembourg, Denmark, Finland, France, Germany, Greece, Iceland, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Turkey, and the United Kingdom including colonial dependencies (competition for the latter countries closes October 1, 1958). In both cases awards are for the academic year 1959–60, but in the former group of countries the academic year begins in the spring or summer instead of the autumn. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards; Naval Research Laboratory; Oak Ridge National Laboratory.* Postdoctoral resident research associateships are available in a variety of sciences including mathematics, and are tenable at the Washington, D. C. and Boulder, Colorado laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; and at the Oak Ridge National Laboratory in Oak Ridge, Tennessee. Necessary facilities and equipment incident to the research of the Associate will be provided. For further information write to Fellowship Office, National Academy of Sciences-National Research Council, 2101 Constitution Avenue, Washington 25, D. C. Applications for the 1958–59 program must be filed on or before January 13, 1958.

7. *Atomic Energy Commission.* The Division of Research of the Atomic Energy Commission through contracts with universities and other organizations supports research in the fields of numerical analysis, digital computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, 1901 Constitution Avenue, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc. under contract with the Atomic Energy Commission offers postdoctoral research appointments in mathematics. Appointments are for one year, and may be renewed for one additional year. U. S. citizenship is not required, although Atomic Energy Commission approval is a prerequisite. The appointee may work in numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches including reactor theory, hydrodynamics, and orbit theory. Computational facilities are available. Letters from candidates should give details of personal history, scientific background, and qualifications; two letters of recommendation, one from the applicant's research professor, are required. Applications for the academic year 1958–59 must be received by August 15, 1958 and should be directed to M. E. Rose, Head, Applied Mathematics Division, Brookhaven National Laboratory, Upton, Long Island, New York.

PERSONAL ITEMS

Professor G. B. Price, University of Kansas, represented the Association at the Fortieth Annual Meeting of the American Council on Education in Washington, D. C., on October 10–11, 1957.

Boston College: A Workshop for High School Teachers of Mathematics was conducted during the summer; the speakers included Dean A. E. Meder, Jr., J. B. Adkins, John Kemeny, Max Beberman, and Rev. S. J. Bezuszka.

Butler University: Capt. F. A. Graf (USN, Ret.), formerly Director of the Naval Observatory, and Miss Joyce Cimelus, Statistics Laboratory, Purdue University, have been appointed Instructors.

Florida State University: Dr. M. B. Smith, Jr., University of North Carolina, has been appointed Assistant Professor; Mr. J. F. Brooks and Mr. G. W. Polites, Graduate Assistants at the University, have been appointed Instructors; Assistant Professors J. W. Ellis and H. C. Griffith have been promoted to Associate Professors.

Fordham University: Mr. N. C. Mitrowsis, Graduate Assistant, University of South Carolina, has been appointed Instructor; Assistant Professor W. T. Shields has retired.

Georgetown University: Mr. Henry Beiman, University of Maryland, has been appointed Instructor; Mr. J. E. Houle has been promoted to Assistant Professor; Rev. F. W. Sohon has retired with the title Professor Emeritus.

Hampton Institute: Assistant Professor Rosalind Eagleson has been promoted to Associate Professor; Miss Harriett R. Junior, Assistant, Syracuse University, has returned to the Institute.

Indiana University: Assistant Professor Louis Auslander, University of Pennsylvania, and Dr. Werner Gautschi, Ohio State University, have been appointed Assistant Professors; Dr. Avner Friedman, Research Associate, University of Kansas, has been appointed Lecturer; Professor Ratip Berker, Technical Institute of Istanbul, has been appointed Visiting Professor for one semester.

Iowa State College: Dr. M. F. Ruchte, Yale University, has been appointed Assistant Professor; Mrs. William Proett, Assistant, University of Texas, has been appointed Instructor.

Kansas State College: Dr. B. G. Pearson, Illinois Institute of Technology, and Assistant Professor Evelyn Kinney, Illinois Wesleyan University, have been appointed Assistant Professors; Miss Lorraine Schwartz, Graduate Student, University of California, has been appointed Instructor; Assistant Professors A. M. Feyerherm, and W. L. Stamey have been promoted to Associate Professors.

Kent State University: Mr. Kenneth L. Cummins, Teacher, New Washington High School, Ohio, has been appointed Assistant Professor; Miss Grace L. Abhau and Mr. R. F. Liskovec have been appointed Instructors.

McMaster University: Dr. J. H. H. Chalk, Lecturer, Bedford College, University of London, has been appointed Assistant Professor; Assistant Professors F. R. Britton, Bernard Banaschewski, and N. D. Lane have been promoted to Associate Professors.

New Mexico College of Agriculture and Mechanic Arts: Assistant Professor E. A. Walker, University of Kansas, and Dr. K. R. Lucas, Graduate Student, University of Kansas, have been appointed Assistant Professors; Assistant Professor R. M. Conkling has been promoted to Associate Professor.

Princeton University: Associate Professor E. A. Coddington, University of California, Los Angeles, has been appointed Visiting Associate Professor; Dr. J. P. Roth, International Business Machines Corporation, Poughkeepsie, New York, has been appointed Visiting Assistant Professor; Dr. G. O. Losey, Teaching Fellow, University of Michigan, Dr. R. D. McWilliams, University of Tennessee, and Dr. James Munkres, University of Michigan, have been appointed Instructors; Dr. Ralph Gomory, Office of Naval Research, Washington, D. C., has been appointed Higgins Lecturer; Assistant Professor I. P. Guttman, who is on leave of absence from the University of Alberta, has been appointed Research Associate; Associate Professor Valentin Barmann has been promoted to Professor; Assistant Professor John Milnor and John Moore have been promoted to Associate Professors.

State University of Iowa: Associate Professor Byron Cosby, Jr., who was on leave of absence during 1956–57, has returned; Professor Emeritus E. W. Chittenden has been granted a leave of absence for 1957–58 in order to continue his appointment as mathematician in the Diamond Fuse Laboratory, Washington, D. C.; Professor Emeritus C. C. Wylie has been granted a leave of absence for the academic year 1957–58 to accept an appointment at Park College.

Stevens Institute of Technology: Dr. Lawrence Goldman, Columbia University, has been appointed Assistant Professor; Mr. Henry Polowy and Mr. M. E. White have been promoted to Assistant Professors.

Technical Operations, Monterey, California: Mr. W. S. Cox, Mr. W. J. Harrington, and Miss Eleanor C. Shlifer have been appointed Operations Analysts.

United States Naval Academy: Mr. R. E. Walters, University of Louisville, has been appointed Assistant Professor; Associate Professors E. E. Betz and S. S. Saslaw have been promoted to Professors; Assistant Professors M. V. Gibbons, E. C. Gras, and W. J. Strange have been promoted to Associate Professors.

University of Alabama: Assistant Professors J. H. Hornback, B. M. Seelbinder, H. C. Filgo, and C. C. Buck have been promoted to Associate Professors.

University of Alberta: Dr. A. P. Guinand, University of New England, Australia,

Dr. S. S. Gupta, Bell Telephone Laboratories, Pennsylvania, and University of North Carolina, and Dr. H. F. J. Lowig, University of Tasmania, have been appointed Associate Professors; Dr. I. N. Baker, Assistant Professor G. C. Cree, University of Nebraska, and Dr. E. L. Whitney, Operational Research Group, Ottawa, have been appointed Assistant Professors; Miss Z. M. Hyduk, Graduate Student at the University, has been appointed Sessional Lecturer; Assistant Professor H. Helfenstein, who is on leave of absence for 1957–58, is a staff member of the University of Ottawa.

University of Arizona: Assistant Professor Fay Farnum, Iowa State College, has been appointed Assistant Professor; Mr. H. L. Hancock, E. B. Hoff, G. M. Jones, J. D. Simley, and A. H. Tellez have been appointed Instructors; Mr. J. E. Lee, Mr. J. E. Strang, and Mrs. E. Virginia Prebula have been appointed Teaching Assistants; Assistant Professor Deonisie Trifan has been promoted to Associate Professor.

University of Arkansas: Dr. J. E. Scroggs has been appointed Assistant Professor; Miss Marion Brashears has been appointed Instructor; Assistant Professor W. R. Orton has been promoted to Associate Professor.

University of British Columbia: Dr. D. W. Bressler, Graduate Student, University of California, and Dr. C. A. Swanson, Teaching Assistant, California Institute of Technology, have been appointed Instructors; Miss Charlotte Froese, and Assistant Professor R. R. Struik, Drexel Institute of Technology, have been appointed Lecturers; Dr. R. R. Christian and Dr. H. F. Davis have been promoted to Assistant Professors.

University of Colorado: Dr. Irwin Fischer, Dartmouth College, has been appointed Assistant Professor; Mrs. Dina G. Thomas, Part-time Lecturer, Boston University, and Part-time Instructor, Suffolk University, has been appointed Instructor; Associate Professor W. J. Thron has been promoted to Professor; Dr. Aboulghassem Zirakzadeh has been promoted to Assistant Professor; Professor Claribel Kendall has retired with the title Professor Emeritus; Professor W. W. Rogosinski, King's College, University of Durham, England, is replacing Professor Thron, who is on leave of absence at the University of Munich, Germany, under an Air Force contract; Professor Sigmund Selberg, Institute of Technology, Trondheim, Norway, is replacing Professor Sarvadaman Chowla, who is on leave of absence at the Institute for Advanced Study.

University of Connecticut: Mrs. Helen G. Brown has been appointed Instructor; Assistant Professor C. G. A. Nordling has been promoted to Associate Professor.

University of Oklahoma: Assistant Professors R. C. Dragoo and J. B. Giever have been promoted to Associate Professors; Professor Casper Goffman is on leave for one year as Visiting Professor at Purdue University.

University of Oregon: Associate Professor F. C. Andrews, University of Nebraska, has been appointed Associate Professor; Assistant Professor F. W. Anderson, University of Nebraska, has been appointed Assistant Professor; Mr. J. A. Dubay, Graduate Student, University of Chicago, has been appointed Instructor; Associate Professor Paul Civin has been promoted to Professor; Dr. L. W. Anderson has been promoted to Assistant Professor.

University of Pennsylvania: Dr. Morikuni Goto, Tulane University, has been appointed Assistant Professor; Dr. Robert Ellis, Pennsylvania State University, has been appointed Visiting Assistant Professor; Dr. Mandakini Rohatgi has been appointed Instructor; Dr. J. J. Price has been appointed Lecturer; Associate Professor Bernard Epstein is on leave during 1957–58 at Stanford University; Assistant Professor Murray Gerstenhaber is on leave during 1957–58 at the Institute for Advanced Study.

University of Saskatchewan: Visiting Associate Professor O. P. Aggarwal, University of Alberta, has been appointed Special Lecturer; Professor Peter Scherk who served as Visiting Professor, University of Pennsylvania, during 1956–57, has returned.

University of South Carolina: Assistant Professor R. Z. Vause, University of Kansas City, has been appointed Assistant Professor; Professor W. L. Williams has been elected a Fellow in the American Association for the Advancement of Science.

University of Tennessee: Mrs. Mary F. Dubose and Mr. R. P. Savage have been appointed Instructors; Professor A. A. Grau, University of Oklahoma, and Professor Ky Fan, University of Notre Dame, have been appointed Lecturers; Mr. P. H. Doyle has been promoted to Acting Assistant Professor; Professor O. G. Harrold, Jr., has received a Guggenheim Fellowship and is on leave of absence at the Institute for Advanced Study and Oxford University, England.

University of Texas: Dr. Alfred Schild, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, has been appointed Professor; Associate Professor R. E. Greenwood has been promoted to Professor; Dr. R. C. Osborn and Dr. Richard Kelisky have been promoted to Assistant Professors.

University of Tulsa: Mr. W. B. Garrison, Douglas Aircraft Company, Tulsa, Oklahoma, has been appointed Assistant Professor; Mr. R. G. Laatsch, Graduate Assistant, University of Missouri, has been appointed Instructor.

University of Washington: Dr. H. J. Bremermann, Member, Institute for Advanced Study, Research Assistant Professor J. M. G. Fell of the University, Dr. J. R. Isbell, Tulane University, Assistant Professor J. P. Jans, Ohio State University, and Dr. R. A. Macauley, University of California, have been appointed Assistant Professors; Dr. H. S. Bear, Teaching Associate, University of California, Dr. J. M. Gonzalez-Fernandez, Teaching Assistant, Northwestern University, and Dr. August Newlander, Teaching Assistant, Institute of Mathematical Sciences, have been appointed Instructors; Mr. K. R. Stromberg, Research Assistant at the University, has been appointed Acting Instructor; Associate Professors V. L. Klee and D. G. Chapman have been promoted to Professors; Dr. R. M. Blumenthal has been promoted to Assistant Professor; Professor R. A. Beaumont is Acting Executive Officer during 1957–58; Professor M. G. Arsove has received a Fulbright Award and is on leave in Paris, France; Professor J. H. Walter has received a National Science Foundation Postdoctoral Fellowship and is spending the year at the Institute for Advanced Study.

University of Wisconsin: Mr. B. L. Foster, University of Washington, Dr. J. M. Osborn, Teaching Assistant, University of Connecticut, and Dr. E. C. Posner, Graduate Student, University of Chicago, have been appointed Instructors; Assistant Professor C. W. Curtis has been promoted to Associate Professor; Professor R. H. Bing is on leave during 1957–58 as Visiting Professor at the Institute for Advanced Study.

University of Wyoming: Miss Esther E. Guerin, Teacher, Roxbury Township High School, New Jersey, has been appointed Instructor; Assistant Professor W. N. Smith has been promoted to Associate Professor.

Western Illinois University: Dr. G. H. Miller has been appointed Associate Professor; Mr. R. L. Suhr has been appointed Instructor.

Western Michigan University: Assistant Professor H. F. Becksfort, Ohio University, and Mr. Stanislaw Leja, Cornell University, have been appointed Assistant Professors; Assistant Professor J. H. Powell has been promoted to Associate Professor.

West Virginia University: Mrs. Gladis Loehr, Arkansas Agricultural and Mechanical College, and Captain R. C. Strong (USN Ret.) have been appointed Instructors; Assistant Professor I. D. Peters has been promoted to Associate Professor; Mr. C. N. Cochran has been promoted to Assistant Professor.

Wisconsin State College, River Falls: Mr. R. W. Willson, State University of Iowa, has been appointed Instructor; Mr. G. D. Bisbey has been promoted to Assistant Professor.

Mr. Walter Abramowitz, Analytical Engineer, Pratt & Whitney Aircraft, East Hartford, Connecticut, has a position as Mathematician at the Control Instrument Company, Brooklyn, New York.

Dr. W. E. Barnes, State College of Washington, has been promoted to Assistant Professor.

Dr. J. C. Bradford, formerly Graduate Assistant, University of Oklahoma, has been appointed Assistant Professor, Abilene Christian College.

Dean W. H. Bradford, McNeese State College, has a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Dr. Barron Brainerd, University of British Columbia, has been appointed Assistant Professor at the University of Western Ontario.

Mr. F. C. Calabrese, Junior Development Engineer, Goodyear Aircraft Corporation, Akron, Ohio, has been appointed Associate Engineer, Special Engineering Products Division, International Business Machines Corporation, Poughkeepsie, New York.

Assistant Professor Jean M. Calloway, Carleton College, has been promoted to Associate Professor.

Professor E. L. Canfield, Drake University, has been appointed Dean of the Graduate Division.

Dr. D. I. Caplan, Staff Consultant, Burroughs Corporation, Philadelphia, Pennsylvania, has been appointed Chief Engineer, Navigation Computer Corporation, Philadelphia.

Assistant Professor K. H. Carlson, Valparaiso University, has been promoted to Associate Professor and Chairman of the Department of Mathematics.

Mr. C. R. Carr, Indiana Technical College, has been promoted to Associate Professor.

Assistant Professor W. L. Carter, University of Cincinnati, has been promoted to Associate Professor of Education.

Professor F. L. Celauro, East Tennessee State College, has been appointed Professor at Central Michigan College.

Mr. F. A. Ceney, Jr., Teacher, East Peoria Community High School, Illinois, is teaching at Hillsboro High School, Illinois.

Mr. D. F. Clapp, Graduate Student, University of Technology, Toronto, Ontario, has a position as a staff member at the Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts.

Professor C. L. Clark, Oregon State College, has been appointed Head of the Department of Mathematics, Los Angeles State College.

Mr. G. M. Clough, Mathematician, Battelle Memorial Institute, Columbus, Ohio, has a position as Engineer at Shell Pipe Line Corporation, Houston, Texas.

Mr. P. J. Cocuzza, Graduate Student, Brooklyn College, has a position as Junior Electrical Engineer for the City of New York, Department of Public Works.

Mr. W. J. Cody, Jr., Graduate Assistant at the University of Oklahoma, has been promoted to Instructor.

Mr. S. H. Cohn, Aerodynamicist, AVRO Aircraft Limited, Malton, Ontario, Canada, is now a senior computing specialist.

Mr. W. L. Congleton has a position as a member of the technical staff, Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. G. F. Cramer, Mathematician, Remington-Rand, New York, New York, is Consultant for IBM Research Laboratory, Poughkeepsie, New York.

Dr. T. H. M. Crampton, Mount Holyoke College, has been promoted to Assistant Professor.

Dr. Arno Cronheim, Brandeis University, has been appointed Instructor at Ohio State University.

Dr. C. H. Cunkle, Senior Research Engineer, Convair, San Diego, California, has been appointed Assistant Professor at Colorado State University.

Mr. C. F. Daniel is employed as Research Engineer by North American Aviation "Autonetics," Downey, California.

Dr. H. J. Dark, Chairman of the Department of Mathematics, David Lipscomb College, has been appointed Associate Professor at Middle Tennessee State College.

Associate Professor Marguerite D. Darkow, Hunter College, has been promoted to Professor.

Mr. H. J. Davis, Pomona College, has been appointed Instructor at Long Beach State College.

Mr. Sol. Davis, Mathematician, Naval Air Material Center, Philadelphia, Pennsylvania, has a position as Vibration Engineer, General Electric Company, Philadelphia.

Assistant Professor F. C. DeSua, University of Pittsburgh, has a position as a member of the technical staff, Bell Telephone Laboratories, Whippany, New Jersey.

Mr. R. E. Dowd, Garnac Grain Company, New York, New York, is now Propulsion Engineer, Grumman Aircraft Engineering Corporation, Bethpage, New York.

Mr. S. S. Draeger, Pan American College, has been promoted to Dean of the School of Technology.

Assistant Professor D. E. Edmondson, Southern Methodist University, has been promoted to Associate Professor.

Dr. J. L. Ericksen, Mathematician, Naval Research Laboratory, Washington, D. C., has been appointed Associate Professor of Theoretical Mechanics, Johns Hopkins University.

Mr. H. L. Farris, Mathematician, Douglas Aircraft Company, Tulsa, Oklahoma, has a position as Mathematician at Black, Sivalls, & Bryson, Tulsa.

Mr. G. F. Feeman, Muhlenberg College, is on leave of absence as Danforth Teacher, Danforth Foundation, St. Louis, Missouri.

Assistant Professor A. B. Finkelstein, Long Island University, has been appointed Associate Professor at Pratt Institute.

Associate Professor Walter Fleming, Mankato State Teachers College, has been appointed Professor and Head of the Department of Mathematics, Hamline University.

Dr. H. J. Fletcher, Brigham Young University, has been promoted to Associate Professor.

Mr. A. J. Flynn, Engineering Assistant, General Electric Company, Bloomington, Illinois, has a position as Computing Technician, Remington Rand UNIVAC, Chicago.

Dr. J. E. Forbes, Purdue University, has been promoted to Assistant Professor.

Mr. Allen Fox has accepted a position as Electrical Engineer, IBM Product Development Laboratory, Poughkeepsie, New York.

Mr. G. C. Fraser, Assistant Director, United Nations Program of American Friends Service Committee, New York, New York, has a position at the George School, Bucks County, Pennsylvania.

Assistant Professor N. S. Free, Rensselaer Polytechnic Institute, has been promoted to Associate Professor.

Dr. T. C. Fry, Retired Assistant to Director, Bell Telephone Laboratories, New York, New York, is Vice President, Sperry Rand Corporation, Remington Rand Division, Stamford, Connecticut.

Mr. B. A. Fusaro, Middlebury College, has been appointed Instructor, University of Maryland.

Mr. C. L. Gape, University of Buffalo, has been appointed Research Assistant, Syracuse University.

Mr. F. W. Gibson, Research Engineer, Radioplane Company, Van Nuys, California, is Engineer, Douglas Aircraft Company, Santa Monica, California.

Associate Professor David Gilbarg, Indiana University, has been appointed Professor at Stanford University.

Dr. Seymour Ginsburg, Research Analyst, Northrop Aircraft Corporation, Hawthorne, California, has a position as Senior Research Engineer at National Cash Register Company, Hawthorne.

Associate Professor A. M. Gleason, Harvard University, has been promoted to Professor.

Mr. W. H. Glenn, Jr., Assistant Principal, John Muir High School, Pasadena, California, has been appointed Assistant Coordinator, Mathematics, Pasadena City Schools.

Mr. Isidore Goldman, Laboratory Director, Weba, New Hyde Park, New York, is Project Engineer, American Machine and Foundry Company, Brooklyn, New York.

Mr. E. G. Goman, College of Puget Sound, has been promoted to Associate Professor and Chairman of the Department of Mathematics.

Assistant Professor W. T. Graybeal, Emory and Henry College, has been promoted to Associate Professor.

Assistant Professor L. J. Green, Case Institute of Technology, has been promoted to Associate Professor.

Mr. Arnold Grudin, Assistant, Syracuse University, has been appointed Assistant Professor, Denison University.

Mr. W. J. Harrold, Research Fellow, Polytechnic Institute of Brooklyn, has a position as Associate Engineer, Sperry Gyroscope Company, Great Neck, New York.

Professor G. E. Hay, University of Michigan, has been appointed Acting Chairman of the Department of Mathematics.

Professor Camilla Hayden, Chairman, Department of Mathematics, St. Mary-of-the-Woods College, has been appointed Assistant Professor at the University of Toledo.

Mr. L. A. Kenna, University of Arizona, has a position with the Radio Corporation of America, Tucson, Arizona.

Dr. John Killeen, New York University, has accepted a position as Mathematician, University of California Radiation Laboratory, Livermore, California.

Assistant Professor R. M. Kozelka, University of Nebraska, has been appointed Assistant Professor, Williams College.

Dr. W. W. Leutert, Manager, Mathematical and Computer Services Department, Lockheed Missile Systems Division, is Mathematician, Electronic Computer Applications Staff, Tidewater Oil Company, San Francisco, California.

Mr. J. L. Lund, Miramonte High School, Orinda, California, has a position as Supervisor, Special Program in Teacher Education, University of California, Berkeley.

Mr. D. C. McGarvey, Yale University, has accepted a position as Mathematician, RAND Corporation, Santa Monica, California.

Associate Professor W. P. Morse, Southeastern Louisiana College, has been appointed Associate Professor at the University of Florida.

Assistant Professor J. I. Northam, Kansas State College, is employed by the Upjohn Company, Kalamazoo, Michigan.

Mr. J. A. Riley, Boston College, has a position as Research Associate, Parke Mathematical Laboratories, Carlisle, Massachusetts.

Dr. G. B. Robison, University of Connecticut, has been appointed Associate Professor, State University of New York, Teachers College at New Paltz.

Mr. R. F. Shaw, Underwood Corporation, has a position as Vice President, Digitronics Corporation, Albertson, New York.

Assistant Professor H. D. Sprinkle, University of Arizona, has a position with the Radio Corporation of America, Tucson, Arizona.

Dr. S. L. Warner, Duke University, has been promoted to Assistant Professor.

Mr. Conrad White, Teacher, Kodaikanal School, South India, has been appointed Assistant Professor, Pacific University.

Mr. C. H. Wilcox, California Institute of Technology, has been promoted to Assistant Professor.

Dean W. E. Wilson, School of Engineering, Pratt Institute, has been named Acting President of the Institute.

Mr. B. H. Youell, West Virginia University, has been appointed Instructor, West Virginia Institute of Technology.

Mr. A. F. Bentley, Paoli, Indiana, died on May 21, 1957. He was a member of the Association for nine years.

Professor H. L. Black, North Dakota Agricultural College, died in February, 1957. He was a member of the Association for fifteen years.

President Emeritus E. O. Lovett, Rice Institute, died on August 13, 1957. He was a charter member of the Association.

Mr. C. C. Massimiano, Head, Department of Mathematics, Pittsfield High School, Massachusetts, died on March 1, 1957.

Lt. L. L. McCelvey, United States Air Force, died on August 6, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

TELEVISION AND FILMS

It is proposed to collect in one issue of the MONTHLY as many articles as possible on the use of television and of films in the teaching of mathematics. Articles intended for this special issue should be submitted to the Editor, Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada, as soon as possible and, in any event, by March 15, 1958. Since space is limited, articles should be concise and should not include unnecessarily elaborate statistical data.

ACKNOWLEDGEMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: C. B. Allendoerfer, R. P. Boas, J. L. Brenner, R. H. Bruck, W. B. Carver, Noam Chomsky, R. R. Christian, K. L. Chung, R. V. Churchill, Paul Civin, A. H. Clifford, Nathaniel Coburn, N. A. Court, H. F. Davis, J. A. Dieudonné, L. P. Edwards, F. A. Ficken, Tomlinson Fort, J. S. Frame, E. T. Frankel, Orrin Frink Jr., F. M. C. Goodspeed, H. W. Gould, R. T. Gregory, Marshall Hall, Jr., Edwin Hewitt, Alfred Horn, G. B. Huff, T. E. Hull, S. A. Jennings, P. S. Jones, J. L. Kelley, J. G. Kemeny, F. L. Kiokemeister, M. S. Klamkin, D. H. Lehmer, Eugene Leimanis, N. H. McCoy, M. D. Marcus, Karl Menger, W. E. Milne, Leo Moser, B. N. Moyls, D. C. Murdoch, S. W. Nash, W. V. Parker, Edmund Pinney, W. V. Quine, R. A. Rankin, R. A. Restrepo, D. E. Richmond, R. A. Rosenbaum, R. L. San Soucie, A. C. Schaeffer, Peter Scherk, M. F. Smiley, A. L. Whiteman, R. M. Winger, R. J. Wisner, Max Wyman, H. J. Zassenhaus.

Mathematical Notes: C. B. Allendoerfer, T. M. Apostol, B. H. Arnold, R. A. Beaumont, R. E. Bellman, D. W. Blackett, David Blackwell, R. P. Boas, Leonard Carlitz, W. B. Carver, Paul Civin, E. A. Coddington, K. W. Crain, Arthur Erdélyi, Ky Fan, William Feller, N. J. Fine, Bernard Friedman, Casper Goffman, Michael Golomb, P. R. Halmos, O. H. Hamilton, L. A. Henkin, Fritz Herzog, Edwin Hewitt, I. I. Hirschman, A. S. Householder, T. E. Hull, J. A. Jenkins, B. W. Jones, Mark Kac, V. L. Klee Jr., Gordon Latta, D. H. Lehmer, Joseph Lehner,

D. C. Lewis, A. E. Livingston, Charles Loewner, A. T. Lonseth, C. C. MacDuffee, Wilhelm Magnus, E. A. Maier, J. E. Maxfield, Margaret Maxfield, A. F. Moursund, Zeev Nehari, C. D. Olds, Sam Perlis, J. W. Peters, W. T. Reid, Herbert Robbins, J. B. Roberts, P. C. Rosenbloom, R. L. San Soucie, W. R. Scott, R. G. Selfridge, H. N. Shapiro, Wen-Hou Shia, C. E. Springer, D. J. Struik, Olga Taussky, G. B. Thomas, Jr., L. J. Warren, M. S. Webster, L. M. Weiner, Lionel Weiss, L. B. Williams.

Classroom Notes: C. B. Allendoerfer, R. W. Ball, P. T. Bateman, E. E. Betz, B. H. Bissinger, Tomlinson Fort, Samuel Goldberg, H. J. Hamilton, E. K. Haviland, R. T. Ives, R. C. James, F. B. Jones, Mark Kac, H. L. Krall, R. E. Langer, O. V. McBrien, E. M. Michalup, S. W. Nash, C. S. Ogilvy, R. C. Osborn, Francis Parker, Hans Rademacher, G. E. Raynor, D. E. Richmond, Daniel Shanks, L. J. Snell, T. H. Southard, W. L. Strother, H. S. Thurston, E. P. Vance, G. C. Vedova, R. M. Winger, R. J. Wisner, J. W. Woll.

The Editor is especially appreciative of the help given to him by the former editor, Carl Allendoerfer, and his able editorial assistant, Mrs. Helen Zuckerman, in making the transfer of editorial duties as smooth as possible. In addition, Mrs. Zuckerman undertook most of the work in connection with the publication of the two supplements to the MONTHLY in 1957.

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 30–31, 1958.

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 10, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D.C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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